SLE に関する話題

中央大学理工学部　香取眞理 (かとりまこと)

Ref. [1] W. Werner : Girsanov's transformation for
SLE(**κ**, **ρ**) processes, intersection exponents
and hiding exponents,
Ann.Toulouse, **13** (2004) 121-147.

[2] M. Yor : private communication (2007).

勉強会「Loewner 方程式とＳＬＥ」
２０１０年2月１２－１４日
東北大学・鳴子会館研究室

1. **SLE κ and SLE(κ, ρ)**

2. **One-Sided Restriction (片側制限性)**

3. **Application of Girsanov's transformation**

4. **Nonintersecting SLE8/3's (Nonintersecting SAW's)**

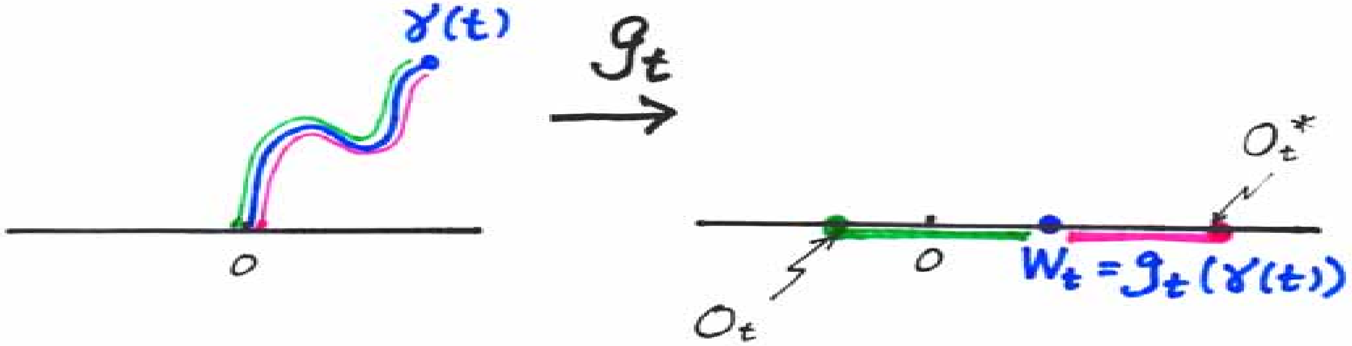
5. **Hitting Time Distribution**

1. SLE κ and SLE(κ , ρ)

Loewner equation with a driving function W_t

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z. \tag{1.1}$$

原点 $0 \in \mathbb{R} = \mathbb{H}$ の境界
原点の像 $O_t = g_t(0)$: "left"-image of 0 under g_t.



(1.1) で $z = 0$ と置くと

$$\frac{d}{dt} O_t = \frac{2}{O_t - W_t}. \tag{1.2}$$

$W_t \in \mathbb{R}$: $t \in [0, \infty)$ の実関数 (SLE の駆動関数) $\Longrightarrow$ 実軸上の 1 粒子の位置
O_t : 実軸上の原点 0 も時間的に変動する.
相対座標 $W_t - O_t$ を考える

$$d(W_t - O_t) \;=\; dW_t - dO_t = dW_t + \frac{2}{W_t - O_t} dt.$$

Loewner eqaution $\Longrightarrow$ SLE_κ: Put $W_t = \sqrt{\kappa} B_t = B_{\kappa t}$, $\kappa > 0$.

$$d(W_t - O_t) = \sqrt{\kappa} dB_t + \frac{2}{W_t - O_t} dt = dB_{\kappa t} + \frac{2}{\kappa} \frac{1}{W_t - O_t} d(\kappa t).$$

$$\text{よって} \quad W_t - O_t = R^{(\nu_0)}_{\kappa t} = \sqrt{\kappa} R^{(\nu_0)}_t. \tag{1.3}$$

ここで, $R^{(\nu)}_t$: index ν の **Bessel 過程**:

$$dR^{(\nu)}_t = dB_t + \left(\nu + \frac{1}{2}\right) \frac{dt}{R^{(\nu)}_t} = dB_t + \frac{d-1}{2} \frac{dt}{R^{(\nu)}_t},$$

$$d = 2(\nu + 1), \qquad \nu = \frac{d-2}{2}. \tag{1.4}$$

ただし (1.3) で

$$\nu_0 + \frac{1}{2} = \frac{2}{\kappa} \quad \Longleftrightarrow \quad \nu_0 = \frac{4-\kappa}{2\kappa}$$

$$\Longleftrightarrow \quad d_0 = 2(\nu_0 + 1) = \frac{4+\kappa}{\kappa}. \tag{1.5}$$

SLE_κ は (Brown 運動の時間変更 $B_{\kappa t} = \sqrt{\kappa} B_t$ ではなく)
index $\dfrac{4-\kappa}{2\kappa}$ $\left(\dfrac{4+\kappa}{\kappa}\text{次元}\right)$ の Bessel 過程の時間変更 $(t \to \kappa t)$ で駆動されると言う方が良いかもしれない.

Loewner eqaution $\Longrightarrow$ SLE$_\kappa$: Put $W_t = \sqrt{\kappa} B_t = B_{\kappa t}$, $\kappa > 0$.

$$d(W_t - O_t) = \sqrt{\kappa} dB_t + \frac{2}{W_t - O_t} dt = dB_{\kappa t} + \frac{2}{\kappa} \frac{1}{W_t - O_t} d(\kappa t).$$

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SLE(κ, ρ)

駆動関数の表す粒子の相対座標を index ν の Bessel 過程とする．ただし，今度は index ν は先の ν_0 とは別物とする．

$$W_t - O_t = \sqrt{\kappa} R_t^{(\nu)} = R_{\kappa t}^{(\nu)} \quad \Longleftrightarrow \quad R_t^{(\nu)} = \frac{W_t - O_t}{\sqrt{\kappa}}. \tag{1.6}$$

$$\begin{aligned} dW_t &= dO_t + \sqrt{\kappa} dR_t^{(\nu)} = \frac{2}{O_t - W_t} dt + \sqrt{\kappa} \left\{ dB_t + \left(\nu + \frac{1}{2}\right) \frac{dt}{R_t^{(\nu)}} \right\} \\ &= -\frac{2}{W_t - O_t} dt + \sqrt{\kappa} dB_t + \sqrt{\kappa} \left(\nu + \frac{1}{2}\right) \frac{\sqrt{\kappa}}{W_t - O_t} dt \\ &= \sqrt{\kappa} dB_t + \left\{ -2 + \left(\nu + \frac{1}{2}\right) \kappa \right\} \frac{dt}{W_t - O_t}. \end{aligned}$$

パラメータ ρ を導入する：

$$\begin{aligned} & \rho = -2 + \left(\nu + \frac{1}{2}\right) \kappa \quad \Longleftrightarrow \quad \nu = \frac{\rho + 2}{\kappa} - \frac{1}{2} \\ \Longleftrightarrow \quad & \rho = -2 + \frac{(d-1)}{2} \kappa \quad \Longleftrightarrow \quad d = 1 + \frac{2(\rho + 2)}{\kappa}. \end{aligned} \tag{1.7}$$

$$r \equiv R_0^{(\nu)}, \qquad a \equiv \sqrt{\kappa} r \quad \textbf{初期値}$$

$$W_t = a + \sqrt{\kappa} B_t + \rho \int_0^t \frac{ds}{W_s - O_s}. \tag{1.8}$$

この W_t で駆動される

$$\frac{\partial}{\partial t} g_t = \frac{2}{g_t - W_t} \tag{1.9}$$

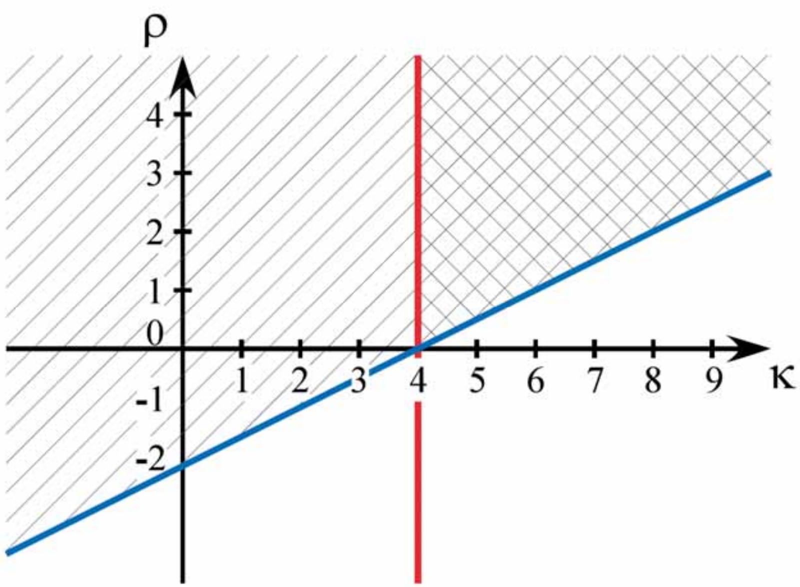
を SLE(κ, ρ) started from $(\gamma(0), W_0) = (O_0, W_0) = (0, a)$ と呼ぶ.

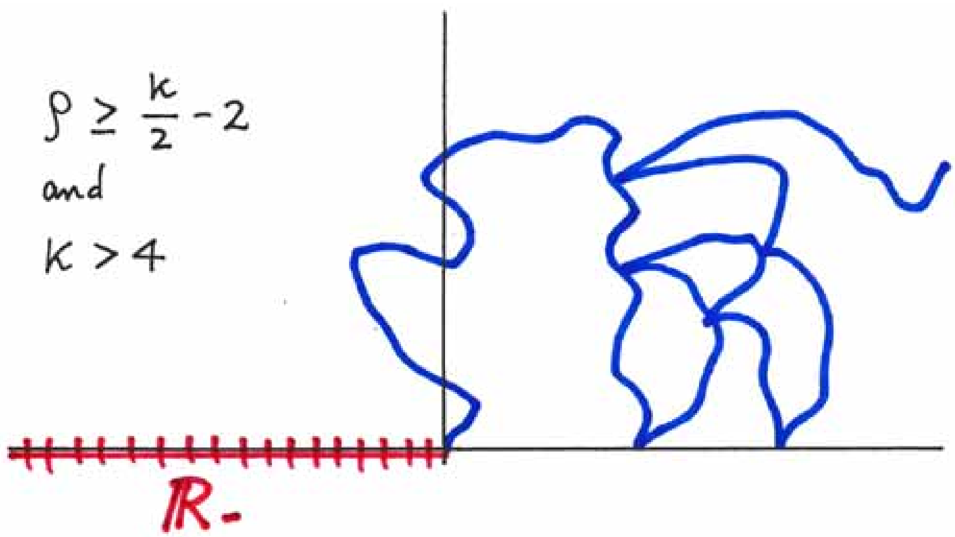
- 以下

$$d \geq 2 \quad \Longleftrightarrow \quad \nu \geq 0 \quad \Longleftrightarrow \quad \rho \geq \frac{\kappa}{2} - 2 \qquad (1.10)$$

を仮定．このとき $\mathbb{R}_t^{(\nu)}$ は確率 1 で 0 に再帰しない．
よって $W_t - O_t = g(\gamma(t)) - g(0) > 0, \forall t > 0$, w.p.1.
$\Longrightarrow O_t \in \mathbb{R}_- \equiv \{x \in \mathbb{R} : x < 0\}, t > 0$ であるが，これは SLE(κ, ρ) 曲線で "swallow" される (呑み込まれる) ことはない．
$\Longleftrightarrow$ SLE(κ, ρ) 曲線は $\mathbb{R}_-$ には接しない．

- (1.10) でかつ $\kappa > 4$ のとき：
SLE (κ, ρ) 曲線は $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ には接する．





2. One-Sided Restriction（片側制限性）

$\mathcal{A} \equiv$ the family of closed subsets A of $\overline{\mathbb{H}}$ s.t.

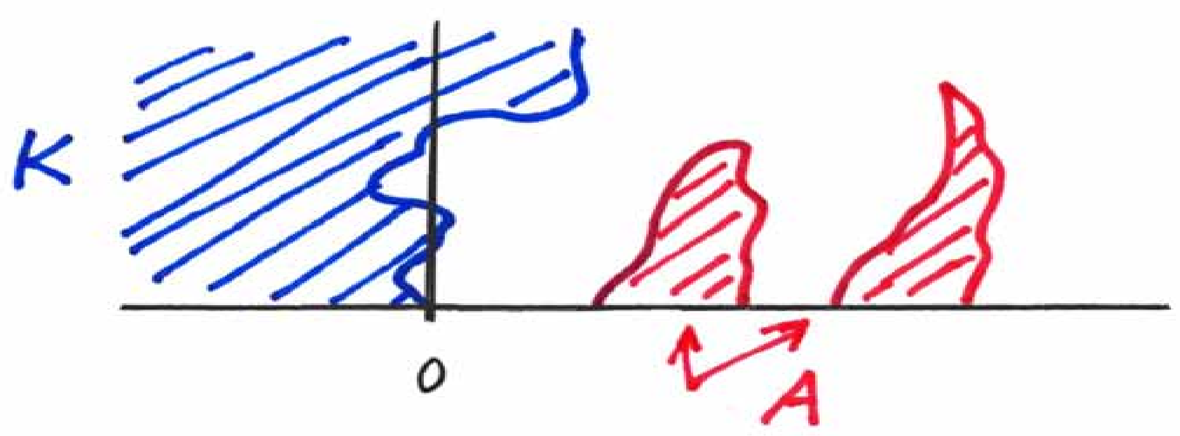
1. $\mathbb{H} \setminus A$ is simply connected.
2. A is bounded, and **bounded away from** $\mathbb{R}_- \equiv \{x < 0 : x \in \mathbb{R}\}$.

To each such A the **conformal mapping** Φ_A from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ is associated such that

$$\Phi_A(0) = 0 \quad \text{and} \quad \Phi_A(z) \sim z \quad \text{when} \quad z \to \infty.$$

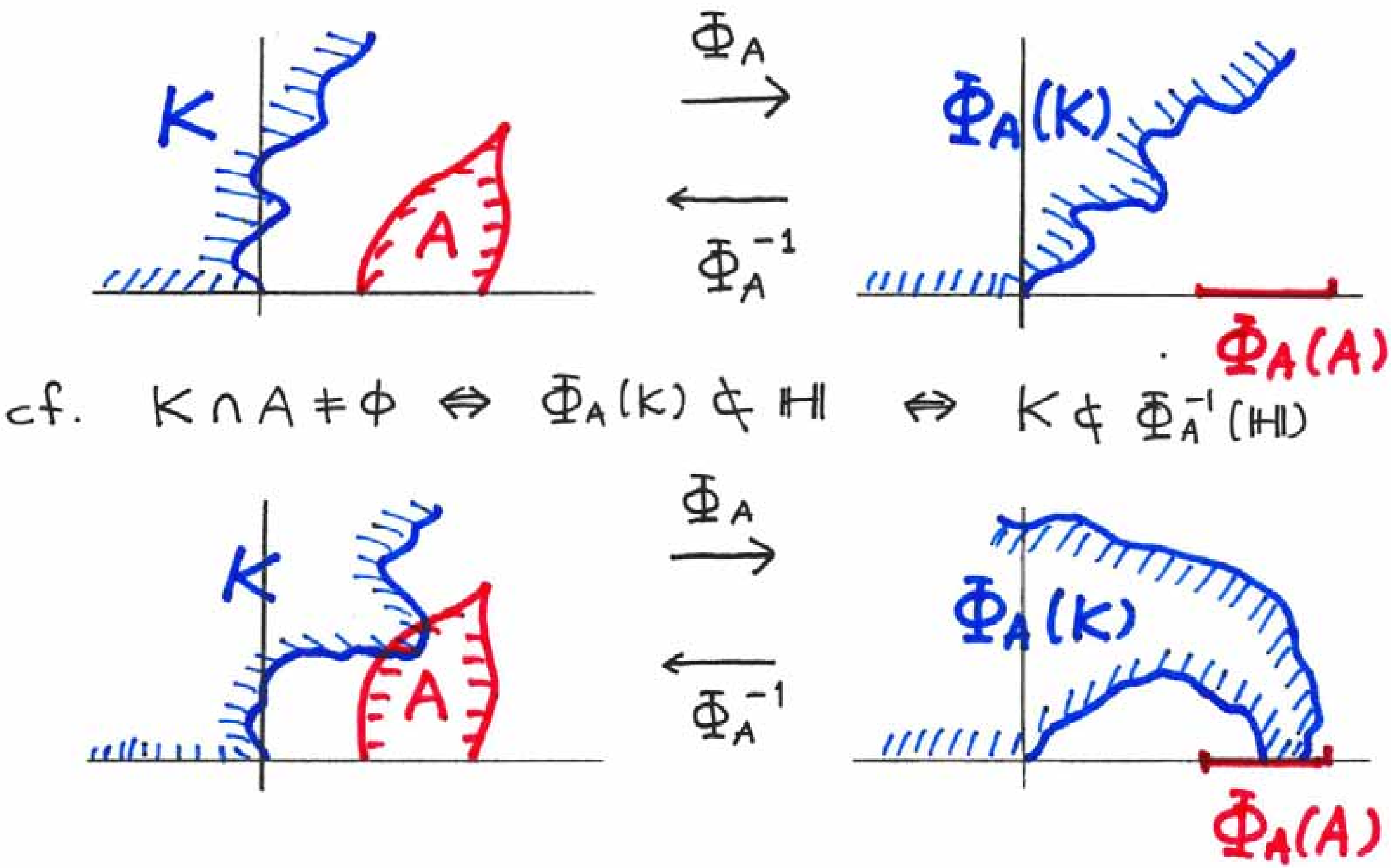
We say that a closed set $K \subset \overline{\mathbb{H}}$ is **left-filled** if

- K and $\mathbb{H} \setminus K$ are both simply connected and unbounded.
- $K \cap \mathbb{R} = \mathbb{R}_-$.



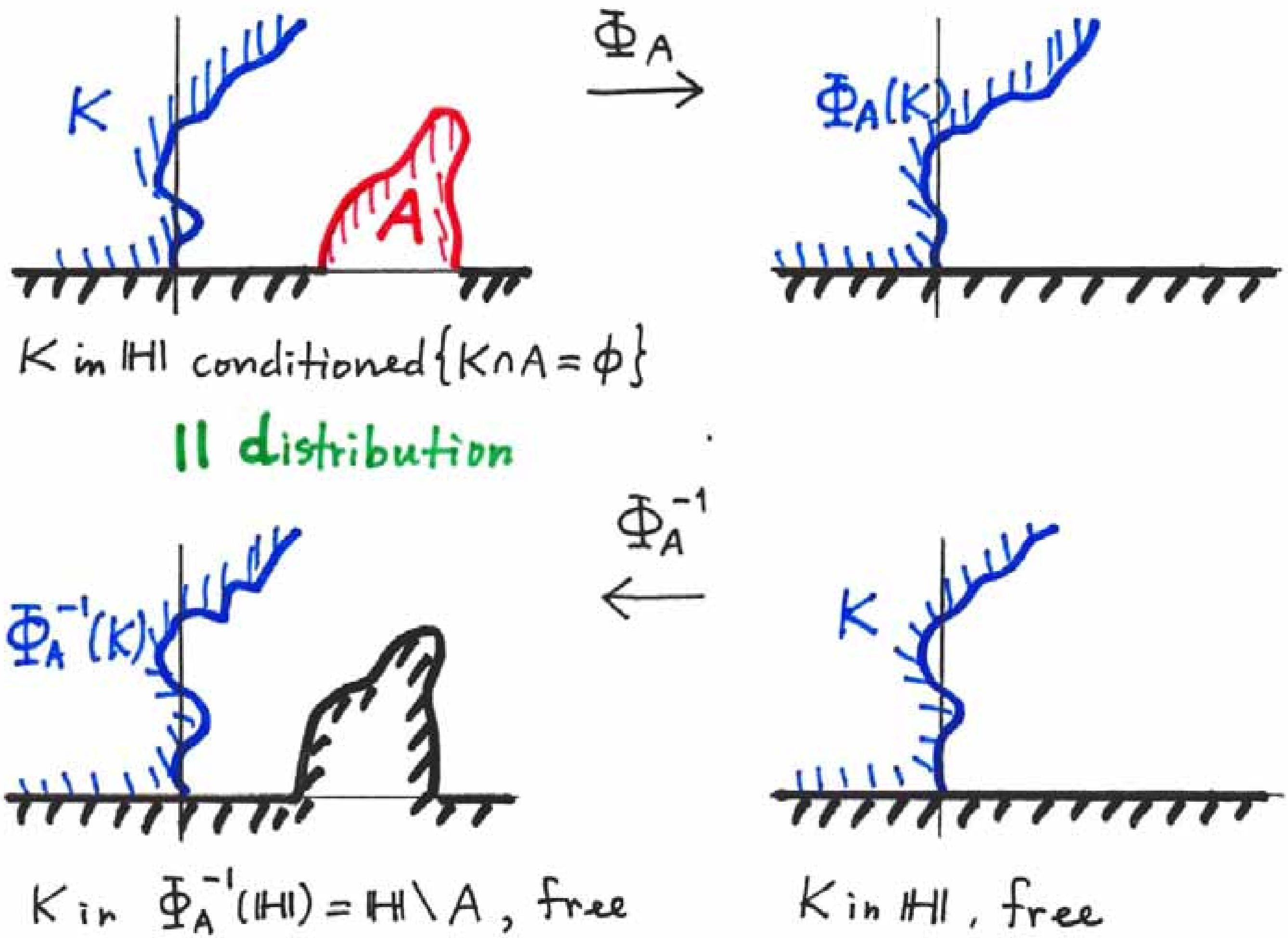
Note that

$$K \cap A = \emptyset \quad \Longleftrightarrow \quad \Phi_A(K) \subset \mathbb{H} \quad \Longleftrightarrow \quad K \subset \Phi_A^{-1}(\mathbb{H})$$



It is said that a random left-filled set satisfies **one-sided restriction** if for all $A \in \mathcal{A}$, the law of K conditioned $\{K \cap A = \emptyset\}$ is identical to that of $\Phi_A^{-1}(K)$;

$$\text{the law of } K \Big| \{K \cap A = \emptyset\} = \text{the law of } \Phi_A^{-1}(K). \tag{2.1}$$



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The one-sided restriction implies that

$$\exists \alpha > 0 \quad \text{s.t.} \quad \mathbf{P}(K \cap A = \emptyset) = \Big(\Phi'_A(0)\Big)^{\alpha}, \qquad (2.2)$$

where

$$\Phi'_A(z) = \frac{d}{dz}\Phi_A(z).$$

The reasoning of the above may be the following.

1. For a fixed left-filled set K satisfying one-sided restriction, consider the probability $\mathbf{P}(K \cap A = \emptyset)$ as a function of the conformal mapping Φ_A, $A \in \mathcal{A}$ and denote it as $f(\Phi_A)$;

$$\mathbf{P}(K \cap A = \emptyset) = \mathbf{P}\Big(K \subset \Phi_A^{-1}(\mathbb{H})\Big) \equiv f(\Phi_A).$$

Then for two different sets $A, A' \in \mathcal{A}$, $A \neq A'$, consider the probability

$$f(\Phi_{A'}(\Phi_A)) = \mathbf{P}\Big(K \subset \Phi_A^{-1}(\Phi_{A'}^{-1}(\mathbb{H}))\Big) = \mathbf{P}\Big(\Phi_A(K) \subset \Phi_{A'}^{-1}(\mathbb{H})\Big).$$

By definition of conditional probability

$$\begin{aligned}\mathbf{P}\Big(\Phi_A(K) \subset \Phi_{A'}^{-1}(\mathbb{H})\Big) &= \mathbf{P}\Big(\Phi_A(K) \subset \Phi_{A'}^{-1}(\mathbb{H}) \Big| K \cap A = \emptyset\Big) \times \mathbf{P}(K \cap A = \emptyset) \\ &= \mathbf{P}\Big(\Phi_A(K) \subset \Phi_{A'}^{-1}(\mathbb{H}) \Big| K \cap A = \emptyset\Big) \times f(\Phi_A).\end{aligned}$$

On the other hand, by the **one-sided restriction property** of K

$$\begin{aligned}\mathbf{P}\Big(\Phi_A(K) \subset \Phi_{A'}^{-1}(\mathbb{H}) \Big| K \cap A = \emptyset\Big) &= \mathbf{P}\Big(\Phi_A(\Phi_A^{-1}(K)) \subset \Phi_{A'}^{-1}(\mathbb{H})\Big) \\ &= \mathbf{P}(K \subset \Phi_{A'}^{-1}(\mathbb{H})) = f(\Phi_{A'}).\end{aligned}$$

So we have

$$f(\Phi_{A'}(\Phi_A)) = f(\Phi_A)f(\Phi_{A'}) \iff f(\Phi_A \circ \Phi_{A'}) = f(\Phi_A)f(\Phi_{A'}) \quad (2.3)$$

2. Loewner's theory shows that it is possible to approximate any conformal mapping Φ_A by the iteration of many conformal maps $\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n$ such that each ϕ_j is a conformal map in a one-parameter family of mapping $\{\varphi^t\}$ satisfying

$$\varphi^{t+s} = \varphi^t \circ \varphi^s \quad \Longleftrightarrow \quad \varphi^{t+s}(z) = \varphi^t(\varphi^s(z)) \quad \forall s, t > 0, \tag{2.4}$$

and

$$\varphi^t(0) = 0. \tag{2.5}$$

(The one-parameter family may be given explicitly using the Loewner chain for a well-defined curve η; $g_t = \Phi_{\mathbb{H} \setminus \eta(0,t]} \cdot$)

3. Differentiate (2.4) with respect to z,

$$(\varphi^{t+s})'(z) = (\varphi^t)'(\varphi^s(z)) \times (\varphi^s)'(z),$$

and set $z = 0$, then we have

$$(\varphi^{t+s})'(0) = (\varphi^t)'(0) \times (\varphi^s)'(0),$$

where (2.5) was used. This relation implies

$$\exists c > 0 \quad \text{s.t} \quad (\varphi^t)'(0) = e^{-ct}. \tag{2.6}$$

Since we assume the one-sided restriction property, we can apply the relation (2.3) to φ^t to have

$$f(\varphi^{t+s}) = f(\varphi^t \circ \varphi^s) = f(\varphi^t) f(\varphi^s).$$

It implies that

$$\exists \theta > 0 \quad \text{s.t.} \quad f(\varphi^t) = e^{-\theta t}. \tag{2.7}$$

Combining (2.6) and (2.7), we see

$$f(\varphi^t) = e^{-ct\times(\theta/c)} = \Big((\varphi^t)'(0)\Big)^\alpha, \quad \alpha = \frac{\theta}{c}.$$

Now following Loewner's theory, we approximate Φ_A as

$$\Phi_A \sim \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n.$$

Then

$$\begin{aligned} f(\Phi_A) &\sim f(\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n) = f(\phi_1)f(\phi_2)\cdots f(\phi_n) \\ &= (\phi_1'(0))^\alpha \times (\phi_2'(0))^\alpha \times \cdots \times (\phi_n'(0))^\alpha = \Big((\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n)'(0)\Big)^\alpha \\ &\sim \Big(\Phi_A'(0)\Big)^\alpha. \end{aligned}$$

- In § 1 we have seen that the SLE(κ, ρ) curve with $\rho \geq \kappa/2 - 2$ never hit $\mathbb{R}_-$. That is

$$\text{SLE}(\kappa, \rho) \text{ curve} \quad \gamma[0,t] \in \mathcal{A}, \quad t > 0, \quad \rho \geq \frac{\kappa}{2} - 2. \qquad (2.8)$$

- Here we should remember the fact that the driving function W_t of the SLE(γ, ρ) is given using the Bessel process with index ν.

  Let $\mathbf{P}^{(\nu)}$ be the probability measure of the Bessel process of index ν, $R^{(\nu)}$.

- Thus we can apply Eq.(2.2) for $A = \gamma[0,t]$ and we have

$$\mathbf{P}^{(\nu)}(K \cap \gamma[0,t] = \emptyset) = \mathbf{E}^{(\nu)}\left[\left(\Phi'_{\gamma[0,t]}(0)\right)^{\alpha}\right], \quad t > 0, \qquad (2.9)$$

  for any independent sample of left-filled K with the one-sided restriction exponent $\alpha > 0$.

By definition

$$\Phi_{\gamma[0,t]}(z) = g_t(z),$$

where $g_t(z)$ is the solution of the SLE(κ, ρ), (1.1):

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

By differentiating this equation with respect to z, we have

$$\frac{\partial}{\partial t} g_t'(z) = -\frac{2g_t'(z)}{(g_t(z) - W_t)^2},$$
$$\frac{\partial}{\partial t} \log g_t'(z) = \frac{1}{g_t'(z)} \frac{\partial}{\partial t} g_t'(z) = -\frac{2}{(g_t(z) - W_t)^2}.$$

Then

$$\frac{\partial}{\partial t} \log g_t'(0) \quad = \quad -\frac{2}{(g_t(0) - W_t)^2} = -\frac{2}{(O_t - W_t)^2} = -\frac{2}{\kappa} \frac{1}{(R_t^{(\nu)})^2}.$$

It implies that

$$\begin{aligned} \Phi'_{\gamma[0,t]}(0) &= g_t'(0) = \exp\Big(\log g_t'(0)\Big) \\ &= \exp\left(\int_0^t ds \frac{\partial}{\partial s} \log g_s'(0)\right) = \exp\left(-\frac{4}{\kappa} \int_0^t \frac{ds}{2(R_s^{(\nu)})^2}\right). \end{aligned}$$

And thus,

$$\mathbf{E}^{(\nu)}\left[\left(\Phi'_{\gamma[0,t]}(0)\right)^{\alpha}\right] = \mathbf{E}^{(\nu)}\left[\exp\left(-\frac{4\alpha}{\kappa}\int_0^t \frac{ds}{2(R_s^{(\nu)})^2}\right)\right]$$
$$= \mathbf{E}^{(\nu)}\left[\left(\frac{r}{R_t^{(\nu)}}\right)^{\mu-\nu} \times \left(\frac{R_t^{(\nu)}}{r}\right)^{\mu-\nu} \exp\left(-\frac{4\alpha}{\kappa}\int_0^t \frac{ds}{2(R_s^{(\nu)})^2}\right)\right].$$

That is

$$\mathbf{P}^{(\nu)}(K \cap \gamma[0,t] = \emptyset) = \mathbf{E}^{(\nu)}\left[\left(\frac{r}{R_t^{(\nu)}}\right)^{\mu-\nu} \times \left(\frac{R_t^{(\nu)}}{r}\right)^{\mu-\nu} \exp\left(-\frac{4\alpha}{\kappa}\int_0^t \frac{ds}{2(R_s^{(\nu)})^2}\right)\right]. \tag{2.10}$$

3. Application of Girsanov's transformation

Let $\mathbf{P}^{(\nu)}$ be the probability measure of the Bessel process of index ν, $R^{(\nu)}$. Consider the probability measure $\mathbf{P}_t^{(\mu)}$ for $\mu \neq \nu$, whose Radon-Nikodym derivative with respect to $\mathbf{P}^{(\nu)}$ is given by

$$\frac{d\mathbf{P}_t^{(\mu)}}{d\mathbf{P}^{(\nu)}} = \left(\frac{R_t^{(\nu)}}{r}\right)^{\mu-\nu} \exp\left\{-(\mu^2-\nu^2)\int_0^t \frac{ds}{2(R_s^{(\nu)})^2}\right\}, \qquad (3.1)$$

for $r > 0$. Under $\mathbf{P}_t^{(\mu)}$, the process $R_t^{(\nu)}$ is a Bessel process of index μ (instead of ν) started from r. This is called **Girsanov's transformation.**

Compare it with

$$\mathbf{P}^{(\nu)}(K \cap \gamma[0,t] = \emptyset) = \mathbf{E}^{(\nu)}\left[\left(\frac{r}{R_t^{(\nu)}}\right)^{\mu-\nu} \times \left(\frac{R_t^{(\nu)}}{r}\right)^{\mu-\nu} \exp\left(-\frac{4\alpha}{\kappa}\int_0^t \frac{ds}{2(R_s^{(\nu)})^2}\right)\right].$$

$$\frac{d\mathbf{P}_t^{(\mu)}}{d\mathbf{P}^{(\nu)}} = \left(\frac{R_t^{(\nu)}}{r}\right)^{\mu-\nu} \exp\left\{-(\mu^2-\nu^2)\int_0^t \frac{ds}{2(R_s^{(\nu)})^2}\right\},$$

$$\mathbf{P}^{(\nu)}(K \cap \gamma[0,t] = \emptyset) = \mathbf{E}^{(\nu)}\left[\left(\frac{r}{R_t^{(\nu)}}\right)^{\mu-\nu} \times \left(\frac{R_t^{(\nu)}}{r}\right)^{\mu-\nu} \exp\left(-\frac{4\alpha}{\kappa}\int_0^t \frac{ds}{2(R_s^{(\nu)})^2}\right)\right].$$

If

$$\frac{4\alpha}{\kappa} = \mu^2 - \nu^2 \iff \alpha = \frac{(\mu^2-\nu^2)}{4}\kappa$$

$$\iff \quad \mu = \sqrt{\frac{4\alpha}{\kappa} + \nu^2} \iff \mu = \sqrt{\frac{4\alpha}{\kappa} + \left(\frac{\rho+2}{\kappa} - \frac{1}{2}\right)^2}, \qquad (3.2)$$

we have the equality [[1] Werner (2004)]

$$\mathbf{P}^{(\nu)}(K \cap \gamma[0,t] = \emptyset) = \mathbf{E}^{(\mu)}\left[\left(\frac{r}{R_t^{(\mu)}}\right)^{\mu-\nu}\right], \qquad (3.3)$$

where $R_t^{(\mu)}$ is the Bessel process of index μ started from r.

$$\frac{d\mathbf{P}_t^{(\mu)}}{d\mathbf{P}^{(\nu)}} = \left(\frac{R_t^{(\nu)}}{r}\right)^{\mu-\nu} \exp\left\{-(\mu^2-\nu^2)\int_0^t \frac{ds}{2(R_s^{(\nu)})^2}\right\},$$

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where $R_t^{(\mu)}$ is the Bessel process of index μ started from r.

Interpretations

A. For fixed $t > 0$ and fixed $\kappa > 0$, we can let $a = \sqrt{\kappa}r \to 0$. Then (3.3) gives

$$\mathbf{P}^{(\nu)}(K \cap \gamma[0,t] = \emptyset) \simeq c a^{\mu-\nu},$$

where

$$c = \mathbf{E}^{(\mu)}\left[(\sqrt{\kappa}R_t^{(\mu)})^{\nu-\mu}\right] = \mathbf{E}^{(\mu)}\left[\left(R_{\kappa t}^{(\mu)}\right)^{\nu-\mu}\right] = (\kappa t)^{(\nu-\mu)/2}\mathbf{E}^{(\mu)}\left[\left(R_1^{(\mu)}\right)^{\nu-\mu}\right].$$

That implies

$$\mathbf{P}^{(\nu)}(K \cap \gamma[0,t] = \emptyset) \sim t^{-\sigma/2} \quad \text{in} \quad t \to \infty \quad \text{with fixed } a \tag{3.4}$$

and

$$\mathbf{P}^{(\nu)}(K \cap \gamma[0,t] = \emptyset) \sim a^{\sigma} \quad \text{in} \quad a \to 0 \quad \text{with fixed } t, \tag{3.5}$$

where

$$\begin{aligned} \sigma &= \mu - \nu \\ &= \sqrt{\frac{4\alpha}{\kappa} + \left(\frac{\rho+2}{\kappa} - \frac{1}{2}\right)^2} - \left(\frac{\rho+2}{\kappa} - \frac{1}{2}\right). \end{aligned} \tag{3.6}$$

In other words, the **intersection exponent** between a one-sided restriction measure with exponent α and the SLE(κ, ρ) is σ given by (3.6).

B. An SLE(κ, ρ) conditioned to avoid a one-sided restriction sample of exponent α is an SLE$(\kappa, \overline{\rho})$, where

$$\begin{aligned}
\overline{\rho} &= -2 + \left(\mu + \frac{1}{2}\right)\kappa \\
&= -2\left(\nu + \sigma + \frac{1}{2}\right)\kappa \qquad \left(\text{by} \quad \sigma = \mu - \nu\right) \\
&= -2 + \left\{\frac{\rho+2}{\kappa} - \frac{1}{2} + \sigma + \frac{1}{2}\right\}\kappa \quad \left(\text{by} \quad \nu = \frac{\rho+2}{\kappa} - \frac{1}{2}\right) \\
&= \rho + \kappa\left\{\sqrt{\frac{4\alpha}{\kappa} + \left(\frac{\rho+2}{\kappa} - \frac{1}{2}\right)^2} - \left(\frac{\rho+2}{\kappa} - \frac{1}{2}\right)\right\} \quad \left(\text{by} \quad (3.6)\right) \\
&= \kappa\sqrt{\frac{4\alpha}{\kappa} + \left(\frac{\rho+2}{\kappa} - \frac{1}{2}\right)^2} + \frac{\kappa}{2} - 2. \qquad (3.7)
\end{aligned}$$

C. If we set $\rho = 0$ above, we can say that an SLE_κ conditioned to avoid a one-sided restriction sample of exponent α is an $\mathrm{SLE}(\kappa, \overline{\rho})$, where

$$\overline{\rho} = \kappa\sqrt{\frac{4\alpha}{\kappa} + \left(\frac{2}{\kappa} - \frac{1}{2}\right)^2} + \frac{\kappa}{2} - 2. \tag{3.8}$$

Conversely, an $\mathrm{SLE}(\kappa, \rho)$ can be viewed as an SLE_κ conditioned not to intersect with a one-sided sample of exponent

$$\begin{aligned} \alpha &= \frac{\kappa}{4}(\mu^2 - \nu^2) \\ &= \frac{\kappa}{4}\left[\frac{4\alpha}{\kappa} + \left(\frac{\rho+2}{\kappa} - \frac{1}{2}\right)^2 - \left(\frac{2}{\kappa} - \frac{1}{2}\right)^2\right] \quad \Big(\text{by} \quad (3.2)\Big) \\ &= \frac{1}{4\kappa}\rho(\rho + 4 - \kappa). \end{aligned} \tag{3.9}$$

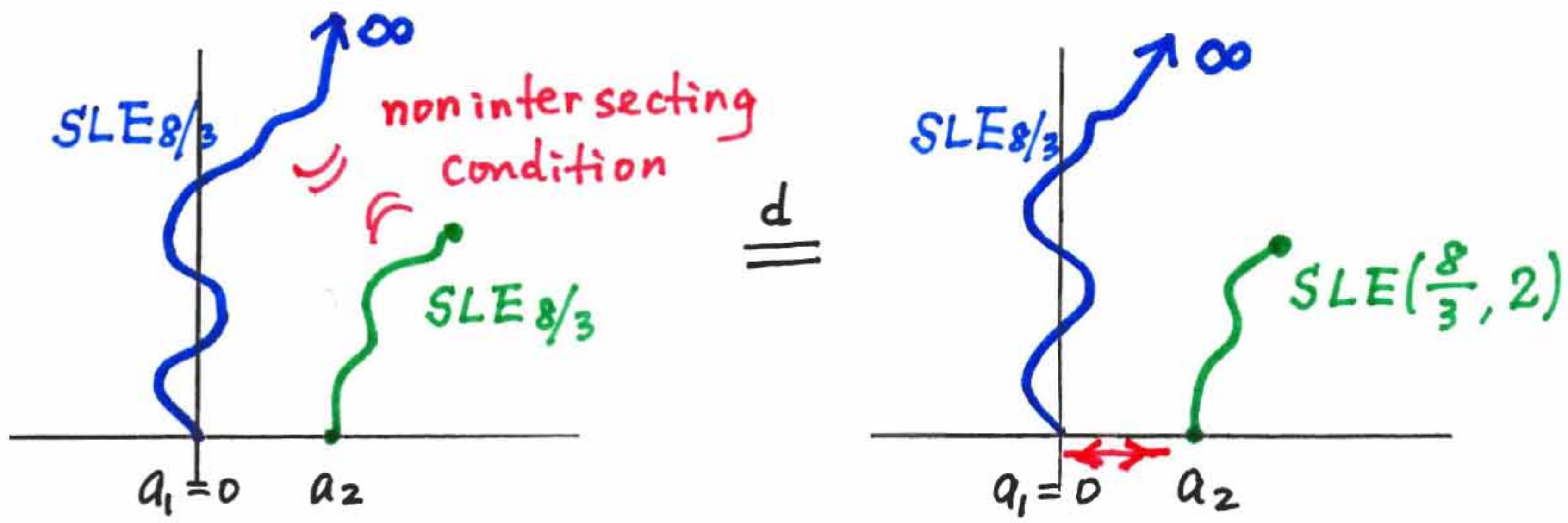
4. Nonintersecting $SLE_{8/3}$'s (Nonintersecting SAW's)

$SLE_{8/3}$ has (two-sided restriction property).
$\Longrightarrow$ one-sided restriction property is also satisfied.
The exponent is $\alpha = \frac{5}{8} = b_{\mathrm{SAW}}$.

In (3.9) $\alpha = \frac{1}{4\kappa}\rho(\rho + 4 - \kappa)$, set $\alpha = \frac{5}{8}, \kappa = \frac{8}{3}$. Since $\rho > 0$, we have $\rho = 2$.

$\mathrm{SLE}_{8/3}$ curve $\gamma_2[0,t]$ starting from $a_2 > 0$
conditioned not to intersect with $\mathrm{SLE}_{8/3}$ curve $\gamma_1[0,\infty)$ starting from $a_1 = 0$

$\stackrel{\mathrm{d}}{=}$ SLE(8/3,2) starting from $a_2 > 0$.



[3] G. F. Lawler, O. Schramm, W. Werner :
Conformal restriction. The chordal case,
J. Amer. Math. Soc. **16** (2003) 917-955.

The boundary of sample of one-sided restriction measure of exponent α

$\stackrel{\mathrm{d}}{=}$ SLE(8/3,ρ) curve, where

$$\alpha = \frac{(\rho+2)(3\rho+10)}{32}. \tag{4.1}$$

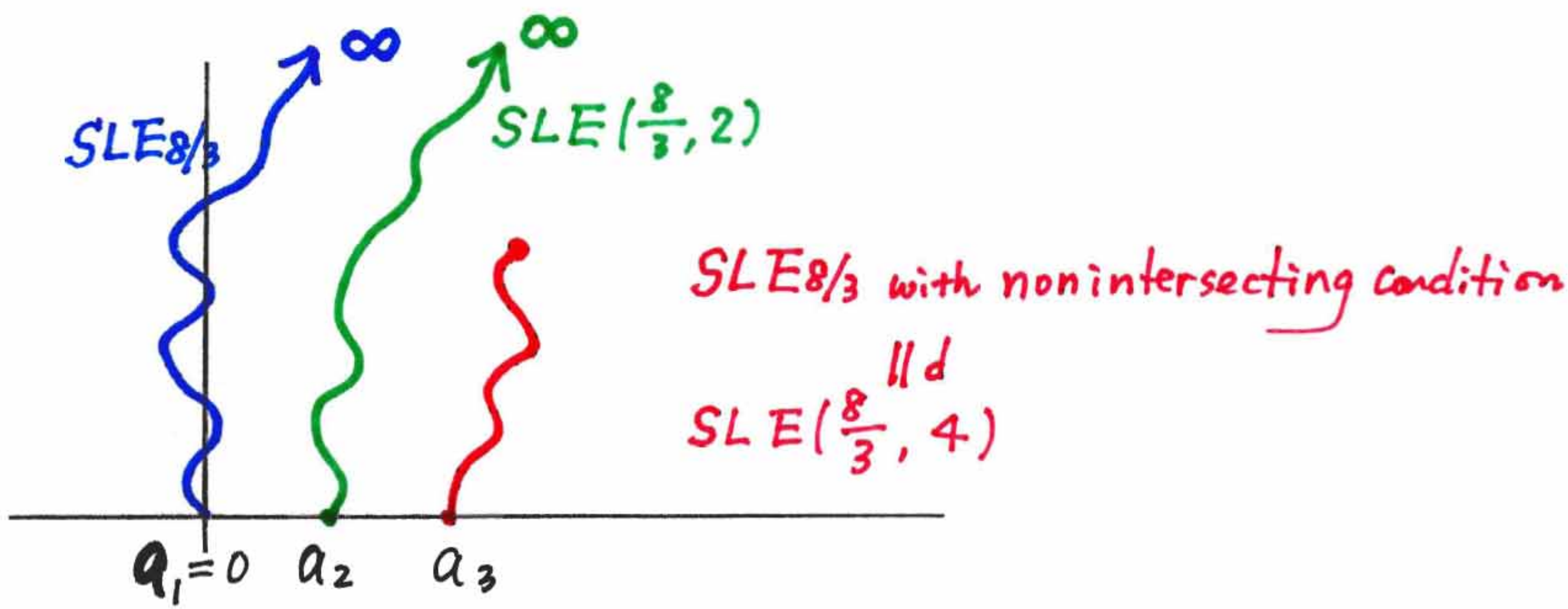
SLE(8/3,2) = one-sided restriction measure of exponent

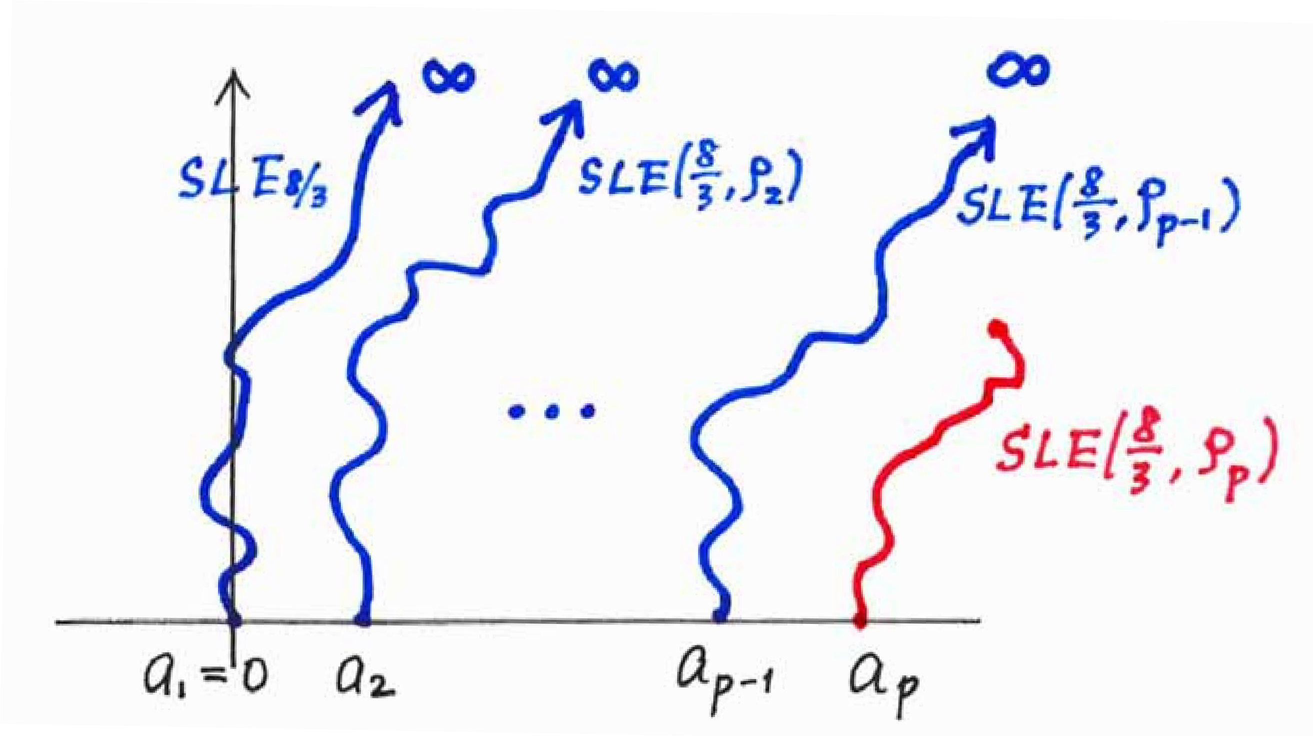
$$\alpha = \frac{(2+2)(3\times 2+10)}{32} = 2.$$

Let $\alpha = 2$ in the equation $\alpha = \frac{1}{4\kappa}\rho(\rho+4-\kappa)$ with $\kappa = \frac{8}{3}$.
Then we have $\rho = 4$.

$\mathrm{SLE}_{8/3}$ curve $\gamma_3[0,t]$ starting from $a_3 > a_2$
conditioned not to intersect with SLE(8/3,2) curve $\gamma_2[0,\infty)$ starting from $a_2 > 0$

$\stackrel{\mathrm{d}}{=}$ SLE(8/3,4) starting from $a_3 > 0$.





Consider the situation shown in the above figure.
Let p be the number of SLE(8/3, ρ)'s. Then

$$\rho_p = 2(p-1). \tag{4.2}$$

The one-sided restriction exponent of SLE(8/3, ρ_p) is

$$\alpha_p = \frac{1}{8}p(3p+2). \tag{4.3}$$

Conformal Field Theory vs. SLE

Kac Formula

$$\text{central charge} \quad c = 1 - \frac{6}{m(m+1)}, \quad m \in \mathbb{C},$$

$$\text{highest weight} \quad h^{(m)}_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)}, \quad r, s = 1, 2, 3, \ldots. \quad (4.4)$$

Consider the case

$$c = 0 \quad \Longrightarrow \quad m = 2.$$

In this case

$$h_{r,s} = h^{(0)}_{r,s} = \frac{(3r - 2s)^2 - 1}{24}.$$

B. Duplantier and H. Saler,
Exact Surface and Wedge Exponents for Polymers in Two Dimensions,
Phys. Rev. Lett. **57** (1986) 3179-3182.

"Surface Dimension" for p chanins of polymers

$$\chi'_p = h_{p+1,1} = \frac{(3(p+1) - 2)^2 - 1}{24} = \frac{1}{8} p(3p+2). \quad (4.5)$$

5. Hitting Time Distribution

Lamperti's relation

Let B_t be the one-dimensional standard Brownian motion started from x. Consider the geometric Brownian motion with drift ν, $\exp(B_t + \nu t), t > 0$. The following relation is established.

Lemma 5.1 *Let $R_t^{(\nu)}$ be the Bessel process with index ν started from, $r = e^x$. Then*

$$\exp(B_t + \nu t) = R^{(\nu)}_{A_t^{(\nu)}} \tag{5.1}$$

with

$$A_t^{(\nu)} = \int_0^t \exp\{2(B_s + \nu s)\} ds = \int_0^t \Big\{ \exp(B_s + \nu s) \Big\}^2 ds \tag{5.2}$$

Lemma 5.1 *Let $R_t^{(\nu)}$ be the Bessel process with index ν started from, $r = e^x$. Then*

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Remark. By (5.2),

$$dA_t^{(\nu)} = \exp\{2(B_t + \nu t)\} dt = \left(R^{(\nu)}_{A_t^{(\nu)}}\right)^2 dt \iff dt = \frac{dA_t^{(\nu)}}{(R^{(\nu)}_{A_t^{(\nu)}})^2}.$$

Then if we set

$$H_t^{(\nu)} = \int_0^t \frac{ds}{(R_s^{(\nu)})^2}, \tag{5.3}$$

we find the correspondence

$$u \le A_t^{(\nu)} \iff H_u^{(\nu)} \le t. \tag{5.4}$$

Asian option discussion

Let T_λ be an exponential variable, with parameter λ, which is independent of $(B_t, t \geq 0)$. Consider the distribution $\mathbf{P}(A_t^{(\nu)} \geq u)$;

$$\begin{aligned}
\mathbf{P}(A_{T_\lambda}^{(\nu)} \geq u) &= \int_0^\infty dt\, \lambda e^{-\lambda t} \mathbf{P}(A_t^{(\nu)} \geq u) \\
&= \lambda \int_0^\infty dt\, e^{-\lambda t} \mathbf{P}(H_u^{(\nu)} \leq t) \\
&= \lambda \int_0^\infty dt\, e^{-\lambda t} \int_0^t dh\, \mathbf{E}^{(\nu)} \left[\mathbf{1}_{\{H_u^{(\nu)} = h\}} \right] \\
&= \int_0^\infty dh\, \mathbf{E}^{(\nu)} \left[\mathbf{1}_{\{H_u^{(\nu)} = h\}} \right] \lambda \int_h^\infty dt\, e^{-\lambda t} \\
&= \int_0^\infty dh\, \mathbf{E}^{(\nu)} \left[\mathbf{1}_{\{H_u^{(\nu)} = h\}} \right] e^{-\lambda h} \\
&= \mathbf{E}^{(\nu)} \left[\exp(-\lambda H_u^{(\nu)}) \right], \qquad (5.5)
\end{aligned}$$

where $\mathbf{1}(\omega)$ is the indicator function of the event ω and (5.4) was used. By (5.3),

$$\mathbf{P}(A_{T_\lambda}^{(\nu)} \geq u) = \mathbf{E}^{(\nu)} \left[\exp\left(-\lambda \int_0^u \frac{ds}{(R_s^{(\nu)})^2} \right) \right].$$

Set

$$\mu = \sqrt{2\lambda + \nu^2} \quad \Longleftrightarrow \quad \lambda = \frac{\mu^2 - \nu^2}{2}. \tag{5.6}$$

Then

$$\begin{aligned} & \mathbf{E}^{(\nu)}\left[\exp\left(-\lambda \int_0^u \frac{ds}{(R_s^{(\nu)})^2}\right)\right] \\ = \; & \mathbf{E}^{(\nu)}\left[\left(\frac{r}{R_u^{(\nu)}}\right)^{\mu-\nu}\left(\frac{R_u^{(\nu)}}{r}\right)^{\mu-\nu} \exp\left(-\lambda \int_0^u \frac{ds}{(R_s^{(\nu)})^2}\right)\right]. \end{aligned}$$

Therefore, by applying Girsanov's transformation (3.1), we have the equality

$$\mathbf{P}\left(A_{T_\lambda}^{(\nu)} \geq u\right) = \mathbf{E}^{(\mu)}\left[\left(\frac{r}{R_u^{(\mu)}}\right)^{\mu-\nu}\right]. \tag{5.7}$$

Now we compare the result (5.7) with the equality (3.3) of Werner. Let

$$\tau_\alpha = \inf\left\{t : K \cap \gamma[0,t] \neq \emptyset\right\} \tag{5.8}$$

for the sample of left-filled K with the one-sided restriction exponent α and the SLE(κ, ρ) curve γ. Then (3.3) is written as

$$\mathbf{P}^{(\nu)}(K \cap \gamma[0,t] = \emptyset) \;=\; \mathbf{P}(\tau_\alpha > t) = \mathbf{E}^{(\mu)}\left[\left(\frac{r}{R_t^{(\mu)}}\right)^{\mu-\nu}\right]. \tag{5.9}$$

Then we have arrived at the following.

Theorem 5.2 *Assume that the left-filled K is under the one-sided restriction measure with exponent α and γ is the SLE(κ, ρ) curve. Set (5.8). Then*

$$\tau_\alpha \stackrel{\mathrm{d}}{=} A^{(\nu)}_{T_\lambda}, \tag{5.10}$$

when

$$\nu = \frac{\rho+2}{\kappa} - \frac{1}{2}, \quad \lambda = \frac{2\alpha}{\kappa}. \tag{5.11}$$

The distribution of $A_{T_\lambda}^{(\nu)}$ has been well studied by M. Yor.

[4] M. Yor, *Exponential Functionals of Brownian Motion and Related Processes*, Springer, 2001.

Let

Z_γ : a gamma variable with parameter γ,

$$\mathbf{P}(Z_\gamma \in du) = \frac{e^{-u}u^{\gamma-1}}{\Gamma(\gamma)}du, \quad \Gamma(\gamma) = \int_0^\infty e^{-s}s^{\gamma-1}ds.$$

$Z_{\alpha,\beta}$: a beta variable with parameters (α, β),

$$\mathbf{P}(Z_{\alpha,\beta} \in du) = \frac{u^{\alpha-1}(1-u)^{\beta-1}}{B(\alpha,\beta)}du, \quad B(\alpha,\beta) = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}ds.$$

Let $\lambda > 0$; define $\mu = \sqrt{2\lambda + \nu^2}$, then

$$A_{T_\lambda}^{(\nu)} \overset{\mathrm{d}}{=} \frac{Z_{1,a}}{2Z_b}, \quad a = \frac{\mu+\nu}{2}, \quad b = \frac{\mu-\nu}{2}. \tag{5.12}$$

The above gives

$$\begin{aligned}\mathbf{P}(A_{T_\lambda}^{(\nu)} \in du) &= \frac{du}{2^b\Gamma(b)u^{b+1}}a\int_0^1 ds\,(1-s)^{a-1}s^b e^{-s/2u} \\ &= -\frac{\Gamma(a+1)}{\Gamma(a+b+1)}e^{-1/2u}F\left(a, a+b+1; \frac{1}{2u}\right)d\left(\frac{1}{2u}\right)^b, \\ F(\alpha,\gamma;z) &= \sum_{n=0}^\infty \frac{(\alpha)_n}{(\gamma)_n}\frac{z^n}{n!} \quad \text{Kummer's confluent hypergeometric series.}\end{aligned} \tag{5.13}$$