International Workshop on Conformal Dynamics and Loewner Theory

Date: November 22 – 24, 2014
Venue: Tokyo Institute of Technology, Ookayama Campus
Main Bldg. Room H121

PROGRAM

November 22 (Saturday)

10:30 – 10:40	Opening
10:40 – 11:30	**Ikkei Hotta** (Tokyo Institute of Technology, Japan) *Introduction to Loewner Theory*
11:40 – 12:30	**Santiago Díaz-Madrigal** (University of Seville, Spain) *Fixed points in Loewner theory*
14:30 – 15:20	**Toshiyuki Sugawa** (Tohoku University, Japan) *An application of the Loewner theory to trivial Beltrami coefficients*
15:30 – 16:20	**Andrea del Monaco** (University of Rome "Tor Vergata", Italy) *Geometry and Loewner Theory*
16:40 – 17:30	**Michiaki Onodera** (Kyushu University) *On a deformation flow for an inverse problem in potential theory*

November 23 (Sunday)

10:00 – 10:50	**Pavel Gumenyuk** (University of Stavanger, Norway) *Loewner-type Parametric Representation of univalent self-maps with given boundary regular fixed points*
11:00 – 11:50	**Takashi Takebe** (National Research University - HSE, Russia) *Loewner equations and dispersionless integrable hierarchies*
13:30 – 14:20	**Hiroyuki Suzuki** (Chuo University, Japan) *Convergence of loop erased random walks on a planar graph to a chordal SLE(2)*
14:30 – 15:20	**Ikkei Hotta** (Tokyo Institute of Technology, Japan) L^d*-Loewner chains with quasiconformal extensions*
15:20 – 15:30	Closing

November 24 (Monday) : Free discussion

ABSTRACT OF TALKS

Introduction to Loewner Theory

Ikkei Hotta (Tokyo Institute of Technology, Japan)

In the early 1900s, mathematicians became interested in the fine structure of the family of conformal maps. One conjecture in particular, posed by Ludwig Bieberbach in 1916, attracted special attention, so-called the Bieberbach conjecture. In 1923, Karl Löwner introduced a method to solve a part of the conjecture with the differential equation which was later named after him. His approach was innovative, and actually provided a key ingredient in the complete proof of the conjecture by de Branges in 1985. Nowadays, this theory which centers around time parameterized conformal maps and ordinary/partial differential equations such maps satisfy is called Loewner theory. It made remarkable advances in the last decade, including the celebrated Schramm-Loewner evolution.

The main aim of this introductory talk is to outline the history of Loewner theory in the 20th century with the works due to Löwner, Pommerenke (radial case) and Kufarev (chordal case). We also mention some related topics, in particular semigroups of holomorphic maps on the unit disk which is closely concerned with the Loewner ordinary differential equation.

Fixed points in Loewner theory

Santiago Díaz-Madrigal (University of Seville, Spain)

Starting from the case of semigroups, we analyze (and compare) fixed points of evolution families as well as critical points of the associated vector fields. A number of examples are also shown to clarify the role of the different conditions assumed in the main theorems.

An application of the Loewner theory to trivial Beltrami coefficients

Toshiyuki Sugawa (Tohoku University, Japan)

A Beltrami coefficient on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is an element μ of $L^\infty(\mathbb{D})$ with $\|\mu\|_\infty < 1$. The measurable Riemann mapping theorem implies that there is a unique quasiconformal automorphism $f = f_\mu$ of $\mathbb{D}$ such that $\partial_{\bar{z}} f = \mu \partial_z f$ a.e. and that $f(1) = 1, f(i) = i, f(-1) = -1$. (Note that a quasiconformal automorphism of $\mathbb{D}$ is known to extend homeomorphically in a unique way to its closure.) A Beltrami coefficient μ on $\mathbb{D}$ is called *trivial* if $f_\mu(\zeta) = \zeta$ for all $\zeta \in \mathbb{T} = \partial\mathbb{D}$. We denote by $M(\mathbb{D})$ and $M_0(\mathbb{D})$ the sets of Beltrami coefficients and trivial Beltrami coefficients on the unit disk, respectively. Recall that the universal Teichmüller space is defined as the quotient space $M(\mathbb{D})/M_0(\mathbb{D})$. The tangent space of $M_0(\mathbb{D})$ at 0 in $L^\infty(\mathbb{D})$ is called the set of *infinitesimally trivial* Beltrami differentials on $\mathbb{D}$.

Though the structure of the set of infinitesimally trivial Beltrami coefficients is well understood, the set $M_0(\mathbb{D})$ of trivial Beltrami coefficients is not much understood so far.

By making use of the Löwner equation techniques employed by T. Betker (*Löwner chains and quasiconformal extensions*, Complex Variables **20** (1992), 107–111), we will construct a large family of trivial Beltrami coefficients.

Geometry and Loewner Theory

Andrea del Monaco (University of Rome “Tor Vergata”, Italy)

Geometric Function Theory is the *deus ex machina* of Loewner Theory. In particular, the notion of "slit" plays a strategic rôle: if on the one hand slits yield the classical Loewner equations, on the other hand slits provide an example that marks the difference between dimension $n \geq 2$ and dimension $n = 1$.

On a deformation flow for an inverse problem in potential theory

Michiaki Onodera (Kyushu University)

The rigidity of the shape of a curve (or hypersurface in higher-dimensional spaces) on which harmonic functions satisfy the arc length mean value formula is studied by a new flow approach. We prove that, if a measure is sufficiently close to the Dirac measure, then there exists a unique smooth closed curve admitting a generalized mean value formula: the integral of any harmonic function with respect to the given measure has the same value as the standard integral on the curve.

The proof is based on the investigation of the dynamical structure of a deformation flow which describes the behavior of the domain when the corresponding measure approaches the Dirac measure. It is worth mentioning that the corresponding deformation flow for the area mean value formula is the Hele-Shaw flow, and thus our flow has an analogous property: all the complex moments, except for the length, of the evolving curve are preserved under the flow; while the Hele-Shaw flow preserves complex moments of the evolving domain.

Loewner-type Parametric Representation of univalent self-maps with given boundary regular fixed points

Pavel Gumenyuk (University of Stavanger, Norway)

It is well-known that the classical Loewner Theory provides the so-called *Parametric Representation* for the much studied class $\mathcal{S}$ of all normalized univalent holomorphic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$ via solutions of a controllable ODE, known as the (radial) *Loewner differential equation*. It is less widely known that the radial Loewner equation also gives a representation for all univalent holomorphic self-maps $\varphi \colon \mathbb{D} \to \mathbb{D}$ with a fixed point at the origin. This representation is based on the fact that such maps form a semigroup w.r.t. the composition operation. It follows an abstract scheme going back to Ch. Loewner, which in principle might be applied to other semigroups with (in some sense) compatible diffeology and which can be considered as a non-autonomous extension of Lie Group Theory. The main problem is that in general there are no known criteria to determine whether the subsemigroup formed by all *representable* elements coincides with the original semigroup. Hence it would be interesting to analyze Loewner's scheme in many different concrete examples.

In this talk we consider semigroups of univalent holomorphic self-maps with certain asymptotic behavior at given boundary points. Thanks to the Schwarz Lemma, a holomorphic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ can have at most one fixed point in $\mathbb{D}$. An important role in the study of holomorphic self-maps is played by the so-called *boundary regular fixed points*, *i.e.* points $\sigma \in \partial\mathbb{D}$ such that $\varphi(r\sigma) \to \sigma$ as $r \to 1^-$ and $\alpha_\varphi(\sigma) := \liminf_{z\to\sigma} \frac{1-|\varphi(z)|}{1-|z|} < +\infty$.

Probably the first attempt to construct a kind of parametric representation for univalent holomorphic self-maps with one boundary regular fixed point at $\sigma = 1$ and to apply it to extremal problems for such maps with given $\alpha_\varphi(1)$ was made by H. Unkelbach [*Math. Z.* **46** (1940), 329–336]. Such a parametric representation in a rigorous way was established in 2011 by V.V. Goryainov [to appear in *Mat. Sb.*].

In this talk we consider the problem of parametric representation of univalent holomorphic self-maps of $\mathbb{D}$ for the case of *several given boundary regular fixed points*. We use the general framework and recent results that will be discussed in a more detail in the talk of Santiago Díaz-Madrigal.

Loewner equations and dispersionless integrable hierarchies

Takashi Takebe (National Research University - HSE, Russia)

Dispersionless integrable hierarchies are a class of integrable systems obtained as quasi-classical limits of integrable hierarchies like the KP hierarchy, the Toda lattice hierarchies and so on. It turned out that Loewner type equations appear from the reduction of dispersionless integrable hierarchies. Examples of this phenomenon are presented together with very brief minimum (for the main topic) introduction to integrable systems for non-experts.

Convergence of loop erased random walks on a planar graph to a chordal SLE(2)

Hiroyuki Suzuki (Chuo University, Japan)

In this talk, we consider the natural random walk on a planar graph and scale it by a small positive number δ. Given a simply connected domain D and its two boundary points a and b, we start the scaled walk at a vertex of the graph nearby a and condition it on its exiting D through a vertex nearby b, and prove that the loop erasure of the conditioned walk converges, as $\delta \to 0$, to the chordal SLE_2 that connects a and b in D, provided that an invariance principle is valid for both the random walk and the dual walk of it. For the proof a suitably chosen martingale associated with the evolving random curve, called martingale observable, plays a dominant role. We take the martingale observable given by the ratio of harmonic measures of a (random) point relative to two points, the starting site of the walk and a suitably chosen site in a random domain defined by the loop erasure. This martingale is suggested in [1] as a suitable candidate of a martingale observable but we need to normalize it in an appropriate way; moreover we must change the normalization as the loop erasure grows.

[1] Gregory F. Lawler, Oded Schramm and Wendelim Werner, *Conformal invariance of planar loop-erased random walks and uniform spanning trees*, Ann. Probab. **32**, **1B** (2004), 939-995.

L^d-Loewner chains with quasiconformal extensions

Ikkei Hotta (Tokyo Institute of Technology, Japan)

Recently, a new approach in Loewner theory has been proposed by F. Bracci, M. Contreras, S. Díaz-Madrigal and P. Gumenyuk which gives a unified treatment of both the radial and chordal version of the Loewner equations. In this framework, a generalized Loewner chain, called a Loewner chain of order d, satisfies the differential equation

$$\partial_t f_t(z) = (z - \tau(t))(1 - \overline{\tau(t)}z)\partial_z f_t(z) p(z, t),$$

where $\tau : [0, \infty) \to \overline{\mathbb{D}}$ is measurable and p is called the Herglotz function of order d. In this paper we show that if there exists a $k \in [0, 1)$ such that the above Herglotz function p satisfies

$$|p(z, t) - 1| \leq k|p(z, t) + 1|$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, then f_t has a k-quasiconformal extension to the Riemann sphere for each $t \in [0, \infty)$. In the theorem, any superfluous assumption is not imposed on τ. As a key idea of the proof, an approximation approach by means of the continuous dependence of evolution families is proposed.