INTERNATIONAL WORKSHOP ON
DYNAMICAL SYSTEMS AND LOEWNER THEORY

Geometry and Loewner Theory

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Tokyo, Japan - November 22nd, 2014

arcs...

Let D be a domain in $\mathbb{C}^n$, $n \geq 1$.

definition

A subset σ of D is called a **Jordan arc in D** if there exists a continuous injective function $\Sigma : [0, 1) \longrightarrow D$ such that $\Sigma\big([0, 1)\big) = \sigma$

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- if σ is an arc in D and Σ is a function as given by the above definition, we will say that Σ is a **parameterization** of the arc σ
- if $\overline{\sigma}$ is the closure of σ in $\overline{D}$, we shall also say that $\Sigma(0)$ and $\Sigma(1)$ are the **endpoints** of σ in D

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Notation: σ will denote both the arc and its parameterization

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The set

$$\bigcap_{\varepsilon>0} \overline{\sigma\big([1-\varepsilon,1)\big)} =: \Omega(\sigma)$$

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A Jordan arc σ in D is said to be a **Jordan boundary path in D** if $\Omega(\sigma) \subset \partial D$

Note that if σ is a boundary path in some domain D of the complex plane $\mathbb{C}$, then $\Omega(\sigma)$ is either a subarc of ∂D or a point.

...and slits

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Given a slit γ in D, we will say that

- the endpoint $\gamma(0)$ is its **tip**
- the endpoint $\gamma(1)$ is its **root**

We shall also say that the slit γ **lands** at the point $p = \gamma(1)$, which is then called its **landing point**

...and slits

Let $f : \mathbb{D} \xrightarrow{\text{into}} \hat{\mathbb{C}}$ be a conformal mapping and let γ be a slit in $D := f(\mathbb{D})$.

Theorem

The set $f^{-1}(\gamma)$ is a slit in $\mathbb{D}$.

...and slits

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Theorem

The set $f^{-1}\left(\gamma\right)$ is a slit in $\mathbb{D}$.

Note that this means that if $\Gamma\colon [0, T] \longrightarrow \overline{\mathbb{D}}$ is any parametrization of γ, then $f^{-1} \circ \Gamma\big|_{[0,T)}$ has a continuous extension to the point $t = T$.

why slits?!

our setting...

Let $g : \mathbb{H} \longrightarrow H$ be a conformal map with $H := \mathbb{H} \setminus \gamma$ and $g(\infty) = \infty$.

Theorem

Let $w \in \partial H$ and $\mathcal{W} := g^{-1}(\{w\})$.
Then the map g establishes a bijective correspondence between the connected components of $\partial\mathbb{H} \setminus \mathcal{W}$ and those of $\partial H \setminus \{w\}$.

In particular, the set $\mathcal{W}$ consists of $\nu \in \mathbb{N}$ pairwise distinct points if and only if $\partial H \setminus \{w\}$ has exactly ν connected components.

our setting...

Moreover, letting ξ_0 and ω_0 be respectively the root and the tip of the slit γ, we have:

- the preimage $g^{-1}(\xi_0)$ of ξ_0 consists exactly of two points $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$
- the preimage $g^{-1}(\omega_0)$ of ω_0 consists of a unique point $\lambda \in (\alpha, \beta)$
- g maps $\hat{\mathbb{R}} \setminus [\alpha, \beta]$ homeomorphically onto $\hat{\mathbb{R}} \setminus \{\xi_0\}$
- each of the segments $[\alpha, \lambda]$ and $[\lambda, \beta]$ is mapped homeomorphically onto $\bar{\gamma} := \gamma \cup \{\xi_0\}$ by g

how to find such a function?!

Theorem

Let γ be a slit in the upper half-plane $\mathbb{H}$ landing at some point $\xi_0 \in \mathbb{R}$. Set $\bar{\gamma} := \gamma \cup \{\xi_0\}$.
There exists a unique single-slit mapping $g_\gamma : \mathbb{H} \xrightarrow{\text{onto}} H := \mathbb{H} \setminus \gamma$ such that

$$\lim_{z \to \infty} g(z) - z = 0 \qquad (1)$$

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Moreover, if $C := g_\gamma^{-1}(\bar{\gamma})$ and $\bar{\gamma}^*$ is the reflection of $\bar{\gamma}$ w.r.t. $\mathbb{R}$, then

- $g_\gamma\big|_{\mathbb{H}}$ extends to a conformal map $g_\gamma^* : \hat{\mathbb{C}} \setminus C \xrightarrow{\text{onto}} \hat{\mathbb{C}} \setminus (\bar{\gamma} \cup \bar{\gamma}^*)$
- g_γ^* has a Laurent expansion at ∞ of the form

$$g_\gamma^*(z) = z + \sum_{n=1}^{\infty} c_n z^{-n},$$

with $c_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and $c_1 < 0$

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Then, for any $t \in [0,T)$, we have

- a slit γ_t in $\mathbb{H}$ defined as $\gamma_t := \Gamma[0,t]$
- a domain $H_t := \mathbb{H} \setminus \gamma_t$
- a conformal map $g_t: \mathbb{H} \longrightarrow \mathbb{H} \setminus \gamma_t$, with $g_t(z) = z + \sum_{n=1}^{\infty} c_n(t) z^{-n}$,

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Remark: to include the case $t = T$ we set $\gamma_T \coloneqq \varnothing$ and $g_T \ :=\mathrm{id}_{\mathbb{H}}$

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- a slit γ_t in $\mathbb{H}$ defined as $\gamma_t := \Gamma[0,t]$
- a domain $H_t := \mathbb{H} \setminus \gamma_t$
- a conformal map $g_t: \mathbb{H} \longrightarrow \mathbb{H} \setminus \gamma_t$, with $g_t(z) = z + \sum_{n=1}^{\infty} c_n(t) z^{-n}$,

Note that for any $t < s$

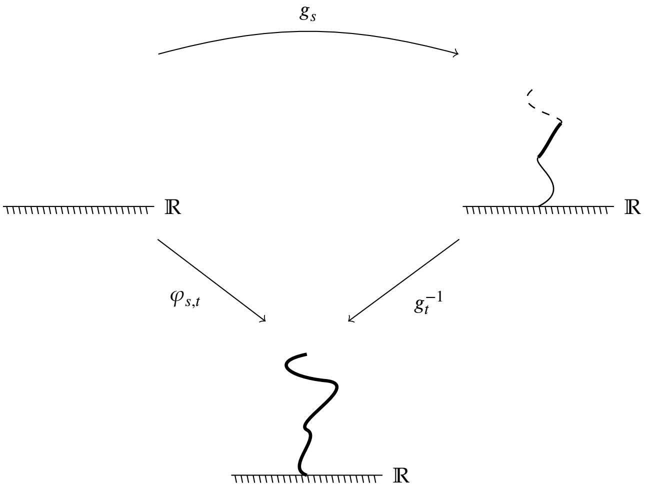
- the slits γ_t and γ_s share the same root ξ_0
- $\gamma_s \subset \gamma_t$
- $H_t \subset H_s$

and how to use this fact?!

Now, for $s \leq t$ in $[0, T]$ and $z \in \mathbb{H}$, define

$$\phi_{s,t}(z) := \left(g_t^{-1} \circ g_s\right)(z)$$

and how to use this fact?!



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Now, for $s \le t$ in $[0, T]$ and $z \in \mathbb{H}$, define

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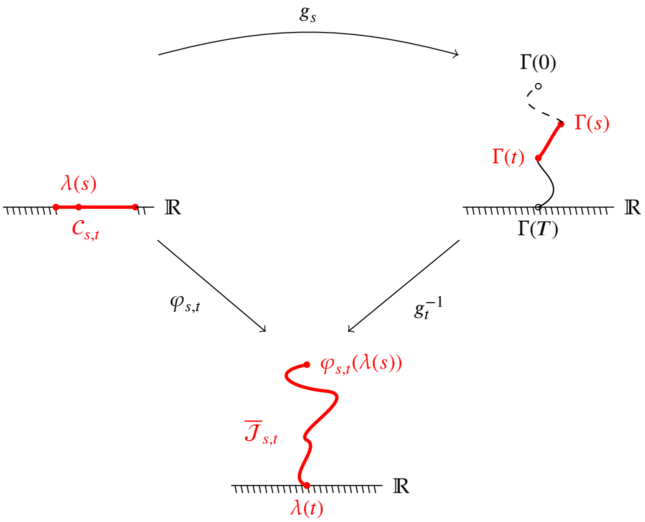
Furthermore, set

- $\lambda(t) := g_t^{-1}\left(\Gamma(t)\right) \in \mathbb{R}$
- $\mathcal{J}_{s,t} := g_t^{-1}\left(\Gamma([s,t))\right) \subset \mathbb{H}$
- $\bar{\mathcal{J}}_{s,t} := g_t^{-1}\left(\Gamma([s,t])\right) = \mathcal{J}_{s,t} \cup \{\lambda(t)\}$
- $C_{s,t} := g_s^{-1}\left(\Gamma([s,t])\right) \subset \mathbb{R}$

and how to use this fact?!



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Note that for any $0 \leq s \leq u \leq t \leq T$ we have

- $\phi_{s,t}$ is a conformal map from $\mathbb{H}$ onto $\mathbb{H} \setminus \mathcal{J}_{s,t}$
- $\lim_{z\to\infty} \phi_{s,t}(z) - z = 0$
- $\phi_{s,t}|_{\mathbb{H}}$ extends to a conformal map $\phi^*_{s,t} : \hat{\mathbb{C}} \setminus C_{s,t} \xrightarrow{\text{onto}} \hat{\mathbb{C}} \setminus \left(\bar{\mathcal{J}}_{s,t} \cup \bar{\mathcal{J}}^*_{s,t}\right)$ with $\bar{\mathcal{J}}^*_{s,t}$ being the reflection of $\bar{\mathcal{J}}_{s,t}$ w.r.t. $\mathbb{R}$
- $\phi^*_{s,t}(z) = z + \sum_{n=1}^{+\infty} c_n(s,t) z^{-n}$ with $c_1(s,t) = c_1(s) - c_1(t) < 0$

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and we also have that

- $\phi_{s,t} = \phi_{u,t} \circ \phi_{s,u}$
- $\phi_{s,t}(\zeta) = \zeta + \frac{1}{\pi} \int_{C_{s,t}} \frac{\mathbb{I}\text{m}\{\phi_{s,t}(\xi)\}}{\xi - \zeta} \, d\xi$ for all $\zeta \in \mathbb{H}$
- $t - s = \frac{1}{\pi} \int_{C_{s,t}} \mathbb{I}\text{m}\left\{\phi_{s,t}(\xi)\right\} \, d\xi$

how to use this fact

Furthermore,

- for any fixed $t \in (0, T]$, the arc $\bar{\mathcal{J}}_{u,t}$ shrinks to the point $\lambda(t)$ and the segment $\mathcal{C}_{u,t}$ tends to the same point as $u \uparrow t$
- for any fixed $s \in [0, T)$, the segment $\mathcal{C}_{s,u}$ shrinks to the point $\lambda(s)$ and the arc $\bar{\mathcal{J}}_{s,u}$ tends to the same point as $u \downarrow s$
- the function $[0, T] \ni t \mapsto \lambda(t)$ is continuous
- the function $[0, T] \ni t \mapsto c_1(t)$ is continuous and strictly increasing

definition

a parametrization $\Gamma : [0, T] \longrightarrow \overline{\mathbb{H}}$ of the slit γ is said to be a **standard parametrization of** γ if $c_1(t) = t - T$ for all $t \in [0, T]$

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proposition

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There exists a unique standard parametrization Γ *of the slit* γ.

Remark: in some applications, it seems to be convenient to rescale the standard parametrization in such a way that $c_1(t) = 2(t - T)$.

how to describe the evolution now?

Theorem

There exists a unique continuous function $\lambda : [0, T] \to \mathbb{R}$ *such that, for every* $s \in [0, T)$ *and every* $z \in \mathbb{H}$*, the function*

$$[s, T] \ni t \longmapsto w_{z,s}(t) := \phi_{s,t}(z)$$

is the unique solution to the Cauchy problem

$$\begin{cases} \dot{\mathrm{F}}(t) = \dfrac{1}{\lambda(t) - \mathrm{F}(t)}, & t \in [s, T] \\ \mathrm{F}(s) = z \end{cases}$$

The equation above is called the **chordal Loewner ODE**.

Recall that we have

- $\phi_{s,t}(\zeta) - \zeta = \frac{1}{\pi} \int_{C_{s,t}} \frac{\mathbb{I}\mathrm{m}\{\phi_{s,t}(\xi)\}}{\xi - \zeta} \, d\xi \quad \text{for all} \quad \zeta \in \mathbb{H}$

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and

- $t - s = \frac{1}{\pi} \int_{C_{s,t}} \mathbb{Im}\left\{\phi_{s,t}(\xi)\right\} \, d\xi$

Assume $s < t$ and take any $u \in [s, t)$. Taking $\zeta := \phi_{s,u}(z)$, we get:

$$\frac{\phi_{s,t}(z) - \phi_{s,u}(z)}{t-u} = \frac{\phi_{u,t}(\zeta) - \zeta}{t-u} = \frac{\displaystyle\int_{C_{u,t}} \frac{\mathbb{Im}\{\phi_{u,t}(\xi)\}}{\xi - \phi_{s,u}(z)} \, d\xi}{\displaystyle\int_{C_{u,t}} \mathbb{Im}\left\{\phi_{u,t}(\xi)\right\} \, d\xi}$$

Applying the Integral Mean Value Theorem, separately for the real and imaginary parts of $\frac{1}{\xi - \phi_{s,u}(z)}$, we may write

$$\frac{\phi_{s,t}(z) - \phi_{s,u}(z)}{t-u} = \frac{1}{\xi_{u,t} - \phi_{s,u}(z)}$$

for some $\xi_{u,t} \in C_{u,t}$.

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for some $\xi_{u,t} \in \mathcal{C}_{u,t}$.

Since both $\mathcal{C}_{u,t}$ and $\mathcal{J}_{u,t}$ tend to $\lambda(t)$ as $u \uparrow t$, we see

- $\phi_{s,u}(z) \to \phi_{s,t}(z)$
- $\xi_{u,t} \to \lambda(t)$

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So we finally get that

$$\frac{\phi_{s,t}(z)-\phi_{s,u}(z)}{t-u}\longrightarrow\frac{1}{\lambda(t)-\phi_{s,t}(z)}$$

and $\phi_{s,t}(z)$ is differentiable from the left.

Analogously, assuming $t < T$ and taking $u \in [t, T)$, we see that

$$\frac{\phi_{s,u}(z) - \phi_{s,t}(z)}{u - t} \longrightarrow \frac{1}{\lambda(t) - \phi_{s,t}(z)}$$

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Thus $\phi_{s,t}(z)$ is also differentiable from the right.

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Thus $\phi_{s,t}(z)$ is also differentiable from the right.

Since the function λ is continuous from $[0, T]$ to $\mathbb{R}$, we have done!

As for the uniqueness, notice that the right hand side is of the form $G\big(\mathrm{F}(t), t\big)$ where the vector field

$$G(w) = \frac{1}{\lambda(t) - w}$$

is Lipschitz continuous in its first variable locally uniformly in $\mathbb{H}$ and the Lipschitz constant does not depend on t.

So, the uniqueness of the solution follows now from the Cauchy Theorem for ODE.

Recall that $\phi_{s,T} = g_s$ for all $s \in [0, T]$.

Then, since $g_t(z)$ is differentiable jointly in z and t, we see that it follows from the chordal Loewner ODE that

$$\frac{\partial g_t(z)}{\partial t} = -\frac{{g_t}'(z)}{\lambda(t) - z}$$

The last equation is know as the **chordal Loewner PDE**.

Consider now the family of the inverse conformal mappings $(h_t)_{t\in[0,T]}$ with

$$h_t := g_t^{-1} : \mathbb{H} \setminus \gamma_t \longrightarrow \mathbb{H}$$

Then $g_t \circ h_t = \mathrm{id}$.

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Differentiating both sides by t we get that $t \mapsto h_t(z)$ solves the chordal Loewner ODE, i.e.

$$\frac{\partial h_t(z)}{\partial t} = \frac{1}{\lambda(t) - h_t(z)}$$

for all $t \in [0, T]$ and $z \in H_t$, with initial conditions given by

$$h_t|_{t=T} = \mathrm{id}$$

Set now $\sigma = T - t$. Then we see that the previous equation becomes

$$\frac{\partial h_\sigma(z)}{\partial \sigma} = \frac{1}{h_\sigma(z) - \lambda(\sigma)}$$

with the new initial data given by

$$h_\sigma|_{\sigma=0} = \mathrm{id}$$

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This allows us

- to consider all $\sigma \geq 0$ and thus Jordan arcs $\Gamma : [0, +\infty] \to \overline{\mathbb{H}}$
- to give a reasonable meaning to the word “chordal”
- to get SLE_k

Let $\mathscr{C} : \mathbb{D} \longrightarrow \mathbb{H}$ the Cayley map, i.e. let

$$\mathscr{C}(z) = i\frac{z+1}{1-z}$$

and consider $\psi_{s,t}(z) \coloneqq \mathscr{C}^{-1} \circ \phi_{s,t} \circ \mathscr{C}(z)$.
Then

$$\frac{\partial}{\partial t}\left[\psi_{s,t}(z)\right] = \frac{\partial}{\partial t}\left[\mathscr{C}^{-1}\left(\phi_{s,t}(\mathscr{C}(z))\right)\right]$$

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Then

$$\frac{\partial}{\partial t}\left[\psi_{s,t}(z)\right] = \frac{\partial}{\partial t}\left[\mathscr{C}^{-1}\left(\phi_{s,t}(\mathscr{C}(z))\right)\right]$$

from which we get that

$$\frac{\partial}{\partial t}\left[\psi_{s,t}\right] = -\left(1-\psi_{s,t}\right)^2 \frac{1}{\frac{\psi_{s,t}+1}{1-\psi_{s,t}} + i\lambda(t)}$$

Note that $\mathbb{R}\mathrm{e}\dfrac{1}{\dfrac{w+1}{1-w}+i\lambda(t)} \geq 0$

So, calling $p(w,t) = \dfrac{1}{\frac{w+1}{1-w}+i\lambda}$, we can write

$$\frac{\partial}{\partial t}\left[\psi_{s,t}\right] = -\left(1-\psi_{s,t}\right)^2 p(\psi_{s,t}, t)$$

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So, calling $p(w,t) = \dfrac{1}{\frac{w+1}{1-w}+i\lambda}$, we can write

$$\frac{\partial}{\partial t}\left[\psi_{s,t}\right] = -\left(1-\psi_{s,t}\right)^2 p(\psi_{s,t}, t)$$

and we see that the (classical) chordal equation is a particular case of the general Loewner equation given by

$$\dot{w} = (w-\tau)(1-\overline{\tau}w)p(w,t)$$

back to geometric function theory

Recall that to prove that $\phi_{s,t}$ map $\mathbb{H}$ onto the slit domains $\mathbb{H} \setminus \mathcal{J}_{s,t}$ we made use of the following fact

theorem

If f is conformal, then $f^{-1}(\gamma)$ is a slit in $\mathbb{D}$.

Recall that to prove that $\phi_{s,t}$ is map $\mathbb{H}$ onto the slit domain $\mathbb{H} \setminus \mathcal{J}_{s,t}$ we made use of the following fact

theorem

If f is a biholomorphism, then $f^{-1}(\gamma)$ is a slit in $\mathbb{D}$.

that is the preimage under a univalent function in $\mathbb{D}$ of a slit in its image is a slit in $\mathbb{D}$.

what about preimages of slits in higher dimensions?

Unfortunately,

There exist univalent functions $f : \mathbb{B}^n \longrightarrow \mathbb{C}^n$ such that, given a slit γ in the image $f(\mathbb{B}^n)$, the set $f^{-1}(\gamma)$ is not a slit in $\mathbb{B}^n$.

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We will prove the existence by construction:

There exist an unbounded univalent function $f : \mathbb{B}^2 \longrightarrow \mathbb{C}^2$ and a slit γ in $f(\mathbb{B}^2)_{\subset \mathbb{C}^2}$ landing at ∞ such that $f^{-1}(\gamma)$ is not a slit in $\mathbb{B}^2$.

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and to do that, we will make use of the following

proposition

Let σ be a boundary path in $\mathbb{D}$. There exists a non-constant holomorphic function $g : \mathbb{D} \longrightarrow \mathbb{C}$ such that $g \to \infty$ along σ.

Let σ be a Jordan boundary path in $\mathbb{D}$ and suppose it is not a slit.
Let $g : \mathbb{D} \longrightarrow \mathbb{C}$ be the function given by the previous proposition.
Define the map $f : \mathbb{B}^2 \longrightarrow \mathbb{C}^2$ as

$$f(z, w) \coloneqq \left(z + w^2 + g^2(z) + wg(z) \, , \; w + g(z) \right)$$

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f is the function we are looking for!

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Let $g : \mathbb{D} \longrightarrow \mathbb{C}$ be the function given by the previous proposition.
Define the map $f : \mathbb{B}^2 \longrightarrow \mathbb{C}^2$ as

$$f(z, w) := \left(z + w^2 + g^2(z) + wg(z) \, , \; w + g(z) \right)$$

f is the function we are looking for! Indeed:

- it is univalent
- $D := f(\mathbb{B}^2)$ is an unbounded domain of $\mathbb{C}^2$
- $f \longrightarrow \infty$ along σ
- γ is a slit in D
- $f^{-1}(\gamma)$ is not a slit in $\mathbb{B}^2$

Note that:

- the construction does not rely on any peculiar property of the unit ball $\mathbb{B}^2$
- it only depends on Jordan boundary paths in $\mathbb{D}$ and automorphisms of $\mathbb{C}^2$

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- it only depends on Jordan boundary paths in $\mathbb{D}$ and automorphisms of $\mathbb{C}^2$

As a consequence:

There exist univalent functions $f : \mathbb{D}^n \longrightarrow \mathbb{C}^n$ such that, given a slit γ in the image $f(\mathbb{D}^n)$, the set $f^{-1}(\gamma)$ is not a slit in $\mathbb{D}^n$.

Grazie a tutti per l'attenzione!

(arigatōgozaimasu)