Löwner Equations
and
Dispersionless Integrable Hierarchies

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§0. Introduction

- Integrable hierarchies = 'solvable' systems with infinitely many variables (e.g., $t = (t_1, t_2, t_3, \dots)$).
- Dispersionless integrable hierarchies = quasi-classical limits of certain integrable hierarchies.
- One-variable reduction: solutions depend on ∞-many variables only through one function, e.g., $\lambda(t)$.

Today's topic

one-variable reduction of the dispersionless KP
(resp. Toda, BKP, DKP) hierarchy

$$\Updownarrow$$

the chordal (resp. radial, quadrant, annulus) Löwner equation.

Plan of the talk:

1. Brief introduction to integrable systems.
2. KP hierarchy and Toda lattice hierarchy.
3. Dispersionless hierarchies.
4. Dispersionless Hirota equations.
5. dKP hierarchy and chordal Löwner equation.
6. Other examples.

Disclaimer: In this talk everything is quite “algebraic”:

- “functions” = formal power series
- “operators” = elements of non-commutative rings

Only algebraic structure is studied.

(& “genericity conditions” often omitted, ...)

§1. What are "integrable systems"?

For systems with *finite* degrees of freedom,

$\exists$ well established/defined geometric criteria of integrability.

- Frobenius integrability condition
- Liouville integrability condition (for Hamiltonian systems)
  = "existence of sufficiently many conserved quantities"

Examples: Kepler motion, Tops (Euler, Lagrange, Kowalevski)

How about "integrable systems" with infinite degrees of freedom?

Modern theory of integrable systems began with the discovery of *remarkable solutions of non-linear partial differential equations* = "SOLITONS" in 1960's.

Soliton = particle-like stable solitary wave

Examples of soliton equations:

- KdV equation (1895): $u = u(x,t)$, $u_t - 3uu_x - \frac{1}{4}u_{xxx} = 0$.
- KP equation (1970): $u = u(x,y,t)$,
$$\frac{3}{4}u_{yy} - (u_t - 3uu_x - \frac{1}{4}u_{xxx})_x = 0.$$
- Sine-Gordon equation : $u = u(x,t)$, $u_{tt} - u_{xx} - \sin u = 0$.
- Toda lattice (1967): $u_n = u_n(t)$, $u_{n,tt} = e^{u_{n-1}-u_n} - e^{u_n - u_{n+1}}$.

- Surprisingly, such soliton equations are *solvable* in spite of its nonlinearity!
  - inverse scattering method, Lax pairs
  - algebro-geometric solutions
  - Hirota's bilinear method

  $\Longrightarrow$ various generalisation
- Why are they solvable? $\Longrightarrow$ discovery of
  - infinitely many conserved quantities/ symmetries
  - moduli space of solutions (e.g., ∞-dimensional Grassmann manifold for KP hierarchy)

  $\Longrightarrow$ relation to algebra (e.g., representation theory of ∞-dimensional Lie algebras).

Let us examine the KP and the Toda lattice hierarchies as examples.

§2 KP hierarchy and Toda lattice hierarchy

KP hierarchy: integrable nonlinear system for $u_i(t)$ $(i = 2, 3, \dots)$ w.r.t. $t = (t_1, t_2, t_3, \dots)$. ($x = t_1$, $\partial = \partial/\partial x$.)

The *Lax operator*: $L = \partial + u_2(t)\partial^{-1} + u_3(t)\partial^{-2} + \cdots$.

Notation: symbols $f(x)\partial^m$ for $m \in \mathbb{Z}$ span an algebra:

$$\left(f(x)\partial^m\right)\left(g(x)\partial^n\right) = \sum_{r=0}^{\infty} \binom{m}{r} f g^{(r)} \partial^{m+n-r}. \quad \left(\binom{m}{r} = \frac{m(m-1)\cdots(m-r+1)}{r!}\right)$$

KP hierarchy: (Lax representation)

$$(\text{KP}) \qquad \frac{\partial L}{\partial t_n} = [B_n, L] \qquad (n = 1, 2, \dots; B_n = (L^n)_{\geq 0}).$$

Notation: $P = \sum_{n \in \mathbb{Z}} a_n \partial^n \ \to P_{\geq 0} := \sum_{n \geq 0} a_n \partial^n$.

This includes the KP equation for $u = u_2$:

$$\tfrac{3}{4}u_{t_2t_2} - \left(u_{t_3} - 3uu_x - \tfrac{1}{4}u_{xxx}\right)_x = 0$$

$\because$) First two equations $\dfrac{\partial L}{\partial t_2} = [B_2, L]$ and $\dfrac{\partial L}{\partial t_3} = [B_3, L]$ are expanded as

$$\frac{\partial u_2}{\partial t_2}\partial^{-1} + \frac{\partial u_3}{\partial t_2}\partial^{-2} + \cdots = (u_2'' + 2u_3')\partial^{-1} + (u_3'' + 2u_4' + 2u_2u_2')\partial^{-2} + \cdots .$$
$$\frac{\partial u_2}{\partial t_3}\partial^{-1} + \frac{\partial u_3}{\partial t_3}\partial^{-2} + \cdots = (3u_3'' + 3u_4' + 6u_2u_2' + u_2''')\partial^{-1} + \cdots .$$

($(\cdot)' = \partial(\cdot)/\partial x$.) Comparing the coefficients of ∂^{-1} and ∂^{-2} we have

$$\frac{\partial u_2}{\partial t_2} = u_2'' + 2u_3', \qquad \frac{\partial u_3}{\partial t_2} = u_3'' + 2u_4' + 2u_2u_2',$$
$$\frac{\partial u_2}{\partial t_3} = 3u_3'' + 3u_4' + 6u_2u_2' + u_2''',$$

Eliminating u_3 and u_4 we obtain the KP equation. $\square$

- KP hierarchy

  = set of compatibility conditions for the linear problem for $\Psi = \Psi(t; z)$: $L\Psi = z\Psi$, $\dfrac{\partial \Psi}{\partial t_n} = B_n \Psi$. ($z$: spectral parameter)

- L satisfies (KP) $\Leftrightarrow \exists\, \tau(t)$ (tau function) such that

$$\Psi(t; z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} e^{\sum t_n z^n},$$

$\left(t = (t_n)_{n=1,2,\dots},\ t - [z^{-1}] = \left(t_n - \frac{z^{-n}}{n}\right)_{n=1,2,\dots}.\right)$

and $\tau(t)$ satisfies a series of bilinear differential equations (the Hirota equations).

- Solutions of the KP hierarchy are parametrised by ∞-dimensional Grassmann manifold (the Sato Grassmann manifold).

- Hirota equations = defining equations of the Grassmann manifold (Plücker relations)

- ∞-dimensional symmetry:

  $GL(\infty)$ acts on the Sato Grassmann manifold $= GL(\infty)/P_{\infty/2}$.

  $\Big($cf. finite dimensional Grassmann manifold $= GL(N)/P$,

$$P = \left\{ \left(\begin{array}{ccc|ccc} * & \cdots & * & & & \\ \vdots & \ddots & \vdots & & * & \\ * & \cdots & * & & & \\ \hline & & & * & \cdots & * \\ & 0 & & \vdots & \ddots & \vdots \\ & & & * & \cdots & * \end{array} \right) \right\}. \Big)$$

Variants:

- (KP) + constraint $L^2 = \partial^2 + 2u$

  $\Longrightarrow$ KdV hierarchy, which contains the KdV equation for u.

  This has the symmetry of $sl(2, \mathbb{C}[t]) \oplus$ (central extension), i.e., $A_1^{(1)}$-type affine Lie algebra.

- (KP) + constraint $L^* = -\partial L \partial^{-1}$

  (Notation: $(a(x)\partial^n)^* := (-\partial)^n a(x)$ is the formal adjoint operator.)

  $\Longrightarrow$ BKP hierarchy, which has the symmetry of $so(2\infty + 1)$ (B_∞-type).

- There are CKP and DKP hierarchies corresponding to C_∞ and D_∞ type symmetries, but the definitions are involved.

  (Usually defined by the Hirota bilinear equations.)

Toda lattice hierarchy: $\phi, u_n, \bar{u}_n$: unknown functions of s, $t = (t_n)_{n\in\mathbb{Z}, n\neq 0}$.

$$L = e^{\phi} e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + u_3 e^{-2\partial_s} + \cdots ,$$
$$\bar{L}^{-1} = e^{\phi} e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \bar{u}_3 e^{2\partial_s} + \cdots ,$$
$$B_n = \begin{cases} (L^n)_{>0} + \frac{1}{2}(L^n)_0, & (n > 0), \\ (\bar{L}^{-n})_{<0} + \frac{1}{2}(\bar{L}^{-n})_0, & (n < 0). \end{cases}$$

Notation:

- $e^{n\partial_s} f(s) = f(s+n)$: difference operator.
- $A = \sum_{n\in\mathbb{Z}} a_n e^{n\partial_s} \to A_S = \sum_{n\in S} a_n e^{n\partial_s}$ for $S =$ "> 0", "< 0" and "0".

Toda lattice hierarchy: (Lax representation)

(Toda) $$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad \frac{\partial \bar{L}}{\partial t_n} = [B_n, \bar{L}], \qquad (n \in \mathbb{Z},\ n \neq 0).$$

- Parametrisations of solutions, τ function etc. are known.
- $n = \pm 1 \Longrightarrow$ the 2d Toda equation:
  $\phi_{t_1 t_{-1}}(s) = e^{\phi(s-1)-\phi(s)} - e^{\phi(s)-\phi(s+1)}$.
- 2d Toda eq. $+$ constraint $\phi(s+2) = \phi(s)$
  ($+$ change of variables) $\Longrightarrow$ Sine-Gordon eq.
- (Toda) $+$ constraint: $L = \bar{L}^{-1}$
  $\Longrightarrow$ 1d Toda hierarchy (which contains the Toda lattice for ϕ).

§4 Dispersionless hierarchies

Replace

- ∂, e^{∂_s} $\to$ commutative symbols.
- commutator $[,]$ $\to$ Poisson bracket $\{,\}$.

$\Longrightarrow$ dispersionless KP/Toda lattice hierarchies.

dispersionless KP hierarchy: $\partial^n \to w^n$, $\{w, x\} = 1$.

$$\mathcal{L} = w + u_2(t)w^{-1} + u_3(t)w^{-2} + \cdots , \qquad \mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}.$$

$(\mathcal{P} = \sum_{n\in\mathbb{Z}} a_n w^n \ \to \mathcal{P}_{\geq 0} := \sum_{n\geq 0} a_n w^n.)$

dKP hierarchy: $$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\} \qquad (n = 1, 2, \dots).$$

dispersionless Toda lattice hierarchy: $e^{n\partial_s} \to w^n$, $\{w, s\} = w$.

$$\mathcal{L} = e^{\phi} w + u_1 + u_2 w^{-1} + u_3 w^{-2} + \cdots ,$$

$$\tilde{\mathcal{L}}^{-1} = e^{\phi} w^{-1} + \bar{u}_1 + \bar{u}_2 w + \bar{u}_3 w^2 + \cdots ,$$

$$\mathcal{B}_n = \begin{cases} (\mathcal{L}^n)_{>0} + \frac{1}{2}(\mathcal{L}^n)_0, & (n > 0), \\ (\tilde{\mathcal{L}}^{-n})_{<0} + \frac{1}{2}(\tilde{\mathcal{L}}^{-n})_0, & (n < 0). \end{cases}$$

$(A = \sum_{n\in\mathbb{Z}} a_n w^n \to A_S := \sum_{n\in S} a_n w^n$ for $S =$ "> 0", "< 0" and "0".)

dToda hierarchy: $\dfrac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}$, $\dfrac{\partial \tilde{\mathcal{L}}}{\partial t_n} = \{\mathcal{B}_n, \tilde{\mathcal{L}}\}$, $(n \in \mathbb{Z},\ n \neq 0)$.

For dKP/dToda hierarchies, ∞-dimensional symmetries (w_∞-algebra), parametrisation of solutions ($\longleftrightarrow$ canonical transformations) are known. ([Takasaki-T.] 1991–1995)

§5 Dispersionless Hirota equations

(Maybe you feel flavour of complex analysis...)

First obtained in [Takasaki-T. (1995)] as a limit of

the differential Fay identity ($\subset$ Hirota eq.) for KP.

Teo's formulation (2002)

$\mathcal{L}(t; w) = w + u_1(t) w^{-1} + u_2(t) w^{-2} + \cdots.$

$k(t; z)$: inverse fuction of $\mathcal{L}(t; w)$ with respect to w:

$\mathcal{L}(t; k(t; z)) = z, \;\; k(t; \mathcal{L}(t; w)) = w.$

Grunsky coefficients b_{mn} of $k(t; z)$ (... for the Bieberbach conjecture):

(dH1) $$\log \frac{k(t; z_1) - k(t; z_2)}{z_1 - z_2} = - \sum_{m,n=1}^{\infty} b_{mn}(t) z_1^{-m} z_2^{-n}.$$

In other words,

$$\mathcal{L}^n + \sum_{m=1}^{\infty} n b_{nm}(t) \mathcal{L}^{-m} = (\text{polynomial in } w) = (\mathcal{L}^n)_{\geq 0}.$$

In particular

(dH2) $$k(t;z) = z + \sum_{m=1}^{\infty} b_{1,m} z^{-m}.$$

Theorem

$\mathcal{L}(t;w)$: solution of dKP

$\iff$ There exists $\mathcal{F}(t)$ such that $\dfrac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} = -mn b_{mn}(t)$.

(dH1&2) rewritten in terms of $\mathcal{F}(t)$ $\Rightarrow$

dispersionless Hirota eq.:

(dH) $$e^{D(z_1)\,D(z_2)\mathcal{F}} = -\frac{\partial_1\big(D(z_1) - D(z_2)\big)\mathcal{F}}{z_1 - z_2},$$

which $\mathcal{F}(t)$ should satisfy. (Notations: $D(z) := \sum \frac{z^{-n}}{n}\frac{\partial}{\partial t_n}$.)

Remark: τ of KP (with $\hbar$) $= \exp\big(\hbar^{-2}\mathcal{F} + O(\hbar^{-1})\big)$.

($\exists$ similar theorem for dToda.)

§6 Dispersionless KP and Löwner equation

Unexpected relation of the (chordal) Löwner equation and the dispersionless KP hierarchy was found by

- Gibbons-Tsarev (1999): for t_1 and t_2.
- Yu-Gibbons (2000): in general (direct computation).
- Mañas-Martínez Alonso-Medina (2002): proof by "S function" ($\doteqdot \log$ (solution of the auxiliary linear problem of KP))).
- T.-Teo-Zabrodin (2006): proof by dHirota eq.

(Radial (i.e., original) Löwner equation corresponds to the dispersionless Toda.)

Chordal Löwner equation:

$H = \{\mathrm{Im}\, z > 0\}$: the upper half plane.

$\cup$

K_λ ($\lambda \in [0, a]$): growing hull of H, $K_0 = \varnothing$.

$g(\lambda; z) : H \setminus K_\lambda \xrightarrow{\sim} H$: conformal mapping normalised as

$$g(\lambda; z) = z + a_1(\lambda) z^{-1} + O(z^{-2}) \quad (z \to \infty), \qquad g(0; z) = z.$$

$\Longrightarrow \exists\, U(\lambda)$ s.t.

$$\frac{\partial g}{\partial \lambda}(\lambda; z) = \frac{1}{g(\lambda; z) - U(\lambda)} \frac{da_1}{d\lambda}$$: Chordal Löwner equation.

One variable reduction of dKP

Theorem

$\mathcal{L}(t;w)$ is a solution of dKP such that:

$\exists$ functions $\lambda(t)$ & $f(\lambda, w)$: $\mathcal{L}(t;w) = f(\lambda(t), w)$.

$\Longrightarrow$

(i) $f(\lambda, w)$ is the inverse function of a solution $g(\lambda, z)$ of the chordal Löwner eq. $(f(\lambda, g(\lambda, z)) = z, \quad g(\lambda, f(\lambda, w)) = w.)$

(ii) $\lambda(t)$ satisfies $\dfrac{\partial \lambda}{\partial t_n} = \dfrac{\partial \Phi_n}{\partial w}(\lambda; U(\lambda))\dfrac{\partial \lambda}{\partial t_1} \quad (n = 1, 2, \dots)$

Here, $\Phi_n(\lambda; w) = (f(\lambda, w)^n)_{\geq 0}$: Faber polynomial of g.

(Polynomial part of $f(\lambda, w)^n$ w.r.t. w.)

Conversely:

Theorem

$g(\lambda, z)$: solution of chordal Löwner equation.

$f(\lambda, w) = w + O(w^{-1})$: inverse function of g,

i.e., $f(\lambda, g(\lambda, z)) = z$, $\quad g(\lambda, f(\lambda, w)) = w$.

$\lambda(t)$: solution of $\dfrac{\partial \lambda}{\partial t_n} = \dfrac{\partial \Phi_n}{\partial w}(\lambda; U(\lambda)) \dfrac{\partial \lambda}{\partial t_1} \quad (n = 1, 2, \dots)$

$\Longrightarrow \mathcal{L}(t, w) := f(\lambda(t), w)$ is a solution of dKP.

Remark: The equation for $\lambda(t)$ is solved implicitly by the relation

$$t_1 + \sum_{n=2}^{\infty} t_n \frac{\partial \Phi_n}{\partial w}(\lambda; U(\lambda)) = R(\lambda).$$

$R(\lambda)$: arbitrary generic function. (Tsarev's generalised hodograph method.)

§7 Other examples

- mKP hierarchy $\longleftrightarrow$ chordal Löwner-like equation
  (Mañas-Martínez Alonso-Medina)
- Toda hierarchy $\longleftrightarrow$ radial Löwner equation (T.-Teo-Zabrodin, ...)
- BKP hierarchy $\longleftrightarrow$ quadrant Löwner equation (T.)
- DKP hierarchy $\longleftrightarrow$ annulus Löwner (Goluzin-Komatu) equation
  (Akhmedova-Zabrodin)

dBKP hierarchy: dKP + constraint: $\mathcal{L}(w) = -\mathcal{L}(-w)$.

Quadrant Löwner equation:

$$\frac{\partial g}{\partial \lambda} = \frac{g}{V^2 - g^2} \frac{du}{d\lambda}.$$

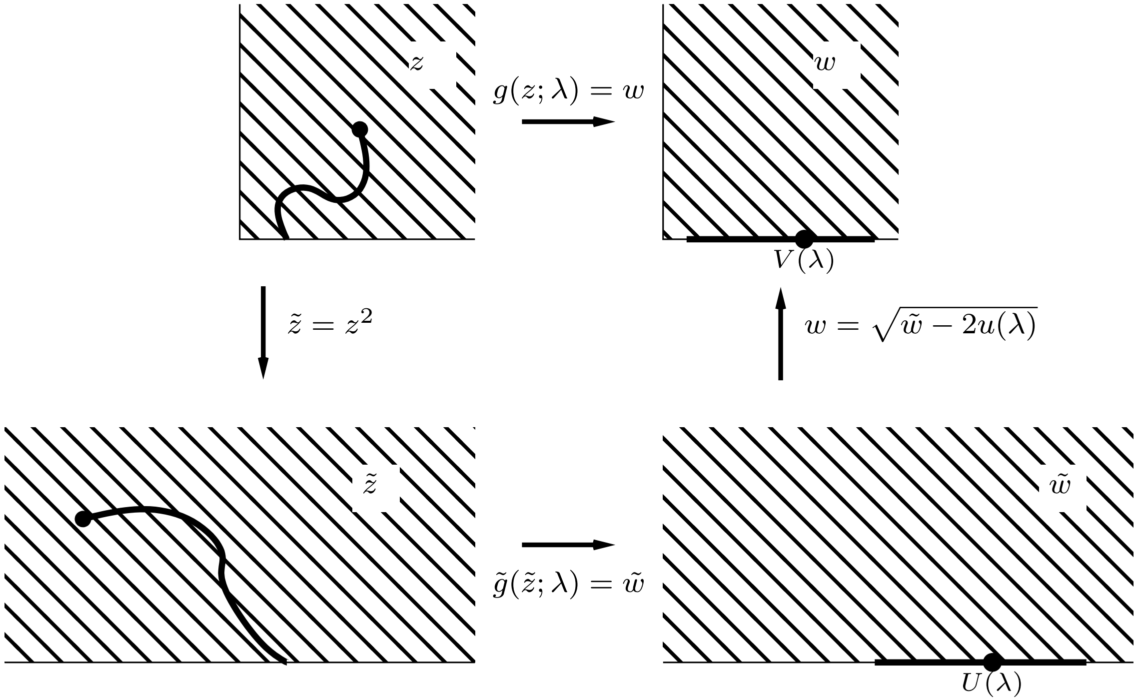

Figure 1: Conformal mapping from a slit domain to the quadrant.

QUESTION

WHY do Löwner type equations give

solutions of dispersionless integrable hierarchies?

Thank you for your attention.

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