Convergence of loop erased random walks on a planar graph to a chordal SLE(2) curve

Hiroyuki Suzuki

Chuo University

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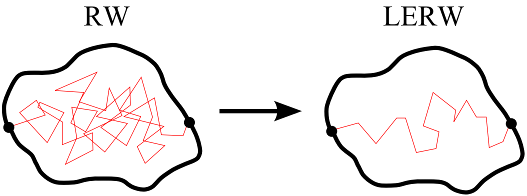
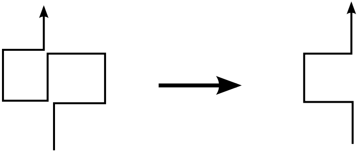
Introduction(1)

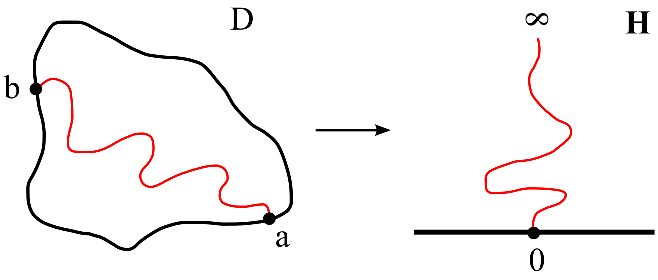
- Model



- Loop erasure



Introduction(2)



- curve in D $\to$ curve in $\mathbb{H}$
- curve $\to$ dynamics of domain
- Represent by Loewner equation

Loewner equation

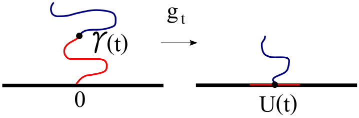
$\gamma : [0,\infty] \to \mathbb{C}$: a simple curve, $\gamma(0) = 0, \gamma(\infty) = \infty, \gamma(0,\infty) \subset \mathbb{H}$,
$g_t : \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$: conformal map, $|g_t(z) - z| \to 0 \quad (z \to \infty)$.
If γ is parametrized by half plane capacity ($\lim_{z\to\infty} z(g_t(z) - z) = 2t$),
g_t satisfies the following differential equation

Loewner equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U(t)}, \; g_0(z) = z,$$

where $U(t) := g_t(\gamma(t))$ and $U(t)$ is a $\mathbb{R}$-valued continuous function.

We call $U(t)$ the driving function of γ.



Rem. We can consider that a curve γ is described by the driving finction $U(t)$.

Candidate for scaling limits

We consider a candidate for scaling limits of the driving function of discrete random curves.
Let γ be the scaling limit of some discrete random curve γ_δ connecting two distinct boundary points a and b of D.
Since there are several conjectures in critical systems, we assume that γ satisfies the following properties.

- Domain Markov property
- Conformal invarinance

Let $\phi : D \to \mathbb{H}$: conformal map, $\phi(a) = 0, \phi(b) = \infty$. Then, the driving function $U(t)$ of $\phi(\gamma)$ satisfies the following properties.

- Stationary increment
- Independent increment
- Scale invariance

Therefore, $U(t)$ must be a Brownian motion $\sqrt{\kappa}B_t$ of variance κ.

Schramm-Loewner evolution

We construct a candidate for scaling limits.
Let $\kappa > 0$, B_t: 1-dim standard Brownian motion with $B_0 = 0$.

chordal SLE$_\kappa$

A chordal Schramm-Loewner evolution with parameter $\kappa > 0$ (chordal SLE$_\kappa$) is the random family of conformal map g_t obtained from the chordal Loewner equation driven by $\sqrt{\kappa}B_t$

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z,$$

The following proposition is very important and basic in SLE theory.

Propositon (existence of chordal SLE$_\kappa$ curve)

With probability 1, we can define the non-self crossing random curve γ which generates SLE$_\kappa$.
We call γ a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞.

SLE in simply connected domains

We define SLE in any simply connected domain.
γ : a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞
$D \subsetneq \mathbb{C}$: simply connected domain, $a \in \partial D$, $b \in \partial D$,
$\phi : D \to \mathbb{H}$: conformal map, $\phi(a) = 0, \phi(b) = \infty$.
Although ϕ is not unique, the distribution of $\phi^{-1}(\gamma)$ is independent of the choice of the map up to time change.
We consider SLE$_\kappa$ curves in D as unparametrized curves.

chordal SLE$_\kappa$ curve in simply connected domains

We call $\phi^{-1}(\gamma)$ a chordal SLE$_\kappa$ curve in D from a to b.

metic on the space of unparametrized curves

$$d_{\mathcal{U}}(\gamma_1, \gamma_2) := \inf_{\alpha} \left[\sup_{0 \le t \le 1} d_*(\gamma_1(t), \gamma_2 \circ \alpha(t)) \right].$$

where d_* は $\widehat{\mathbb{C}}$ is the spherical metric on $\hat{\mathbb{C}}$ and the infimum is taken over all reparametrization α.

Domain Markov and conformal inariance

We consider properties that SLE curves are expected to have.
Let $\mu_D(a, b)$: the law of a chordal SLE_κ curve in D from a to b,
The following two properties immediately follow from the definition of SLE.

- domain Markov property

$$\mu_D(a, b)(\,\cdot\,|\gamma[0, t]) = \mu_{D\setminus\gamma[0,t]}(\gamma(t), b)$$

- conformal invariance
  $f : D \to f(D)$: conformal map.

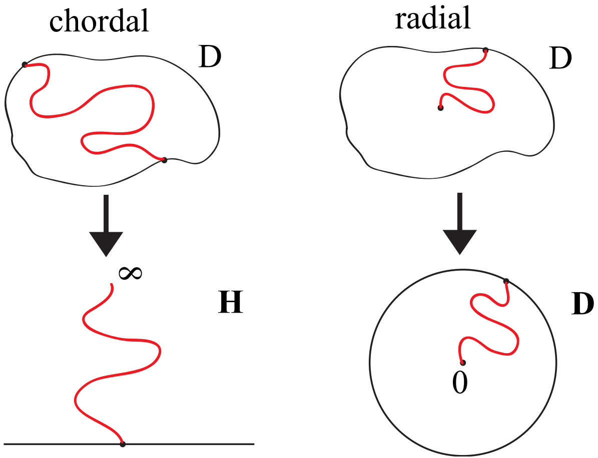
$$f \circ \mu_D(a, b) = \mu_{f(D)}(f(a), f(b))$$

These properties are important to characterize SLE curves.

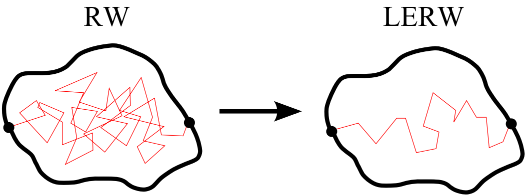
The scaling limit of descrete models

- $\kappa = 2$
  loop-erased random walk (LERW)
- $\kappa = \frac{8}{3}$(conjecture)
  self-avoiding walk
- $\kappa = 3$
  critical Ising model
- $\kappa = 4$
  harmonic explorer, Gaussian free field
- $\kappa = \frac{16}{3}$
  FK Ising model (FK percolation, $q = 2$)
- $\kappa = 6$
  critical percolation
- $\kappa = 8$
  uniform spanning tree Peano curve

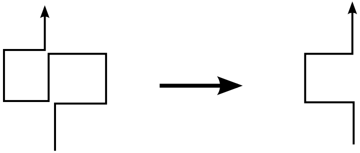
chordal and radial



- Model



- Loop erasure



Known results (radial)

We will introduce known results for radial.

Lawler,Schramm,Werner (2004)

G: square lattice (triangular lattice),
LERW starting from an inner point $\Rightarrow$ radial SLE_2 w.r.t. $d_{\mathcal{U}}$

In above paper, they construct the basic idea of proof of convergence to a SLE curve. So, it is the origin of the research on SLE and scaling limit.

Yadin and Yehudayoff extend to more general graphs.

Yadin, Yehudayoff (2011)

G : planar irreducible graph + invariance principle,
LERW starting from an inner point $\Rightarrow$ radial SLE_2 w.r.t. $d_{\mathcal{U}}$

Known results (chordal)

We will introduce known results for chordal.

Zhan (2008)

G: square lattice ,
LERW connecting two boundary points $\Rightarrow$ chordal SLE_2 w.r.t. $d_{\mathcal{U}}$

I extend Zhan's result in a similar setteing to Yadin and Yehudayoff.
In the rest of this talk, I will talk about my main result precisely.

Planar-irreducible graph

$G = (V, E)$: a directed weighted graph,
$0 \in V \subset \mathbb{C}$: a set of verticies, $E : V \times V \to [0, \infty)$: a set of edges.
We define a planar-irreducible graph G that satisfies the following conditions .

- G is a planar graph.
  (i.e. every two edges do not intersect except for vertices.)
- For any compact set $K \subset \mathbb{C}$, $\sharp\{v \in V : v \in K\} < \infty$.
- For any $u \in V$, $\sum_{w \in V} E(u, w) < \infty$.
- Let $p(u, v) := \dfrac{E(u, v)}{\sum_{w \in V} E(u, w)}$.
  The Markov chain $S(\cdot)$ on V with the transition probability $p(u, v)$ is irreducible.

We call this Markov chain $S(\cdot)$ a natural random walk on G.

Notation

For $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$, let $s_0 := \max\{k \geq 0 : \omega_0 = \omega_k\}$,
$s_m := \max\{k \geq 0 : \omega_{s_{m-1}+1} = \omega_k\}$, $l := \min\{m \geq 0 : \omega_{s_m} = \omega_n\}$.

loop erasure

$$L[\omega] := (\omega_{s_0}, \omega_{s_1}, \ldots, \omega_{s_l}).$$

time-revarsal

$$\omega^- := (\omega_n, \omega_{n-1}, \ldots, \omega_0).$$

dual walk

Suppose that there exists an invariant measure π for a natural random walk $S(\cdot)$ on G such that $0 < \pi(v) < \infty$ for any $v \in V$. Then, we can define the dual walk $S^*(\cdot)$ with the following transition probability p^*.

$$p^*(u, v) := \frac{\pi(v)}{\pi(u)} p(v, u).$$

Invariance principle

For $\delta > 0$, the graph $G_\delta = (V_\delta, E_\delta)$ defined by

$$V_\delta = \{\delta u : u \in V\}, \quad E_\delta = \{(\delta u, \delta v) : E(u, v) > 0\}.$$

Let $S_\delta^x(\cdot)$ be a natural random walk on G_δ starting at $x \in V_\delta$.

In this talk, invariance principle mean that the following.

invariance principle

A natural random walk trajectry weakly converges to a 2-dim Brownian motion trajectry locally uniformly for starting points.

Main result

$D \subsetneq \mathbb{C}$: a bounded simply connected domain, $a \in \partial D$, $b \in \partial D$.
∂D is locally connected and locally analytic at a and b.
$G = (V, E)$: a planar irreducible graph,
$\Gamma_\delta^{a,b}$: a natural random walk on G_δ started at a and stopped on exiting D and conditioned to hit ∂D at b,
$\gamma_\delta^{a,b}$: the loop erasure of $\Gamma_\delta^{a,b}$ (LERW),
$\eta^{a,b}$: a chordal SLE$_2$ curve in D from a to b.

Theorem (S,2014)

Suppose that S_δ^x and $(S^*)_\delta^x$ satisfy invariance principle.
Then,

$$\gamma_\delta^{a,b} \Rightarrow \eta^{a,b} \ (\delta \to 0) \quad \text{w.r.t} \quad d_{\mathcal{U}}$$

Outline of proof

step 1 Estimate for driving function $U(t)$

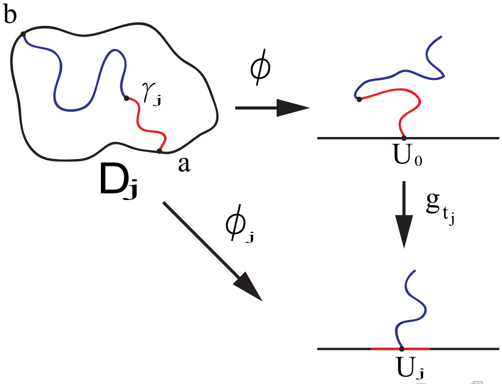
- In order to estimate the driving function of LERW, we must find a ”nice” martingale obsrvable for LERW which converges to some conformal invariant.
- By using martingale observable, we estimate expectation and variance of increment of driving function $U(t)$.

step 2 Convergence w.r.t. driving function $U(t)$

step 3 Convergence w.r.t. $d_{\mathcal{U}}$

Notation

Let $(\gamma_\delta^{b,a})^- = \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$,
$\gamma[0, j]$ is a linear interpolation of $(\gamma_0, \gamma_1, \ldots, \gamma_j)$,
$\phi : D \to \mathbb{H}$ be a conformal map with $\phi(a) = 0, \phi(b) = \infty$,
$U(t)$: the driving function of $\phi(\gamma)$,
g_t : the Loewner chain driven by $U(t)$, $t_j = \text{hcap}(\phi(\gamma[o, j]))$,
$U_j := U(t_j), \phi_j := g_{t_j} \circ \phi, D_j := D \setminus \gamma[0, j]$.



Martingale observable

For $\forall \epsilon > 0$, $m = m(\epsilon) := \inf\{ j \geq 1 : t_j \geq \epsilon^2 \text{ or } |U_j - U_0| \geq \epsilon\}$.
$w \in V_\delta \cap D$, $A = \phi^{-1}([-1, 1])$,
$H_j^{(\delta)}(x, \cdot)$: the hitting probability of RW starting at x in D_j.

martingale observable for LERW

Let

$$M_j := \frac{H_j^{(\delta)}(w, \gamma_j)}{H_j^{(\delta)}(b, \gamma_j)} H_0^{(\delta)}(b; A).$$

Then, M_j is a martingale and

$$M_j = -\frac{2}{\pi}\mathrm{Im}\left(\frac{1}{\phi_j(w) - U_j}\right) + O(\epsilon^3), \quad 0 \leq j \leq m$$

Because M_j is a martingale and m is a bounded stopping time,

$$\mathbf{E}[M_m - M_0] = 0$$

By substituting

$$M_j = -\frac{2}{\pi}\mathrm{Im}\left(\frac{1}{\phi_j(w) - U_j}\right) + O(\epsilon^3), \quad 0 \le j \le m,$$

we get

$$\mathbf{E}\left[\mathrm{Im}\left(\frac{1}{\phi_m(w) - U_m}\right) - \mathrm{Im}\left(\frac{1}{\phi(w) - U_0}\right)\right] = O(\epsilon^3). \qquad (1)$$

We consider Taylor expantion of the left hand side of this equation.

Step1-2

Let $f(u,v) = 1/(u-v)$. Using

$$t_m = O(\epsilon^2), \quad U_m - U_0 = O(\epsilon),$$

$$\phi_m(w) - \phi(w) = \frac{2}{\phi(w) - U_0} \cdot t_m + O(\epsilon^3),$$

we Taylor-expand $f(\phi_m(w), U_m) - f(\phi(w), U_0)$ with respect to $\phi_m(w) - \phi(w)$ and $U_m - U_0$, up to $O(\epsilon^3)$. Observing imaginary part of this Taylor expansion, we get by (1)

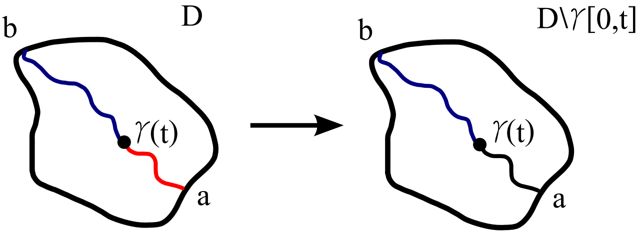
$$\mathrm{Im}\left(\frac{1}{(\phi(w)-U_0)^2}\right)\mathbf{E}[U_m - U_0] + \mathrm{Im}\left(\frac{1}{(\phi(w)-U_0)^3}\right)\mathbf{E}[(U_m - U_0)^2 - 2t_m] = O(\epsilon^3).$$

By two different choices of w, we get the following estimates

$$\mathbf{E}[U_m - U_0] = O(\epsilon^3),$$
$$\mathbf{E}[(U_m - U_0)^2 - 2t_m] = O(\epsilon^3).$$

Step1-3

Now, we can esrimate for expectation and variance of increment of the driving function $U(t)$ at time 0. Because LERW has the domain Markov property, we may consider at the time 0 in another domain $D \setminus \gamma[0, t]$ instead of at time t in D.



Step1-4

Because we shoud estimate uniformly, we introduce a domain class $\mathcal{D}$.
D : a simply connected domain,∂D is locally connected,
a, b : two distinct points on ∂D,
$\phi : D \to \mathbb{H}$: a conformal map with $\phi(a) = 0, \phi(b) = \infty$.
Let $p = \phi^{-1}(i)$, $\mathrm{rad}_p(D) := \inf\{|z - p| : z \notin D\}$.
$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.
$\psi : D \to \mathbb{D}$: a conformal map with $\psi(b) = 1, \psi(p) = 0, \psi(a) = -1$.

class $\mathcal{D}$

Let $\mathcal{D} = \mathcal{D}(r, R, \eta)$ be the collection of all quadruplets (D, a, b, p) such that

- $\mathrm{rad}_p(D) \geq r$
- $D \subset R\mathbb{D}$
- ψ^{-1} has analytic extension in $\{z \in \mathbb{C} : |z - 1| < \eta\}$

Step1-5

key Lemma

For any $r > 0, R > 0, \eta > 0$.
there exists a constant $C > 0$ and a number $\epsilon_0 > 0$ such that for each positive $\epsilon < \epsilon_0$, there exists $\delta_0 > 0$ such that if $(D, a, b, p) \in \mathcal{D}(r, R, \eta)$ and $0 < \delta < \delta_0$,
then we get the following estimates

$$|\mathbf{E}[U_m - U_0]| \leq C\epsilon^3,$$

and

$$|\mathbf{E}[(U_m - U_0)^2 - 2t_m]| \leq C\epsilon^3.$$

Step2 and Step3

Step 2 Convergence w.r.t. driving function $U(t)$

- By Key Lemma and Skorokhod embedding theorem, we can prove that the driving function $U(t)$ weakly converges to $\sqrt{2}B_t$

Step 3 Convergence w.r.t. $d_{\mathcal{U}}$

- We improve to convergence w.r.t. the metric $d_{\mathcal{U}}$ by using Sun and Sheffield's sufficient condition. Then, we need convergnce of γ and γ^- w.r.t. driving function.

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