# An application of the Loewner theory to trivial Beltrami coefficients

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# Contents

# Background

# Quasiconformal mappings

Let $k$ be a constant with $0 \le k < 1$.
A homeomorphism $f : \Omega \to \Omega'$ between plane domains is called a $k$-quasiconformal ($k$-qc, for short) mapping if $f$ is in the Sobolev class $W^{1,2}_{\mathrm{loc}}(\Omega)$ and satisfies the differential inequality

$$\left|\bar{\partial} f\right| \le k \left|\partial f\right|$$

a.e. on $\Omega$. Here,

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

for $z = x + iy$.

In the literature, it is more often called $K$-qc with $K = \dfrac{1+k}{1-k} \ge 1$.

# Löwner chain

Let $f_t(z) = f(z,t),\ t \geq 0$, be a family of analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. It is called a Löwner chain if

1. $f(0,t) = w_0$ is independent of $t$,
2. $a(t) = f_t'(0)$ is (locally) absolutely continuous in $t \geq 0$ and $a(t) \to \infty$ as $t \to +\infty$,
3. $f_t : \mathbb{D} \to \mathbb{C}$ is univalent and $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s \leq t$.

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3. $f_t : \mathbb{D} \to \mathbb{C}$ is univalent and $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s \leq t$.

Then $f(z,t)$ is absolutely continous in $t \geq 0$ for each $z \in \mathbb{D}$ and satisfies the Löwner differential equation

$$\dot{f}(z,t) = zf'(z,t)p(z,t), \quad z \in \mathbb{D}, \text{a.e. } t \geq 0,$$

where

$$\dot{f} = \frac{\partial f}{\partial t} \quad \text{and} \quad f' = \frac{\partial f}{\partial z},$$

$p(z,t) = p_t(z)$ is analytic for each $t$, measurable in $t \geq 0$ for each $z \in \mathbb{D}$ and satisfies $\operatorname{Re} p_t(z) > 0$ on $|z| < 1$ for a.e. $t \geq 0$.

# Existence of Löwner chain

Conversely, if a family of analytic functions $p_t(z) = p(z,t),\ t \geq 0$, on $\mathbb{D}$ are given so that $p(z,t)$ is measurable in $t \geq 0$ and satisfies $\operatorname{Re} p_t > 0$, the following theorem ensures existence of a corresponding Löwner chain under a mild condition.

**Theorem (Löwner, Pommerenke, Becker)**

If $p(z,t)$ is locally integrable in $t$ and if

$$\int_0^\infty \operatorname{Re} p(0,t)dt = +\infty,$$

then there exists a Löwner chain $f_t,\ t \geq 0$, such that

$$\dot{f}(z,t) = zf'(z,t)p(z,t), \quad z \in \mathbb{D}, \text{a.e. } t \geq 0.$$

# Becker's theorem

Theorem (Becker 1972)

Let $0 \leq k < 1$ be a constant. If $p_t$ satisfies

$$\left| \frac{1 - p_t(z)}{1 + p_t(z)} \right| \leq k$$

for $z \in \mathbb{D}$ and a.e. $t \geq 0$, then $f_t$ is continuous and injective on $\overline{\mathbb{D}}$ for each $t \geq 0$ and $f_0$ extends to a $k$-quasiconformal map of the complex plane and its extension $\tilde{f}_0$ is given by

$$\tilde{f}_0(e^t \zeta) = f(\zeta, t), \quad |\zeta| = 1, t \geq 0.$$

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In this case, $\operatorname{Re} p(0,t) \ge \frac{1-k}{1+k} > 0$, which implies $\int_0^\infty \operatorname{Re} p(0,t) dt = +\infty$.

# Betker's theorem

### Theorem (Betker 1992)

Let $f(z,t)$ be a Löwner chain with $\dot{f}(z,t) = zf'(z,t)p(z,t)$ and $\int_0^\infty \operatorname{Re} p(0,t)dt = +\infty$. Suppose that there is a mesurable family of analytic functions $q_t(z) = q(z,t)$ on $\mathbb{D}$ and a constant $k \in [0,1)$ such that

$$\left| \frac{p(z,t) - \overline{q(z,t)}}{p(z,t) + q(z,t)} \right| \le k, \quad z \in \mathbb{D}, \text{a.e. } t \ge 0.$$

Then $f_0(z) = f(z,0)$ has a $k$-qc extension $\tilde{f}_0 : \mathbb{C} \to \mathbb{C}$.

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Then $f_0(z) = f(z,0)$ has a $k$-qc extension $\tilde{f}_0 : \mathbb{C} \to \mathbb{C}$.

We remark that Betker gave a construction of $\tilde{f}_0$ in terms of the inverse Löwner equation, which will be introduced later in this talk.

# Beltrami coefficients

# A framework

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$f = f_\mu : \mathbb{D} \to \mathbb{D}$ quasiconformal (qc) automorphism of $\mathbb{D}$ with

$$\bar{\partial} f = \mu \partial f$$

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$M_0(\mathbb{D}) = \{\mu \in M(\mathbb{D}) : f_\mu = \mathrm{id} \text{ on } \partial\mathbb{D}\}$ the set of trivial Beltrami coefficients
$\mathrm{Teich} = M(\mathbb{D})/M_0(\mathbb{D})$ the universal Teichmüller space
(The set of normalized quasisymmetric self-homeomorphisms of $S^1 = \partial\mathbb{D}$)
Here, $\mu_1 \sim \mu_2$ if $f_{\mu_1} = f_{\mu_2}$ on $\partial\mathbb{D}$.

# Facts about trivial Beltrami coefficients

Theorem (Earle-Eells 1967)

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$M_0(\mathbb{D})$ is a contractible $C^0$ submanifold of $M(\mathbb{D})$.

However, we do not know much about $M_0(\mathbb{D})$. For instance, we cannot find any *simple* sufficient condition for $\mu \in M(\mathbb{D})$ to be trivial.

# Infinitesimally trivial Beltrami differentials

A tangent vector of $M_0(\mathbb{D})$ at $0$ in the space $M(\mathbb{D}) \subset L^\infty(\mathbb{D})$ is called an infinitesimally trivial Beltrami differential on $\mathbb{D}$. The set of those differentials will be denoted by $N(\mathbb{D})$.

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$A(\mathbb{D}) = \{\varphi :$ holomorphic on $\mathbb{D}, \|\varphi\| = \iint_{\mathbb{D}} |\varphi(z)| dxdy < \infty\}$
integrable holomorphic quadratic differentials on $\mathbb{D}$

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**Teichmüller's lemma**

$N(\mathbb{D}) = \{\nu \in L^\infty(\mathbb{D}) : \iint_{\mathbb{D}} \nu(z)\varphi(z) dxdy = 0 \ \forall \varphi \in A(\mathbb{D})\}.$

# Main result

# Harmonic Hardy space $h^\infty$

$\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ the unit circle
For $\phi \in L^\infty(\mathbb{T})$, let

$$w(z) = P[\phi](z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\phi(e^{i\theta})d\theta, \quad z \in \mathbb{D}.$$

Then $w$ is a bounded harmonic function on $\mathbb{D}$ and

$$\lim_{r\to 1-} w(re^{i\theta}) = \phi(e^{i\theta}) \quad \text{a.e. } \theta \in \mathbb{R}.$$

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Through this correspondence, the set $h^\infty(\mathbb{D})$ of (complex valued) bounded harmonic functions on $\mathbb{D}$ can be identified with $L^\infty(\mathbb{T})$.

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This is a closed subspace of $h^\infty(\mathbb{D})$. The boundary values of $H^\infty(\mathbb{D})$ is thus a closed subspace of $L^\infty(\mathbb{T})$, which will be denoted by $H^\infty(\mathbb{T})$. Also, we can describe it by

$$H^\infty(\mathbb{T}) = \{\phi = \phi_1 + i\phi_2 \in L^\infty(\mathbb{T}) : \phi_2 = H[\phi_1] + \text{const.}\},$$

where $H$ is the Hilbert transformation:

$$H[\phi](e^{i\theta}) = \frac{1}{2\pi}\text{p.v.}\int_0^{2\pi} \phi(e^{it}) \cot\left(\frac{\theta - t}{2}\right) dt.$$

# Harmonic extention of a function in $L^\infty(\mathbb{D})$

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$$u_t(z) = P[\psi_t](z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\mu(e^{-t+i\theta})e^{-2i\theta}d\theta.$$

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Then $u(t,z) = u_t(z)$ is measurable in $t \ge 0$ and harmonic in $z \in \mathbb{D}$.
Conversely, if $u : [0,+\infty) \times \mathbb{D} \to \overline{\mathbb{D}}_k = \{|z| \le k\}$ satisfies these conditions, then the radial limit $\psi_t$ of $u_t(z) = u(t,z)$ defines a function $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_\infty \le k$ by $\mu(e^{-t}\zeta) = \zeta^2\psi_t(\zeta)$ for $t \ge 0$ and $\zeta \in \mathbb{T}$.

# Main result

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Let $\mu \in M(\mathbb{D})$ and set $\psi_t(\zeta) = \zeta^{-2}\mu(e^{-t}\zeta)$ for $t \geq 0$ and $\zeta \in \mathbb{T}$. If $\psi_t \in H^\infty(\mathbb{T})$ for almost every $t \geq 0$, then $\mu \in M_0(\mathbb{D})$.

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Remark 1: The above condition is linear in some sense. For instance, $c\mu \in M_0(\mathbb{D})$ for the above $\mu$ and any complex number $c$ with $|c| < 1/\|\mu\|_\infty$.

Remark 2: By Teichmüller's lemma, we see that $\mu \in N(\mathbb{D})$ for the above $\mu$.

# Examples of trivial Beltrami coefficients

Let $N$ be a non-negative integer and $a_j(t),\ j = 0, 1, \ldots, N$, be essentially bounded measurable functions in $t \geq 0$ so that

$$\mu(z) = \sum_{j=0}^{N} a_j(-\log|z|) \left(\frac{z}{|z|}\right)^{j+2}$$

satisifies $\|\mu\|_\infty < 1$. Then $\mu \in M_0(\mathbb{D})$.

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$$\psi_t(\zeta) = \zeta^{-2}\mu(e^{-t}\zeta) = \sum a_j(t)\zeta^j$$

extends to the analytic function $\sum a_j(t)z^j$. For instance, if

$$\sum_{j=0}^{N} \|a_j\|_\infty < 1,$$

then $\mu$ of the above form is in $M_0(\mathbb{D})$.

# A concrete example (picture)

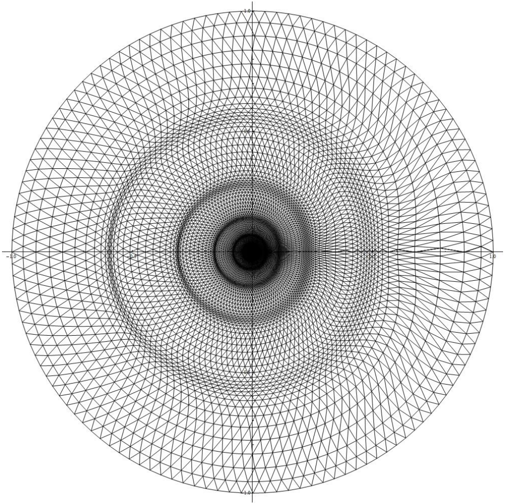

Figure: Drawn by Mr. Shimauchi

# Proof of the main result

# Inverse Löwner chain

Let $\omega_t(z) = \omega(z, t),\ t \geq 0$, be a family of analytic functions on the unit disk $\mathbb{D}$. It is called an inverse Löwner chain if

1. $\omega(0, t) = w_0$ is independent of $t$,
2. $b(t) = \omega_t'(0)$ is (locally) absolutely continuous in $t \geq 0$ and $b(t) \to 0$ as $t \to +\infty$,
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3. $\omega_t : \mathbb{D} \to \mathbb{C}$ is univalent and $\omega_s(\mathbb{D}) \supset \omega_t(\mathbb{D})$ whenever $0 \leq s \leq t$.

Then, similarly, $\omega(z,t)$ satisfies the differential equation

$$\dot{\omega}(z,t) = -z\omega'(z,t)q(z,t), \quad z \in \mathbb{D}, \text{a.e. } t \geq 0,$$

where $q(z,t) = q_t(z)$ is analytic for each $t$, measurable in $t \geq 0$ for each $z \in \mathbb{D}$ and satisfies $\operatorname{Re} q_t(z) > 0$ on $|z| < 1$ for a.e. $t \geq 0$.

# Existence of the Inverse Löwner chain

Conversely, suppose that a measurable family $q_t(z) = q(z, t)$ of analytic functions on $\mathbb{D}$ with $\operatorname{Re} q_t > 0$ is given.

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Conversely, suppose that a measurable family $q_t(z) = q(z,t)$ of analytic functions on $\mathbb{D}$ with $\operatorname{Re} q_t > 0$ is given. If furthermore

$$\int_0^\infty \operatorname{Re} q(0,t)dt = +\infty,$$

then there is an inverse Löwner chain $\omega_t(z) = \omega(z,t)$ satisfying

$$\dot{\omega}(z,t) = -z\omega'(z,t)q(z,t), \quad z \in \mathbb{D}, \text{a.e. } t \geq 0$$

and $\omega(z,0) = z$ for $z \in \mathbb{D}$.
Note that $\omega(0,t) = \omega(0,0) = 0$ for $t \geq 0$.

# Betker's theorem, revisited

Recall:

**Betker's Theorem**

Let $f(z,t)$ be a Löwner chain with $\dot{f}(z,t) = zf'(z,t)p(z,t)$ and $\int_0^\infty \mathrm{Re}\, p(0,t)dt = +\infty$. Suppose that there is a mesurable family of analytic functions $q_t(z) = q(z,t)$ on $\mathbb{D}$ and a constant $k \in [0,1)$ such that

$$\left| \frac{p(z,t) - \overline{q(z,t)}}{p(z,t) + q(z,t)} \right| \le k, \quad z \in \mathbb{D}, \text{a.e. } t \ge 0.$$

Then $f_0(z) = f(z,0)$ has a $k$-qc extension $\tilde{f}_0 : \mathbb{C} \to \mathbb{C}$.

The extension is given in such a way that

$$\tilde{f}_0(1/\overline{\omega(\zeta,t)}) = f(\zeta,t), \quad |\zeta| = 1, t \ge 0,$$

where $\omega(z,t)$ is the inverse Löwner chain for $q(z,t)$.

# Betker's lemma

## Lemma (Betker 1992)

Let $0 \le k < 1$ be a constant. Suppose $q(z,t) = q_t(z)$ is analytic on $|z| < 1$ for each $t$, measurable in $t \ge 0$ for each $z \in \mathbb{D}$ and satisfies

$$\left| \frac{1 - q_t(z)}{1 + q_t(z)} \right| \le k$$

for $z \in \mathbb{D}$ and a.e. $t \ge 0$, then there exists an inverse Löwner chain $\omega_t(z) = \omega(z,t)$ such that $\omega_0(z) = z,\ z \in \mathbb{D}$,

$$\dot{\omega}(z,t) = -z\omega'(z,t)q(z,t), \quad z \in \mathbb{D}, \text{a.e. } t \ge 0,$$

$\omega_t$ is continuous and injective on $\overline{\mathbb{D}}$ for each $t \ge 0$. Moreover, the map $\Omega : \mathbb{C} \to \mathbb{C}$ defined in the following is $k$-qc:

$$\Omega(e^{-t}\zeta) = \omega(\zeta,t), \quad t \ge 0, \zeta \in \mathbb{T}, \quad \text{and} \quad \Omega(z) = z,\ |z| > 1.$$

# Complex dilatation of $\Omega(z)$

The complex dilatation of the map $\Omega$ In Betker's lemma can be computed as

$$\mu_\Omega(z) = \frac{\bar\partial\Omega}{\partial\Omega} = \frac{z}{\bar z}\cdot\frac{q(\zeta,t)-1}{q(\zeta,t)+1} = \zeta^2\psi(\zeta,t), \quad z = e^{-t}\zeta \in \mathbb{D},$$

where

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Note that $|\psi_t| \le k < 1$ for a.e. $t \ge 0$.

# Proof of the main result

Recall the hypothesis. For a $\mu \in M(\mathbb{D})$, set $\psi_t(\zeta) = \zeta^{-2}\mu(e^{-t}\zeta)$ for $t \geq 0$ and $\zeta \in \mathbb{T}$. Suppose that $\psi_t \in H^\infty(\mathbb{T})$ for almost every $t \geq 0$.

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Thank you very much for your attension!