On a deformation flow for an inverse problem in potential theory

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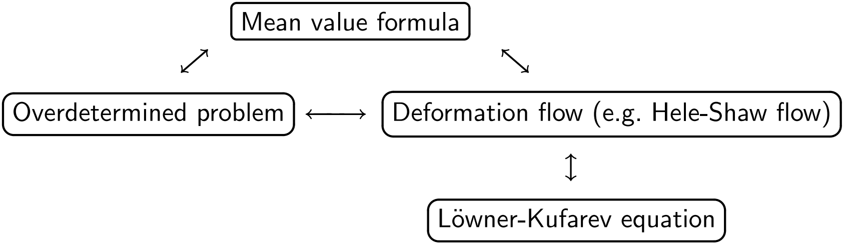
International Workshop on
Conformal Dynamics and Loewner Theory

Tokyo Institute of Technology

November 22, 2014

Introduction

Relationship diagram



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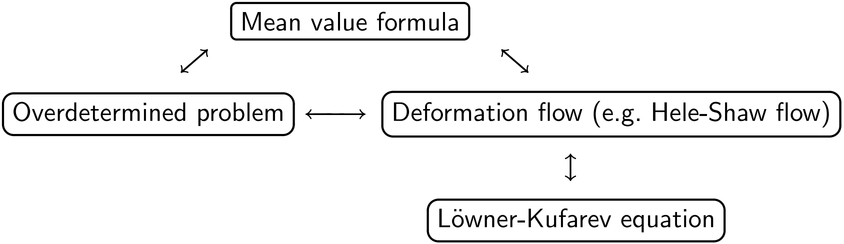

Mean value formula

$$|\mathbb{B}|h(0) = \int_{\mathbb{B}} h\, dx, \quad \text{or} \quad |\mathbb{S}^{N-1}|h(0) = \int_{\mathbb{S}^{N-1}} h\, d\sigma.$$

$h \in H(\overline{\mathbb{B}})$: the space of all harmonic functions on $\overline{\mathbb{B}}$.

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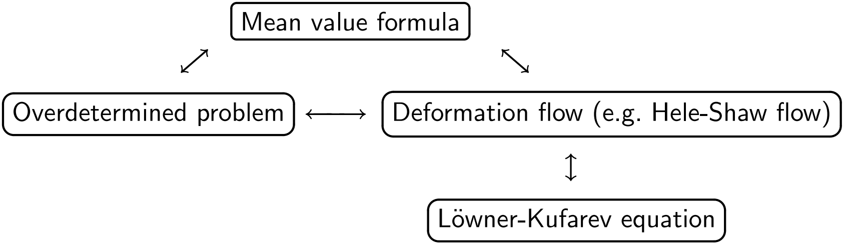

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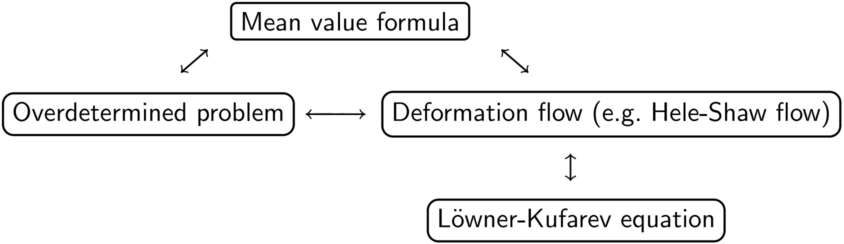

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(A) ↗↙ ↘↖

Overdetermined problem ⟷ Deformation flow (e.g. Hele-Shaw flow)

↕

Löwner-Kufarev equation

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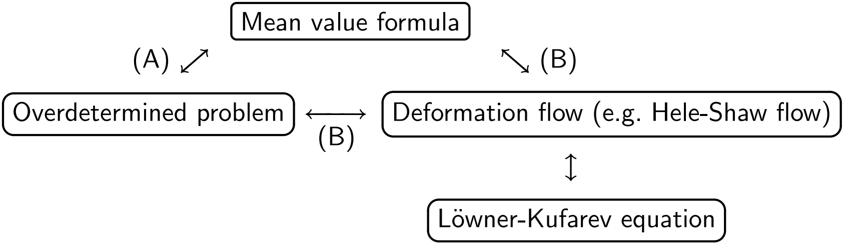

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(B) $\mu(t) \mapsto \Omega(t)$ induces an evolution of domains.

What's known about mean value formula

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- Proof is based on the existence of a continuous family of domains $\Omega(t)$ covering $\mathbb{R}^n$, which is implied by the global-in-time solvability of the flow.

Flow characterization of mean value formulas

(A) Overdetermined problem: reformulation by PDE

Mean value formula

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Overdetermined problem $\longleftrightarrow$ Deformation flow (e.g. Hele-Shaw flow)

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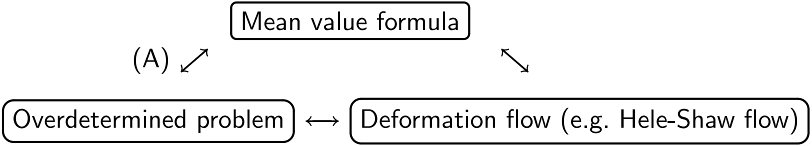
The following are equivalent:



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The following are equivalent:

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$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ -\partial_n u = 1 & \text{on } \partial\Omega. \end{cases}$$

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(Proof.) Indeed, "(1) $\Rightarrow$ (2)" immediately follows by setting

$$u(x) := \int E(x-y)\,d\mu(y) - \int_{\partial\Omega} E(x-y)\,d\sigma(y).$$

Conversely, "(1) $\Leftarrow$ (2)" follows from

$$\int h\,d\mu = \int_{\Omega} h(-\Delta u)\,dx$$

$$= \int_{\partial\Omega} (\partial_n h u - h\partial_n u)\,d\sigma = \int_{\partial\Omega} h\,d\sigma.$$

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As a consequence ...

- Symmetry of Ω, i.e., $\Omega = \mathbb{B}$, holds for $\mu = c\delta_0$ by the method of moving planes.

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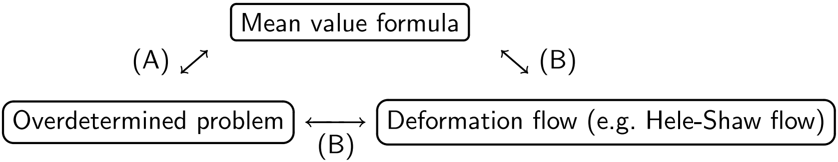
But, it is still unclear if uniqueness is preserved under small perturbation of measure. We derive *deformation flow* of $\Omega(t)$ for a parametrized measure $\mu(t)$ to construct a continuous family of $\Omega(t)$. This deduces the uniqueness!

(B) Deformation flow



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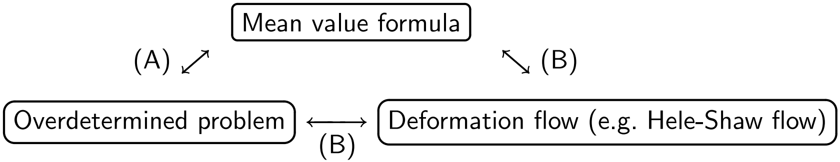
- Let $\mu(t)$ be a (prescribed) parametrized measure, and suppose that $\partial\Omega(0)$ admits the mean value formula for $\mu(0)$. How do we construct $\Omega(t)$ for $\mu(t)$?



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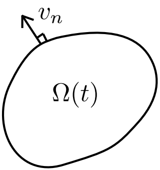
Deformation flow

$$v_n = p \quad \text{on } \partial\Omega(t), \quad \text{with} \begin{cases} -\Delta p = \delta_0 & \text{in} \quad \Omega(t) \\ \partial_n p + H p = 0 & \text{on} \quad \partial\Omega(t) \end{cases}$$

H: mean curvature of $\partial\Omega(t)$



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Theorem

Let $\{\partial\Omega(t)\}_{0\le t<T}$ be a $C^{3+\alpha}$ family of surfaces with positive mean curvature. Then, the following are equivalent:

- $\{\partial\Omega(t)\}_{0\le t<T}$ is called *a $C^{3+\alpha}$ family of surfaces* if $\partial\Omega(t)$ is locally represented as graph of a $C^{3+\alpha}$ function and its time derivative is of $C^{2+\alpha}$.

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(i) $\{\partial\Omega(t)\}$ *is a solution to the deformation flow;*

(ii) *Each $\partial\Omega(t)$ admits the mean value formula*

$$\int h \, d\mu(t) = \int_{\partial\Omega(t)} h \, d\sigma \qquad \left(\forall h \in H(\overline{\Omega(t)})\right).$$

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The infinitesimal deformation of $\partial\Omega(t)$ is (formally) derived by substituting

$$\partial\Omega(t+\varepsilon) = \partial\Omega(t) + \varepsilon v_n \overrightarrow{n} + O(\varepsilon^2),$$
$$u(x) = u_0(x) + \varepsilon p(x) + O(\varepsilon^2)$$

into the overdetermined problem

$$\begin{cases} -\Delta u = \mu(t+\varepsilon) & \text{in } \Omega(t+\varepsilon) \\ u = 0 & \text{on } \partial\Omega(t+\varepsilon) \\ \partial_n u = -1 & \text{on } \partial\Omega(t+\varepsilon) \end{cases}$$

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- $-\Delta u_0 - \varepsilon\Delta p = \mu(t) + \varepsilon\mu'(t) + O(\varepsilon^2)$ for $x \in \Omega(t)$,
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Uniqueness of admissible domain for mean value formula

Main result

Generalized mean value formula

$$\int h \, d\mu = \int_{\partial\Omega} h \, d\sigma$$

- $\mu = \omega\delta_0 \Rightarrow \Omega = \mathbb{B}.$ $(\omega := |\mathbb{S}^{N-1}|)$

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"Distance" is defined by $d(\partial\Omega_\rho, \mathbb{S}^{N-1}) := \|\rho\|_{C^l(\mathbb{S}^{N-1})}$ for $l \in \mathbb{N}$, where

$$\partial\Omega_\rho := \left\{(1+\rho(\zeta))\,\zeta \;\middle|\; \zeta \in \mathbb{S}^{N-1}\right\} \text{ for } \rho \in C(\mathbb{S}^{N-1}).$$

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Theorem (Stability of the mean value formula)

There is $\eta_0 > 0$ *s.t. for* μ *with* $\|\mu - \omega\delta_0\| < \eta_0$ *and* $\operatorname{supp}\mu \subset B(0, \eta_0)$,

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Therefore, we established

- Uniqueness and Stability hold under small perturbation of measure;
- Higher symmetry of μ implies stronger stability of mean value formula.

Outline of proof

Assume $\mu = (\omega - \|\nu\|)\delta_0 + \nu$, where $\|\nu\| + (\mathrm{diam}\,\mathrm{supp}\,\nu)^{N-1} \ll 1$.

1. **Construction of a "good" solution**:

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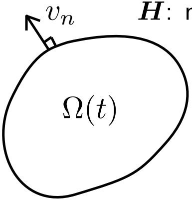
H: mean curvature of $\partial\Omega(t)$



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   - *Characterization of invariant manifolds*:
     Global-in-time solvability for $\tilde{\nu} + t\delta_0$ and $\Omega(0) \sim$ a ball,
     Infinitely many conserved quantities (harmonic moments).

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     Global-in-time solvability for $\tilde{\nu} + t\delta_0$ and $\Omega(0) \sim$ a ball,
     Infinitely many conserved quantities (harmonic moments).
2. **Uniqueness**: Maximum principle applied to $u - \tilde{u}$ where $\tilde{u}$ is a super or subsolution to ODP for $\tilde{\Omega}$ with $\tilde{\Omega} \supset \Omega$ or $\tilde{\Omega} \subset \Omega$. The construction of such $\tilde{u}$ and $\tilde{\Omega}$ relies on the global-in-time solvability of the flow.

Outline of proof: *Uniqueness*

Let us denote by Ω_* the domain we constructed, and assume that there is another different domain Ω satisfying GMVF, i.e.,

$$\begin{cases} -\Delta u_* = \mu & \text{in } \Omega_*, \\ u_* = 0 & \text{on } \partial\Omega_*, \\ \partial_n u_* = -1 & \text{on } \partial\Omega_*, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_n u = -1 & \text{on } \partial\Omega. \end{cases}$$

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- A comparison argument yields $B(0, 1-\varepsilon) \subset \Omega \subset \Omega_*$ with $\varepsilon \ll 1$.
- Now $\Omega \supset \Omega_*$ follows from a similar (but a little more involved) argument with $\Omega(0) = B(0, 1-\varepsilon)$.

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Flow characterization of mean value formulas

- Continuous family of domains admitting mean value formulas is shown to form a flow described by an evolution equation.

Uniqueness and stability of mean value formula

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Thank you for your attention!