$\mathrm{L^d}$-Loewner chains with quasiconformal extensions

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Ordinary differential equations for evolution families

- Radial Loewner equations for $\varphi_{s,t} = f_t^{-1} \circ f_s$:

$$\dot{w}_t = G(w_t, t) \quad \text{with} \quad G(z,t) := -zp(z,t)$$

- Chordal Loewner equations (by transforming everything from $\mathbb{H}^+$ to $\mathbb{D}$):

$$\dot{w}_t = G(w_t, t) \quad \text{with} \quad G(z,t) := (1-z)^2 p(z,t)$$

- Berkson-Porta representation for semigroups $\{\phi_t\} \subset \mathrm{Hol}(\mathbb{D})$:

$$\dot{w}_t = G(w_t) \quad \text{with} \quad G(z) := (z-\tau)(\bar{\tau}z - 1)p(z)$$

$\Longrightarrow$ Unified treatment of the above differential equations!!

Evolution family of order d

Definition 7.1

A family of holomorphic self-maps of the unit disk $(\varphi_{s,t})_{0\le s\le t<\infty}$, is an **evolution family of order** d with $d \in [1,\infty]$, or in short an L^d**-evolution family**, if

1. $\varphi_{s,s}(z) = z$,
2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \le s \le u \le t < \infty$,
3. for all $z \in \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{z,T} \in L^d([0,T],\mathbb{R})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le \int_u^t k_{z,T}(\zeta)d\zeta$$

for all $0 \le s \le u \le t \le T$.

- We denote the family of all evolution families of order d by $\mathtt{EF}^d$.

Herglotz vector fields of order d

Definition 7.3

A weak holomorphic vector field of order $d \in [1, \infty]$ on $\mathbb{D}$ is a function $G : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ with the following properties:

1. For all $z_0 \in \mathbb{D}$, the function $G(z_0, t)$ is measurable on $t \in [0, \infty)$,
2. For all $t_0 \in [0, \infty)$, the function $G(z, t_0)$ is holomorphic on $\mathbb{D}$,
3. For any compact set $K \subset \mathbb{D}$ and for all $T > 0$, there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|G(z, t)| \leq k_{K,T}(t) \tag{1}$$

   for all $z \in K$ and for almost every $t \in [0, T]$.

Furthermore, G is said to be a **Herglotz vector field of order d** if $G(\cdot, t)$ is the infinitesimal generator of a semigroup of holomorphic functions for almost all $t \in [0, \infty)$.

- $\mathtt{HV}^d$: a family of all Herglotz vector field of order d

Theorem 7.5

Let $d \in [1, \infty]$ be fixed. Then, for any $\varphi_{s,t} \in \mathtt{EF}^d$, there exists an essentially unique $G \in \mathtt{HV}^d$ such that

$$\dot{\varphi}_{s,t}(z) = G(\varphi_{s,t}(z), t) \tag{2}$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, where $\dot{\varphi}_{s,t} := \partial\varphi_{s,t}/\partial t$.
Conversely, for any $G \in \mathtt{HV}^d$, a unique solution of (2) with the initial condition $\varphi_{s,s}(z) = z$ is an evolution family of order d.

It determines one-to-one correspondence between $(\varphi_{s,t}) \in \mathtt{EF}^d$ and $G \in \mathtt{HV}^d$.

Definition 7.6

A Herglotz function of order $d \in [1, \infty]$ on the unit disk $\mathbb{D}$ is a function $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ with the following properties:

1. For all $z_0 \in \mathbb{D}$, the function $p(z_0, t)$ belongs to $L^d_{\text{loc}}([0, \infty), \mathbb{C})$ on $t \in [0, \infty)$,
2. For all $t_0 \in [0, \infty)$, the function $p(z, t_0)$ is holomorphic on $\mathbb{D}$,
3. $\operatorname{Re} p(z, t) \geq 0$ for all $z \in \mathbb{D}$ and $t \in [0, \infty)$.

Then, $\mathtt{HF}^d$ denotes the family of all Herglotz functions of order d.

Theorem 7.8

Let $G \in \mathtt{HV}^d$. Then there exist an essentially unique $p \in \mathtt{HF}^d$ and a measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$ s.t.,

$$G(z, t) = (z - \tau(t))(\overline{\tau(t)} z - 1) p(z, t). \tag{3}$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$.
Conversely, for a given $p \in \mathtt{HF}^d$ and a measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$, the equation (3) determines an essentially unique $G \in \mathtt{HV}^d$.

(essentially) 1-to-1 correspondence between;

$(\varphi_{s,t}) \in \mathtt{EF}^d$ $\overset{(A)}{\longleftrightarrow}$ $G \in \mathtt{HV}^d$ $\overset{(B)}{\longleftrightarrow}$ $(p,\tau) \in \mathtt{BP}$

(A) : $\dot{\varphi}_{s,t}(z) = G(\varphi_{s,t}(z), t)$ with the initial condition $\varphi_{s,s}(z) = z$

(B) : $G(z,t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z,t)$

Loewner chains of order d

Definition 7.9

A family of holomorphic maps of the unit disk $(f_t)_{t\geq 0}$ is called a **Loewner chain of order d** with $d \in [1, \infty]$, or in short an **L^d-Loewner chain**, if

1. $f_t : \mathbb{D} \to \mathbb{C}$ is univalent for each $t \in [0, \infty)$,
2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for all $0 \leq s < t < \infty$,
3. for any compact set $K \subset \mathbb{D}$ and all $T > 0$, there exists a non-negative function $k_{K,T} \in L^d([0,T], \mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\zeta) d\zeta$$

for all $z \in \mathcal{K}$ and all $0 \leq s \leq t \leq T$.

$\mathtt{LC}^d$: a family of all Loewner chains of order d

- There exist $(f_t) \in \mathtt{LC}^d$ s.t. $\Omega[(f_t)] := \bigcup_{t\geq 0} f_t(\mathbb{D}) \neq \mathbb{C}$.
  In fact $f_s(\mathbb{D})$ is allowed to be equal to $f_t(\mathbb{D})$ for some $s < t$ (even for all $s < t$)
- For any compact subset $K \in \mathbb{D}$, there exists $(f_t) \in \mathtt{LC}^d$ s.t. $f_s(K) \not\subset f_t(K)$ for some $s < t$,

Evolution families and Loewner chains

Theorem 7.10

For any $(f_t) \in \mathtt{LC}^d$, if we define

$$\varphi_{s,t}(z) := (f_t^{-1} \circ f_s)(z) \qquad (z \in \mathbb{D},\, 0 \le s \le t < \infty)$$

then $(\varphi_{s,t}) \in \mathtt{EF}^d$. Conversely, for any $(\varphi_{s,t}) \in \mathtt{EF}^d$, there exists a $(f_t) \in \mathtt{LC}^d$ such that the following equality holds

$$(f_t \circ \varphi_{s,t})(z) = f_s(z) \qquad (z \in \mathbb{D},\, 0 \le s \le t < \infty).$$

We can deduce that a Loewner chain of order d satisfies the differential equation

$$\dot{f}_t(z) = f_t'(z)(z - \tau(t))(1 - \overline{\tau(t)}z)p(z,t),$$

where $\tau : [0, \infty) \to \overline{\mathbb{D}}$ is a measurable function and $p \in \mathtt{HF}^d$.

$\mathcal{L}[(\varphi_{s,t})]$: a family of $(f_t) \in \mathtt{LC}^d$ associated with $(\varphi_{s,t}) \in \mathtt{EF}^d$ satisfying $f_0 \in \mathcal{S}$

Theorem 7.11

Let $(\varphi_{s,t}) \in \mathtt{EF}^d$. Then there exists a unique $(f_t) \in \mathcal{L}[(\varphi_{s,t})]$ such that $\Omega[(f_t)]$ is $\mathbb{C}$ or an Euclidean disk in $\mathbb{C}$ whose center is the origin. Furthermore;

- The following 4 statements are equivalent;
  1. $\Omega[(f_t)] = \mathbb{C}$,
  2. $\mathcal{L}[(\varphi_{s,t})]$ consists of only one function,
  3. $\beta(z) = 0$ for all $z \in \mathbb{D}$, where
     $$\beta(z) := \lim_{t\to+\infty} \frac{|\varphi_{0,t}'(z)|}{1-|\varphi_{0,t}(z)|^2},$$
  4. there exists at least one point $z_0 \in \mathbb{D}$ such that $\beta(z_0) = 0$.
- On the other hand, if $\Omega[(f_t)] \neq \mathbb{C}$, then the Euclidean disk is written by
  $$\Omega[(f_t)] = \left\{ w : |w| < \frac{1}{\beta(0)} \right\}$$
  and the other $(g_t) \in \mathcal{L}[(\varphi_{s,t})]$ has an expression
  $$g_t(z) = \frac{h(\beta(0) f_t(z))}{\beta(0)} \quad (h \in \mathcal{S}).$$

Quasiconformal extensions for L^d-Loewner chains

In 1972, Becker applied for Loewner's method to derive a quasiconformal extension criterion.

Theorem (Becker 1972)

Let $k \in [0,1)$ be a constant. Suppose that (f_t) is a (classical) radial Loewner chain for which the Herglotz function p in the Loewner PDE satisfies

$$p(z,t) \in \underline{\underline{U(k)}} := \left\{ w \in \mathbb{C} : \left| \frac{w-1}{w+1} \right| \leq k \right\} \subsetneq \mathbb{H}$$

for all $z \in \mathbb{D}$ and almost all $t \geq 0$. Then the function F defined by

$$F(z) := \begin{cases} f_0(z), & z \in \mathbb{D}, \\ f_{\log|z|}\left(\dfrac{z}{|z|}\right), & z \in \mathbb{C} \backslash \overline{\mathbb{D}}, \end{cases}$$

is a k-quasiconformal mapping of $\mathbb{C}$.

Main result

Theorem (H. 2014)

Let $d \in [1, \infty)$ and $k \in [0, 1)$. Let $(f_t) \in \mathtt{LC}^d$ and $p \in \mathtt{HF}^d$ associated with (f_t). If p satisfies

$$p(z,t) \in U(k)$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, then

1. f_t has a k-quasiconformal extension to $\widehat{\mathbb{C}}$ for each $t \in [0, \infty)$.
2. $\Omega[(f_t)] = \mathbb{C}$.

In this theorem, any superfluous assumption is not imposed on τ.

Theorem (Gumenyuk and H, 2014)

Let $d \in [1, \infty)$ and $k \in [0, 1)$. Let $(f_t) \in \mathtt{LC}^d$ and $p \in \mathtt{HF}^d$ associated with (f_t). If $\tau \in \overline{\mathbb{D}}$ is constant and p satisfies

$$p(z, t) \in U(k)$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, then f_t has a k-quasiconformal extension to $\widehat{\mathbb{C}}$ for each $t \in [0, \infty)$.

Then we can also prove it for the case when τ is of the form

$$\tau(t) = \sum_{i=1}^{n} \tau_i \cdot \chi_{I_i}(t),$$

where $\tau_i \in \overline{\mathbb{D}}$, $n \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty$, $I_i := [t_{i-1}, t_i)$ and χ_I is a characteristic function.

Theorem (H. 2014, Roth 1997)

Let $G \in \mathtt{HV}^d$. Consider the family $\{G(z,t)\}$ such that

1. $\{G(\cdot\,,t)\}$ forms a normal family for almost every fixed $t \in [0,\infty)$.
2. $\{G_n(z,t)\}_{n\in\mathbb{N}} \subset \{G(z,t)\}$ is a sequence converging weakly to $G \in \mathtt{HV}^d$

Then, a sequence of evolution families $\{(\varphi^n_{s,t})\}_n$ of order d associated with $\{G_n\}_n$ converges locally uniformly to $(\varphi_{s,t})$ associated with G on $(z,t) \in \mathbb{D} \times [s,\infty)$.

Proposition (Gumenyuk and H, 2014)

Let $(f_t) \in \mathtt{LC}^d$. Let $p \in \mathtt{HF}^d$ and τ be a measurable function associated with (f_t). Suppose that $\tau \in \overline{\mathbb{D}}$ is a constant, and there exist uniform constants $C_1, C_2 > 0$ such that

$$C_1 < \operatorname{Re} p(z,t) < C_2$$

for all $z \in \mathbb{D}$ and almost all $t \in [0,\infty)$. Then $\Omega[(f_t)] = \mathbb{C}$.

Definition

A family $\{g_t\}_{t\geq 0}$ of holomorphic maps of the unit disk is called a **decreasing Loewner chain of order d** with $d \in [1,\infty]$ if it satisfies the following conditions:

1. g_t is univalent on $\mathbb{D}$ for each $t \in [0,\infty)$,
2. $g_0(z) = z$ and $g_s(\mathbb{D}) \supset g_t(\mathbb{D})$ for all $0 \leq s < t < \infty$,
3. for any compact set $K \subset \mathbb{D}$ and all $T > 0$, there exists a non-negative function $k_{K,T} \in L^d([0,T],\mathbb{R})$ such that

$$|g_s(z) - g_t(z)| \leq \int_s^t k_{K,T}(\zeta)d\zeta \qquad (4)$$

for all $z \in \mathcal{K}$ and all $0 \leq s \leq t \leq T$.

- We denoted by $\texttt{DLC}^d$ a family of all decreasing Loewner chain of order d.
- $\partial_t g_t(z) = (z - \sigma(t))(\overline{\sigma(t)}z - 1)\partial_z g_t(z)q(z,t)$
- $\Lambda[(g_t)] := \bigcap_{t\geq 0} \overline{g_t(\mathbb{D})}$

Definition

Let $d \in [1, \infty]$. A family $\{\omega_{s,t}\}_{0 \le s \le t}$ of holomorphic self-maps of the unit disk $\mathbb{D}$ is called a **reverse evolution family of order d** with $d \in [1, \infty]$ (or in short, an L^d-reverse evolution families) if the following conditions are fulfilled:

1. $\omega_{s,s}(z) = z$,
2. $\omega_{s,t} = \omega_{s,u} \circ \omega_{u,t}$ for all $0 \le s \le u \le t < \infty$,
3. for all $z_0 \in \mathbb{D}$ and for all $T_0 > 0$ there exists a non-negative function $k_{z_0,T_0} \in L^d([0, T_0], \mathbb{R})$ such that

$$|\omega_{s,u}(z_0) - \omega_{s,t}(z_0)| \le \int_u^t k_{z_0,T_0}(\zeta) d\zeta$$

for all $0 \le s \le u \le t \le T_0$.

- $\mathtt{REF}^d$: a family of all reverse evolution family of order d.

Theorem (H, 2014)

Let $d \in [1, \infty]$ and $k \in [0, 1)$. Let $(f_t) \in \mathtt{LC}^d$ and $(p, \tau) \in \mathtt{BP}$ associated with (f_t). We denote by $T^* \in [0, \infty]$ the smallest number such that $p(\mathbb{D}, t) \in i\mathbb{R}$ for almost all $t \in (T^*, \infty)$. Suppose that $T^* \neq 0$ and $p \in \mathtt{HF}^d$ satisfies

$$|p(z,t) - \overline{q(z,t)}| \leq k \cdot |p(z,t) + q(z,t)| \qquad (5)$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, where $q \in \mathtt{HF}^d$. Let $(\omega_{s,t}) \in \mathtt{REF}^d$ associated with $(q, \tau) \in \mathtt{BP}$ and $(g_t) \in \mathtt{DLC}^d$ associate with $(\omega_{s,t})$. Then, f_t and g_t has continuous extensions to $\overline{\mathbb{D}}$ for each $t \in [0, T^*)$, and Φ defined by

$$\begin{cases} \Phi(z) = f_0(z), & z \in \mathbb{D}, \\ \Phi\left(\dfrac{1}{\overline{g_t(e^{i\theta})}}\right) = f_t(e^{i\theta}), & \theta \in [0, 2\pi) \quad \text{and} \quad t \in [0, T^*), \end{cases} \qquad (6)$$

is a k-quasiconformal mapping on $\Delta[(g_t)]$ onto $\Omega[(f_t)]$.

- $\Delta[(g_t)] := \left\{ \frac{1}{\bar{w}} : w \in \widehat{\mathbb{C}} \backslash \Lambda[(g_t)] \right\}$

Theorem (H, 2014)

Let $d \in [1, \infty)$ and $k \in [0, 1)$. Let $(g_t) \in \mathtt{DLC}^d$ and $(q, \tau) \in \mathtt{BP}$ associated with (g_t). If q satisfies

$$\left| \frac{q(z,t) - 1}{q(z,t) + 1} \right| \leq k$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, then g_t has a k-quasiconformal extension to $\widehat{\mathbb{C}}$ for each $t \in [0, \infty)$. Further, $\Lambda[(g_t)]$ consists of one point in $\overline{\mathbb{D}}$.

Thank you for your attention!!