# Introduction to Loewner Theory

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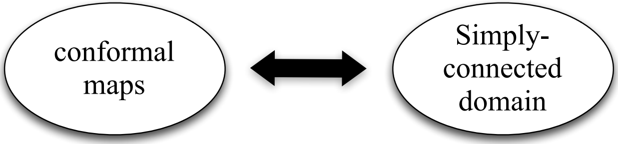
# Conformal maps

**Theorem (The Riemann mapping theorem)**

Let $\Omega \subset \mathbb{C}$ be a simply connected proper subdomain. Then there is a conformal surjection $f : \mathbb{D} \to \Omega$. Moreover, if $g$ is another such mapping, then $g^{-1} \circ f : \mathbb{D} \to \mathbb{D}$ is a linear fractional transformation. In particular, given $z_0 \ in\Omega$, there exists a unique conformal mapping $f : \mathbb{D} \to \Omega$ with $f(0) = 0$ and $f'(0) > 0$.




Geometry:

$\{$hyperbolic simply-connected domains on $\mathbb{C}\}/\{$rotation, expansion, reduction$\}$

Analysis:

### Definition (Class $\mathcal{S}$)

By $\mathcal{S}$ we denote the family of all holomorphic univalent functions $f : \mathbb{D} \to \mathbb{C}$,

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n \ (f(0) = 0,\ f'(0) = 1).$$

### Theorem

$\mathcal{S}$ is compact in the topology of locally uniform convergence.

### Theorem (The Bieberbach conjecture)

For all $f(z) := z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $\mathcal{S}$, we have

$$|a_n| \leq n \qquad (n \geq 2).$$

Equality holds iff $f$ is the koebe function

$$K(z) := \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$$

and its rotation $e^{i\theta} K(ze^{-i\theta})$, $0 \leq \theta < 2\pi$.

# Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln.

Von Prof. Dr. Ludwig Bieberbach
in Frankfurt a. M.

(Vorgelegt von Hrn. Frobenius am 6. Juli 1916 [s. oben S. 775].)

**The article**

[1] Daß $k_n \geq n$ zeigt das Beispiel $\sum nz^n$. Vielleicht ist überhaupt $k_n = n$.

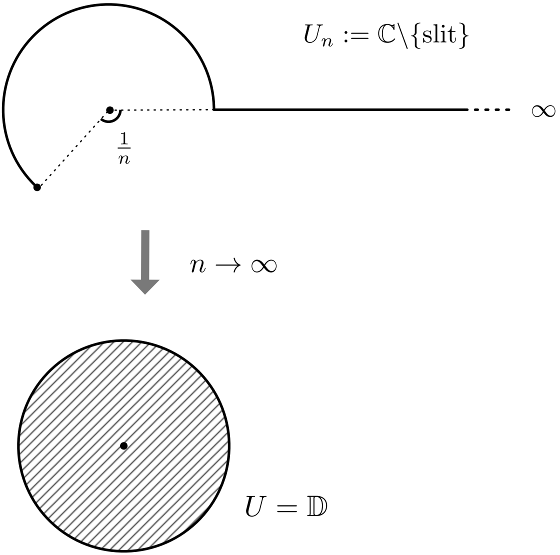
**and the footnote that led to it all.**

# The kernel convergence

## Definition (Carathéodory kernel)

- $a \in \mathbb{C}$ : a point
- $\{U_n\} \ni a$ : a sequence of domains on $\mathbb{C}$.
- $V_n$ : a connected component of the interior of $U_n \cap U_{n+1} \cap \cdots$ in which $a$ is contained

$\Rightarrow$ $U := \bigcup_n^\infty V_n \neq \emptyset$ is called the **Carathéodory kernel** of $\{U_n\}$ w.r.t. $a$.
$\Rightarrow$ If $U$ is empty, then we employ $\{a\}$ as the Carathéodory kernel.


$U_n := \mathbb{C}\backslash\{\text{slit}\}$
$\infty$
$\frac{1}{n}$
$n \to \infty$
$U = \mathbb{D}$

# The kernel convergence

## Definition (Carathéodory kernel)

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## Theorem

Let $\{f_n\}$ be a sequence of conformal maps $\mathbb{D}$ with $f_n(0) = a$ and $f_n'(0) > 0$. Then $f_n$ converges on $\mathbb{D}$ locally uniformly to $f$ iff $U_n = f_n(\mathbb{D})$ converges to its kernel $U \neq \mathbb{C}$.
If the kernel is $\{0\}$, then $f = 0$. Otherwise $f$ is conformal on $\mathbb{D}$ and $f(\mathbb{D}) = U$.

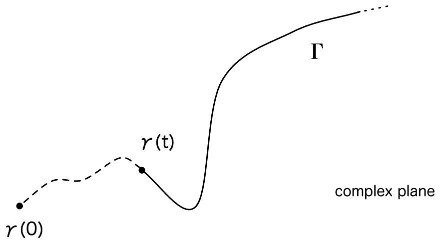
## Theorem

$\mathcal{S}_{\text{slit}} := \{f \in \mathcal{S} : f(\mathbb{D}) = \mathbb{C} \backslash \Gamma$, where $\Gamma \subset \mathbb{C}$ is a Jordan arc extending to $\infty\}$.
Then

$$\mathcal{S}_{\text{slit}} \overset{\text{dense}}{\subset} \mathcal{S}.$$

# Löwner's construction

$\mathcal{S}_{\text{slit}} := \{f \in \mathcal{S} : f(\mathbb{D}) = \mathbb{C}\backslash\Gamma$, where $\Gamma := \gamma[0,\infty)$ is a Jordan arc extending to $\infty\}$.




Löwner's Construction

- Take one function $f \in \mathcal{S}_{\text{slit}}$,
- Consider the domain $\Omega_t := \mathbb{C} \setminus \gamma[t,\infty)$ $(t \geq 0)$,
- There exists a unique conformal mapping $f_t : \mathbb{D} \to \Omega_t$ such that $f_t(0) = 0$ and $f_t'(0) > 0$ (note that $f_0(z) = f(z)$),
- Reparameterize $\Gamma$ as $f_t'(0) = e^t$.

### Theorem (Löwner, 1923)

1. The family $(f_t)$ is of class $C^1$ with respect to $t \in [0, \infty)$ (even $\Gamma$ is not smooth).
2. There exists a continuous real function $\lambda : [0, \infty) \to \mathbb{R}$ such that

$$\dot{f}_t(z) = z f_t'(z) \cdot p(z,t) \qquad (z \in \mathbb{D},\ t \geq 0), \tag{1}$$

where $\dot{f}_t := \partial f_t / \partial t$, $f_t' := \partial f_t / \partial z$ and $p(z,t) := \dfrac{1 + e^{i\lambda(t)} z}{1 - e^{i\lambda(t)} z}$.

- The partial differential equation (1) is called the **radial Loewner PDE** (of the slit case).
- Comparing the coefficient of $z$ of the both sides, we have the equation

$$a_2(t) = -2e^{2t} \int_t^\infty e^{-u} e^{-i\lambda(u)} du.$$

Hence $|a_2| = 2$.

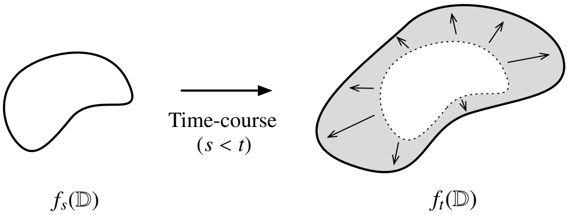
# Pommerenke's generalization

Pommerenke dealt with more general case of that $f_t(\mathbb{D})$ are simply-connected domains.

**Definition**

Let $f_t(z) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$ be a function defined on $\mathbb{D} \times [0, \infty)$. $f_t$ is said to be a (**classical**) **Loewner chain** if $f_t$ satisfies the conditions (Fig. 2);

1. $f_t$ is holomorphic and univalent in $\mathbb{D}$ for each $t \in [0, \infty)$,
2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for all $0 \leq s < t < \infty$.



## Theorem (Pommerenke, 1965)

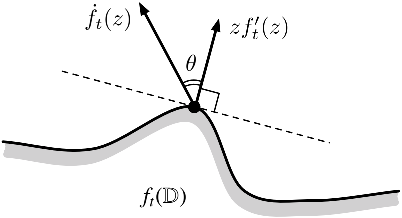
If $f(z,t) := f_t$ is a Loewner chain, then

- for each $z_0 \in \mathbb{D}$, $f(z_0, \cdot)$ is absolutely continuous on $t \in [0,\infty)$,
- $f_t$ satisfies

$$\dot{f}_t(z) = f_t'(z) \cdot zp(z,t) \qquad (z \in \mathbb{D}, \text{ a.e. } t \geq 0), \tag{2}$$

where $p(z,t)$ is a **Herglotz function**, i.e. $p$ satisfies

1. For all $z_0 \in \mathbb{D}$, the function $p(z_0, \cdot)$ is measurable on $t \in [0,\infty)$,
2. For all $t_0 \in [0,\infty)$, the function $p(\cdot, t_0)$ is holomorphic on $z \in \mathbb{D}$,
3. $\operatorname{Re} p(z,t) \geq 0$ for all $z \in \mathbb{D}$ and $t \in [0,\infty)$.




$$|\theta| = |\arg p(z,t)| < \frac{\pi}{2}$$

# Evolution family

To solve the Loewner differential equation, the notion of evolution families plays a key role.

**Definition**

A two-parameter family of holomorphic self-maps of the unit disk $(\varphi_{s,t})$, $0 \le s \le t < \infty$ is called an **evolution family** if;

1. $\varphi_{s,s}(z) = z$,
2. $\varphi_{s,t}(0) = 0$ and $\varphi'_{s,t}(0) = e^{s-t}$,
3. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \le s \le u \le t < \infty$.

$f_t^{-1} \circ f_s$ defines an evolution family. Further, since $\dot{f}_t(\varphi_{s,t}) + f'_t(\varphi_{s,t})\dot{\varphi}_{s,t} = 0$ one can obtain

$$\dot{\varphi}_{s,t}(z) = -\varphi_{s,t}(z)p(\varphi_{s,t}(z), t). \tag{3}$$

## Theorem (Pommerenke, 1965)

Suppose that $p$ is the Herglotz function. Then, for each fixed $z_0 \in \mathbb{D}$ and $s_0 \in [0, \infty)$, the initial value problem

$$\begin{cases} \dot{w}_t = -w_t p(w_t, t) & t \in (s_0, \infty) \\ w_{s_0} = z_0 \end{cases}$$

for almost all $t \in [s, \infty)$ has a unique absolutely continuous solution $w_{z_0, s_0}(t)$ with the initial condition $w(s_0) = z_0$. If we write $\varphi_{s,t}(z) := \{w(t)\}_{z \in \mathbb{D}, s \geq 0}$, then $\varphi_{s,t}$ is an evolution family and univalent on $\mathbb{D}$.
Conversely, if $f_t$ is a Loewner chain and $\varphi_{s,t}$ is an evolution family associated with $f_t$ by $\varphi_{s,t} := f_t^{-1} \circ f_s$. Then for almost all fixed $t \in [s, \infty)$, $\varphi_{s,t}$ satisfies

$$\dot{\varphi}_{s,t}(z) = -\varphi_{s,t}(z) p(\varphi_{s,t}(z), t)$$

for all $z \in \mathbb{D}$.

Further, the function $f_s(z)$ defined by

$$f_s(z) := \lim_{t \to \infty} e^t \varphi_{s,t}(z)$$

exists locally uniformly in $z \in \mathbb{D}$ and is a Loewner chain.

## Chordal Loewner Equations

On the other hand, in 1968 Kufarev, Sobolev, Sporysheva considered the following class ($\mathbb{H}^+ := \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta > 0\}$)

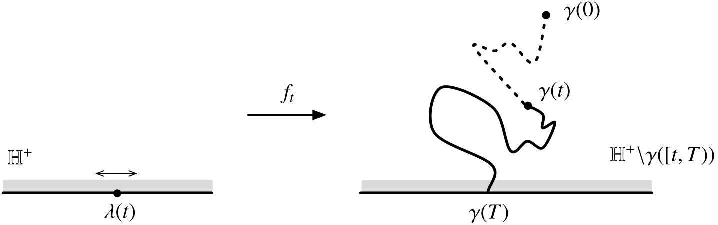
$$\{f : \mathbb{H}^+ \to \mathbb{H}^+, \text{ holomorphic univalent} : f \text{ satisfies } (*)\},$$

where

$$(*) : \text{Hydrodynamic Normalizations } \lim_{\zeta \to \infty} |f(\zeta) - \zeta| = 0,$$

i.e., $f$ has the following Laurent expansion at $\infty$ of the form

$$f(\zeta) = \zeta + \frac{b_1(t)}{\zeta} + \sum_{n=2}^{\infty} \frac{b_n(t)}{\zeta^n}.$$




- $\Gamma$ is a Jordan arc on $\mathbb{H}^+$, parametrize it as $\gamma[0,T]$,
- Let $f_t := \mathbb{H}^+ \to \mathbb{H}^+ \setminus \gamma[t,T)$ be a conformal mapping (exists uniquely),
- We reparametrize $b_1(t)$ as $2t$, then there exists a $\lambda(t) : [0,T] \to \mathbb{R}$ s.t.

$$\dot{f}_t(\zeta) = -f_t'(\zeta)p(\zeta,t) \tag{4}$$

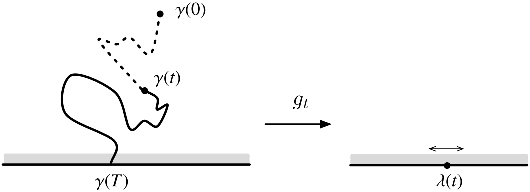
where $p(\zeta,t) = \dfrac{2}{\lambda(t) - \zeta}$.

The equation (4) is called **chordal Loewner PDE**, and the above $(f_t)$ is called a **(classical) chordal Loewner chain**).

⇒ Nov. 22, 15:30 - 16:20 **Andrea del Monaco** (Univ. Rome "Tor Vergata")
*Geometry and Loewner Theory*

# Schramm-Loewner Evolution

Consider the inverse map $g := f^{-1}$, then $\dot{g}_t(w) = \dfrac{2}{g_t(w) - \lambda(t)}$.




Set $\lambda(t) := \sqrt{\kappa}\mathcal{B}_t$, where $\mathcal{B}_t$ is 1-dim Brownian motion and $\kappa > 0$. Then

$$\dot{g}_t(w) = \frac{2}{g_t(w) - \sqrt{\kappa}\mathcal{B}_t}. \tag{5}$$

The unique solution $g_t$ of (5) is called the **Schramm-Loewner Evolution**.

$\Rightarrow$ Nov 23, 13:30 - 14:20 **Hiroyuki Suzuki** (Chuo Univ.)
*Convergence of loop erased random walks on a planar graph to a chordal SLE(2)*

## Semigroup of holomorphic self-maps of $\mathbb{D}$

Let Hol($\mathbb{D}$) be a family of all holomorphic self-maps of $\mathbb{D}$.

**Definition (The Denjoy-Wolff point)**

- By the Schwarz-Pick Lemma, $f \in$ Hol($\mathbb{D}$) may have at most one fixed point in $\mathbb{D}$. If such a point exists, then it is called the **Denjoy-Wolff point** of $f$.
- If $f$ does not have a fixed point in $\mathbb{D}$, then the Denjoy-Wolff theorem claims that there exists a unique boundary fixed point $\angle \lim_{z\to\tau} f(z) = \tau \in \partial\mathbb{D}$ such that the sequence of iterates $\{f^n\}_{n\in\mathbb{N}}$ converges to $\tau$ locally uniformly. In this case $\tau$ is also called the **Denjoy-Wolff point** of $f$.

A family $\{\phi_t\}_{t\geq 0}$ of holomorphic self-mappings of $\mathbb{D}$ is called a **one-parameter semigroup** if

1. $\phi_0(z) = id_{\mathbb{D}}$,
2. $\phi_{s+t} = \phi_t \circ \phi_s$ for all $s, t \in [0, \infty)$,
3. $\lim_{t\to 0^+} \phi_t(z) = z$ locally uniformly on $\mathbb{D}$.

For a semigroup $\phi_t$, there exists a holomorphic function $G \in \text{Hol}(\mathbb{D}, \mathbb{C})$ such that $\phi_t$ is a unique solution of the Cauchy problem

$$\frac{d\phi_t(z)}{dt} = G(\phi_t(z)) \qquad (t \in [0, \infty)) \tag{6}$$

with the initial condition $\phi_0(z) = z$. The above function $G$ is called the **infinitesimal generator** of the semigroup.

Various criteria which guarantee that a homeomorphic function $G \in \text{Hol}(\mathbb{D}, \mathbb{C})$ is the infinitesimal generator are known.

### Berkson and Porta (1978)

A holomorphic function $G \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is the infinitesimal generator if and only if there exists a $\tau \in \overline{\mathbb{D}}$ and a function $p \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ with $\mathrm{Re}\, p(z) \geq 0$ for all $z \in \mathbb{D}$ such that

$$G(z) = (\tau - z)(1 - \bar{\tau} z) p(z) \qquad (7)$$

for all $z \in \mathbb{D}$. This equation is called the *Berkson-Porta representation*.

The point $\tau$ in (7) is the Denjoy-Wolff point of all the functions of the semigroup.

⇒ Nov. 22, 11:40 - 12:30 **Santiago Díaz-Madrigal** (Univ. Seville)
*Fixed points in Loewner theory*

⇒ Nov. 23, 10:00 - 10:50 **Pavel Gumenyuk** (Univ. Stavanger)
*Loewner-type Parametric Representation of univalent self-maps with given boundary regular fixed points*

## Applications

**Quasiconformal Mappings**

- $f$: homeomorphism on $G$ with $f \in W^{1,2}_{\text{loc}}$
- $\mu_f := \partial_{\bar{z}} f / \partial_z f$ : **Beltrami coefficient**
- If $||\mu||_\infty \leq k$ a.e. on $G$, then $f$ is called **k-quasiconformal** on $G$ ($k \in [0, 1)$).

⇒ Nov. 22, 14:30 - 15:20 **Toshiyuki Sugawa** (Tohoku Univ.)
*An application of the Loewner theory to trivial Beltrami coefficients*

**Theorem (Becker, 1972)**

Let $f_t$ be a radial Loewner chain and $k \in [0, 1)$. If the herglotz function $p$ associated with $f_t$ satisfies

$$|1 - p(z,t)| \leq k|1 + p(z,t)| \qquad (z \in \mathbb{D}, \text{a.e. } t \in [0, \infty))$$

then there exists a $k$-quasiconformal map $F$ on $\mathbb{C}$ such that $F\big|_{\mathbb{D}} \equiv f_0$.

⇒ Nov. 23, 14:30 - 15:20 **Ikkei Hotta** (Tokyo Tech.)
*$L^d$-Loewner chains with quasiconformal extensions*

## Application

**Hele-Shaw Flows**

Find a conformal map $f_t$ on the closed unit disk $\overline{\mathbb{D}}$ satisfying $f_t(0) = 0$, $f_t'(0) > 0$ and

$$\text{Re}\left\{ z f_t'(z) \overline{\dot{f}_t(z)} \right\} = \frac{q(t)}{2\pi} \qquad (|z| = 1).$$

By the Poisson-Schwarz formula, it is represented by

$$\dot{f}_t(z) = z f_t'(z) p(z,t) \quad \text{with} \quad p(z,t) := \frac{q(t)}{4\pi^2} \int_0^{2\pi} \frac{1}{|f_t'(e^{i\theta})|} \cdot \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\theta$$

⇒ Nov. 22, 16:40 - 17:30 **Michiaki Onodera** (Kyushu Univ.)
*On a deformation flow for an inverse problem in potential theory*

**Integrable systems**

Chordal Loewner PDE (slit case) $\Longleftrightarrow$ dKP hierarchy

Radial Loewner PDE (slit case) $\Longleftrightarrow$ dToda hierarchy

⇒ Nov. 23, 11:00 - 11:50 **Takashi Takebe** (National Research Univ.)
*Loewner equations and dispersionless integrable hierarchies*

Thank you for your attention!!