RESEARCH REPORTS OF THE NEVANLINNA THEORY AND ITS APPLICATIONS II
(COMPLEX DYNAMICS, COMPLEX DIFFERENTIAL EQUATIONS, P-ADIC NEVANLINNA THEORY)

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Two international mathematical conferences on complex analysis were organized at the Nippon Institute of Technology (NIT) in 1998. One is the meeting "Meromorphic functions; theories and applications" at 19th May. In this meeting, 9 talks were given. The other one is held in November 26–29 titled "p-adic Nevanlinna theory and related topics" in which 19 talks were presented. Many mathematicians participated in them, who research complex analysis, in particular, complex dynamics theory and the value distribution theory. Moreover, NIT is very glad to have guests from foreign countries, Professors C.C. Yang (Honk Kong) and L.H. Zhao (Kyoto) in the first conference, and Professors W. Cherry(Texas), L.C. Hsia (Taiwan), P.C. Hu (Shandong), H. H. Khoai (Hanoi), T-Y. J. Wang (Taiwan) in the second conference. We could have a very precious opportunity to discuss the Nevanlinna theory and its applications with them.

Editors strongly felt that it would be very important to collect summaries of talks in the two conferences. It is very fortunate that the collection is now realized, say this note. The editors express their sincere appreciation to the contributors for their supplies of abstract of his/her talks, new results, timely topics, and open problems.

Contents (in alphabetical order)

Date: May 26, 2005.

1991 *Mathematics Subject Classification.* 30D35 .

This work was supported in part by a Grant-in-Aid for General Scientific Research from the Ministry of Education, Science and Culture 11640164 (Toda) and by a Grant from NIPPON Institute of Technology 0221 (1999).

Introduction to p-adic Analysis

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Introduction.

The purpose of this article is to give some fundamental results on p-adic numbers and p-adic analysis which are used in p-adic Nevanlinna theory. This article consists of two parts. The first three sections are devoted to constructing the non-archimedean field $\mathbb{C}_p$ which is not only complete but also algebraically closed. In the next three sections of this article, we deal with analysis on $\mathbb{C}_p$ and especially give several important theorems about zeros of functions on $\mathbb{C}_p$ defined by power series. All the proofs of these results are found in [**?**], [**?**], [**?**] and [**?**].

§1. Absolute Values on Fields.

Let $\mathbb{k}$ be a field. We begin by defining an absolute value on $\mathbb{k}$.

Definition (1.1). An *absolute value on* $\mathbb{k}$ is a real valued function $|\cdot| : \mathbb{k} \to \mathbb{R}^+$ that satisfy the following three conditions:

(AV 1) $|x| = 0$ if and only if $x = 0$

(AV 2) $|xy| = |x|\,|y|$ for all $x,\ y \in \mathbb{k}$

(AV 3) $|x + y| \le |x| + |y|$ for all $x,\ y \in \mathbb{k}$.

We say that the absolute value $|\cdot|$ is *archimedean* if it satisfies (AV 1) - (AV3). An absolute value on $\mathbb{k}$ is said to be *non-archimedean* if instead of (AV 3) it satisfies the stronger condition

(AV 4) $|x + y| \le \max\{|x|,\ |y|\}$ for all $x,\ y \in \mathbb{k}$.

We say that $|\cdot|$ is a *trivial absolute value* if $|0| = 0$ and $|x| = 1$ for all $x \in \mathbb{k}^\times$. Throughout this note, we assume that an absolute value $|\cdot|$ is nontrivial. The most obvious example of archimedean absolute value is the usual absolute value on $\mathbb{Q}$. This absolute value called the *absolute value at infinity*, and written as $|\cdot|_\infty$. It is clear that condition (AV 4) does not hold for $|\cdot|_\infty$. We say that two absolute values on a field $\mathbb{k}$ is equivalent they defined the same topology on $\mathbb{k}$. We now introduce an example of non-archimedean absolute value, so called p-adic absolute value, which is not equivalent to $|\cdot|_\infty$. Let any prime $p \in \mathbb{Z}$ be fixed.

Definition (1.2). The *p-adic valuation on* $\mathbb{Z}$ is a function

$$v_p : \mathbb{Z} - \{0\} \to \mathbb{R}$$

defined as follows: for each integer $n \in \mathbb{Z}, n \neq 0$, let $v_p(n)$ be the unique positive integer satisfying

$$n = p^{v_p(n)} n' \quad \text{with } p \nmid n'.$$

We extend v_p to the field of rational numbers $\mathbb{Q}$ as follows: if $x = a/b \in \mathbb{Q}^\times$, then

$$v_p(x) = v_p(a) - v_p(b).$$

For the convenience, we often set $v_p(0) = +\infty$. It is in fact easy to see that the p-adic valuation of $x \in \mathbb{Q}^\times$ is determined by the formula

$$x = p^{v_p(x)} \cdot \frac{a}{b}, \quad pa \nmid b.$$

The basic properties of p-adic valuations are the following:

Lemma (1.3). *Let* $x,\ y \in \mathbb{Q}$. *Then*

(1) $v_p(xy) = v_p(x) + v_p(y)$

(2) $v_p(x + y) \ge \min\{v_p(x),\ v_p(y)\}$

with the conventions with respect to $v_p(0) = +\infty$.

Definition (1.4). For $x \in \mathbb{Q}$, we define the *p-adic absolute value* of x by

$$|x|_p = p^{-v_p(x)}$$

if $x \neq 0$, and $|0|_p = 0$.

By making use of Lemma (1.3), we see that our definition really does give an absolute value:

Proposition (1.5) *The function* $|\cdot|_p$ *is a non-archimedean valuation on* $\mathbb{Q}$.

Moreover, we have the following (cf. [**?**, p. 28]):

Theorem (1.6). *Let A be the image of the usual homomorphism $\mathbb{Z} \to \mathbb{k}$. An absolute value $|\cdot|$ is non-archimedean if and only if $|a| \le 1$ for all $a \in A$. In particular, an absolute value on $\mathbb{Q}$ is non-archimedean if and only if $|n| \le 1$ for every $n \in \mathbb{Z}$.*

Proposition (1.7). *Let $|\cdot|$ be a non-archimedean absolute value on $\mathbb{k}$. If $x,\ y \in \mathbb{k}$ and $|x| \neq |y|$, then*

$$|x + y| = \max\{|x|,\ |y|\}.$$

Now we consider a metric space $(\Bbbk,\ d)$ with the distance $d(x,\ y)$ between two element $x,\ y \in \Bbbk$ defined by

$$d(x,\ y) = |x - y|.$$

Let $r \in \mathbb{R}^+$. We define an open ball and a closed ball of radius r and center $a \in \Bbbk$ as the usual way:

$$B(a,\ r) = \{x \in \Bbbk; d(x,\ a) < r\}$$

and

$$\overline{B}(a,\ r) = \{x \in \Bbbk; d(x,\ a) \leq r\}.$$

For non-archimedean absolute values, we get the following properties (cf. [**?**, p. 34]):

Proposition (1.8). *Let $\Bbbk$ be a field with a non-archimedean absolute value.*

(i) *If $b \in B(a,\ r)$, then $B(a,\ r) = B(b,\ r)$.*

(ii) *If $b \in \overline{B}(a,\ r)$, then $\overline{B}(a,\ r) = \overline{B}(b,\ r)$.*

(iii) *The set $B(a,\ r)$ is both open and closed.*

(iv) *If $r \neq 0$, the set $\overline{B}(a,\ r)$ is both open and closed.*

(v) *Any two open balls are either disjoint or contained in one another.*

(vi) *Any two closed balls are either disjoint or contained in one another.*

Proposition (1.9). *Let $\Bbbk$ be a field with a non-archimedean absolute value. Then $(\Bbbk,\ d)$ is a totally disconnected topological space.*

Next we take a more algebraic point of view, and look for connections between non-archimedean absolute values and the algebraic structure.

Proposition (1.10). *Let $\Bbbk$ be a field with a non-archimedean absolute value $|\cdot|$. The set $\mathcal{O} = \{x \in \Bbbk; |x| \leq 1\}$ is a subring of $\Bbbk$. Its subset $\mathfrak{P} = \{x \in \Bbbk; |x| < 1\}$ is a unique maximal ideal in $\mathcal{O}$.*

Definition (1.11). Let $\Bbbk$ be as in Proposition (1.10). The subring $\mathcal{O}$ is called the *valuation ring* of $|\cdot|$. The ideal $\mathfrak{P}$ is called the *valuation ideal* of $|\cdot|$. The quotient $\kappa = \mathcal{O}/\mathfrak{P}$ is called the *residue field* of $|\cdot|$.

Example (1.12). Let $\Bbbk = \mathbb{Q}$, and let $|\cdot| = |\cdot|_p$ be a p-adic absolute value. Then:

(i) the associate valuation ring is $\mathcal{O} = \mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q}; p \nmid b\}$;

(ii) the valuation ideal is $\mathfrak{P} = p\,\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q}; p \nmid b \text{ and } p \mid a\}$;

(iii) the residue field is $\kappa = \mathbb{F}_p$ (the field with p elements).

§2. p-adic Numbers.

In §1, we have constructed two examples of absolute value on the field $\mathbb{Q}$ of rational numbers, that is, the usual absolute value $|\cdot| = |\cdot|_\infty$ and for each prime, the p-adic absolute value $|\cdot|_p$. We notice that these examples have an essential importance for the theory of absolute values on $\mathbb{Q}$. Indeed, we have the following fundamental result (cf. [**?**, p. 44] or [**?**, p. 3]):

Theorem (2.1) (Ostrowski). *Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime number p or $p = \infty$.*

We notice that the field $\mathbb{Q}$ of rational numbers is not complete with respect to any of its nontrivial absolute value (cf. [**?**, p. 50]). The completion of the field $\mathbb{Q}$ of rational numbers with respect to $|\cdot|_\infty$ is the field $\mathbb{R}$ of real numbers. In what follows, we will construct the completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the p-adic absolute value.

Let $\mathcal{C}$ be the set of all Cauchy sequences of elements of $\mathbb{Q}$. Then defining

$$\{x_n\} + \{y_n\} := \{x_n + y_n\} \qquad \text{and} \qquad \{x_n\}\{y_n\} := \{x_n y_n\}$$

makes $\mathcal{C}$ a commutative ring with unity. We define $\mathcal{N} \subset \mathcal{C}$ by

$$\mathcal{N} = \{\{x_n\}; \lim_{n\to\infty} |x_n|_p = 0\}.$$

Then $\mathcal{N}$ is a maximal ideal of $\mathcal{C}$ (cf. [**?**, p. 53]). We define the *field of p-adic numbers* by $\mathbb{Q}_p = \mathcal{C}/\mathcal{N}$. We have a natural inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$. We shall show that the absolute value $|\cdot|_p$ extends to $\mathbb{Q}_p$.

Lemma (2.2). *Let $\{x_n\} \in \mathcal{C} - \mathcal{N}$. Then there is a positive integer N such that $|x_n|_p = |x_m|_p$ whenever $m,\ n \geq N$.*

Definition (2.3). Let $\lambda \in \mathbb{Q}_p$. If $\{x_n\} \in \mathcal{C}$ is representing λ, we define

$$|\lambda|_p = \lim_{n\to\infty} |x_n|_p.$$

By Lemma (2.2), it is well-defined. The following is the main result in this section (cf. [**?**, p. 55]):

Proposition (2.4). *The field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to a non-archimedean absolute value $|\cdot|_p$.*

Notice that the set of values of $\mathbb{Q}$ and $\mathbb{Q}_p$ under $|\cdot|_p$ is the same; specifically the two set are equal to the set $\{p^n; n \in \mathbb{Z}\} \cup \{0\}$. Thus the p-adic valuation v_p is extended to $\mathbb{Q}_p$ by $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{Q}_p$.

Definition (2.5). The *ring of p-adic integers* is the valuation ring

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p;\ |x|_p \le 1\}.$$

Then we have the following (cf. [**?**, pp. 59–62]):

Proposition (2.6). *The ring $\mathbb{Z}_p$ is a local ring whose maximal ideal is the principal ideal $p\mathbb{Z} = \{x \in \mathbb{Q}; |z|_p < 1\}$. Furthermore*

(i) $\mathbb{Q} \cap \mathbb{Z}_p = \mathbb{Z}_{(p)}$.

(ii) *The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ has dense image.*

Corollary (2.7). $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$. *The map $\mathbb{Q}_p \to \mathbb{Q}_p$ given by $x \mapsto px$ is a homeomorphism. The sets $p^n\mathbb{Z}$, $n \in \mathbb{Z}$ forms a fundamental system of neighborhoods of zero in $\mathbb{Q}_p$ which covers all $\mathbb{Q}_p$.*

Corollary (2.8). *For any $n \ge 1$, $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$.*

Corollary (2.9). *$\mathbb{Q}_p$ is a totally disconnected Hausdorff space.*

Corollary (2.10). *$\mathbb{Z}_p$ is compact and $\mathbb{Q}_p$ is locally compact.*

§3. Building up $\mathbb{C}_p$.

In this section, we construct the field $\mathbb{C}_p$ containing $\mathbb{Q}_p$ that is not only complete but also algebraically closed. In order to be able to give the construction, we need to recall a few facts from the theory of fields (for details, see [**?**]). We assume that all fields have characteristic zero. Let K and F be fields, and assume that $F \subseteq K$. We write $[K : F] = \dim_F K$ and call its number the degree of K over F. Assume that $[K : F]$ is finite. We say that the field extension K/F is a finite field extension.

Let $\mathbf{C}$ be any algebraically closed field containing F. We say that the field extension K/F is *normal* if all the homomorphisms $\sigma : K \hookrightarrow \mathbf{C}$ which induce the identity on F have the same image. We call an automorphism $\sigma : k \to K$ which induces the identity on F an *automorphism of the extension K/F*. It is known that when K/F is normal the automorphisms of K/F form a finite group whose order is equal to the degree $[K : F]$. This group $\mathrm{Gal}(K/F)$ is called the *Galois group* of the extension. Further, it is known that given any finite extension K/F, there exists a finite normal extension of F containing K. The smallest such is called the *normal closure* of K/F. In what follows, we assume that K/F is normal extension. Then we define a function

$$N_{K/F} : K \to F,$$

which is called the *norm form* K to F, as follows:

$$N_{K/F}(\alpha) = \prod_{\sigma\in \mathrm{Gal}(K/F)} \sigma(\alpha).$$

Notice that $N_{K/F}(\alpha) = \alpha^n$ for $\alpha \in F$, where $n = [K : F]$.

Now we consider the case $F = \mathbb{Q}_p$. Then we have the following (cf. [**?**, §5.3]):

Proposition (3.1). *Let $K/\mathbb{Q}_p$ be a normal extension of degree n. Then there exists a unique non-archimedean absolute value $|\cdot|$ on K extending the p-adic absolute value on $\mathbb{Q}_p$, which is defined by*

$$|x| = \sqrt[n]{|N_{K/\mathbb{Q}_p}|}.$$

Furthermore, K is complete with respect to $|\cdot|$.

The p-adic valuation v_p on K is now defined by $|x| = p^{-v_p(x)}$ for $x \ne 0$. We extend the definition formally by $v_p(0) = +\infty$. We also know how to compute v_p:

$$v_p(x) = \frac{1}{n} v_p(N_{K/\mathbb{Q}_p}(x)).$$

We notice the following (cf. [**?**, p. 159]):

Proposition (3.2). *The image of p-adic valuation* v_p *is of the form* $\frac{1}{e}\mathbb{Z}$*, where* e *is a divisor of* $[K:\mathbb{Q}_p]$.

We call e the *ramification index* of K over $\mathbb{Q}_p$. We say an element $\pi \in K$ is a *uniformizer* if $v_p(\pi) = 1/e$. Now we describe the algebraic structure of K. Let $\mathcal{O}_K$ be the valuation ring and $\mathfrak{p}_K$ its maximal ideal. Then the ideal $\mathfrak{p}_K$ is principal, and π is a generator. The residue field $\kappa = \mathcal{O}_K/\mathfrak{p}_K$ is a finite extension of $\mathbb{F}_p$ whose degree is less than or equal to $[K:\mathbb{Q}_p]$. Furthermore, $\mathcal{O}_K$ is a compact topological ring. The sets $\pi^n\,\mathcal{O}_K$ $(n \in \mathbb{Z})$ form a fundamental system of neighborhood of zero in K, which is a totally disconnected Hausdorff locally compact space.

We now define the non-archimedean absolute value on the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ as follows: For any $x \in \overline{\mathbb{Q}}_p$, there exists a normal extension $K/\mathbb{Q}_p$ of degree n with $x \in K$. Now define

$$|x| = \sqrt[n]{|N_{K/\mathbb{Q}_p}|_p}.$$

Then the value $|x|$ of the above equality is independent of choice of K. Therefore we obtain the function $|\cdot| : \overline{\mathbb{Q}}_p \to \mathbb{R}^+$ which is equal to the p-adic absolute value on $\mathbb{Q}_p$.

Theorem (3.3). *The function* $|\cdot| : \overline{\mathbb{Q}}_p \to \mathbb{R}^+$ *constructed as above is the unique non-archimedean absolute value which extends the p-adic absolute value on* $\mathbb{Q}_p$.

It is known that $\overline{\mathbb{Q}}_p$ is not complete with respect to this absolute value (cf. [**?**, p. 154]). Hence we need to construct the completion. This is done exactly as in the case of $\mathbb{Q}_p$.

Proposition (3.4). *There exists a field* $\mathbb{C}_p$ *and the non-archimedean absolute value* $|\cdot|$ *on* $\mathbb{C}_p$ *such that*

(i) $\overline{\mathbb{Q}}_p$ *is dense in* $\mathbb{C}_p$ *and and the restriction of* $|\cdot|$ *to* $\overline{\mathbb{Q}}_p$ *coincides with the p-adic absolute value*;

(ii) $\mathbb{C}_p$ *is complete with respect to* $|\cdot|$; *and*

Proposition (3.5). *If* $x \in \mathbb{C}_p,\ x \neq 0$*, then there exists a rational number* v *such that* $|x| = p^{-v}$.

We write $\mathfrak{O}$ the valuation ring of $\mathbb{C}_p$, that is, $\mathfrak{O} = \{x \in \mathbb{C}_p; |x| \leq 1\}$. This contains the valuation ideal $\mathfrak{P} = \{x \in \mathbb{C}_p; |x| < 1\}$. As always, $\mathfrak{O}$ is a local ring. Finally we have the following (cf. [**?**, p. 182]):

Proposition (3.6). $\mathbb{C}_p$ *is algebraically closed.*

We notice that the ideal $\mathfrak{P}$ is not principal and the residue field $\mathbb{F} = \mathfrak{O}/\mathfrak{P}$ is the algebraic closure of $\mathbb{F}_p$.

§4. Elementary Analysis on $\mathbb{C}_p$.

In this section, we give some elementary results on the function defined by power series in the p-adic context. We begin by studying the basic convergence properties of sequences and series.

Lemma (4.1). *A sequence* $\{a_n\}$ *in* $\mathbb{C}_p$ *is a Cauchy sequence if and only if*

$$\lim_{n\to+\infty} |a_{n+1} - a_n| = 0.$$

Lemma (4.2). *Let* $\{a_n\}$ *be a convergent sequence in* $\mathbb{C}_p$ *which does not tend to zero. Then there exists an positive integer* N *such that* $|a_n| = |a_N|$ *for* $n \geq N$.

Lemma (4.3). *An infinite series* $\sum_{n=0}^{+\infty} a_n$ $(a_n \in \mathbb{C}_p)$ *is convergence if and only if*

$$\lim_{n\to+\infty} a_n = 0,$$

in which case the following inequality holds:

$$\left|\sum_{n=0}^{+\infty} a_n\right| \leq \max_n\ |a_n|.$$

These Lemmas are proved by only using the property of non-archimedean absolute value. Now we consider a power series

$$f(X) = \sum_{n=0}^{+\infty} a_n X^n.$$

Given $x \in \mathbb{C}_p$, we already know that $\sum_{n=0}^{+\infty} a_n x^n$ converges if and only if $|a_n x^n| \to 0$. As in the classical case, the set of all such x is the disk (cf. [**?**, p. 94]):

Proposition (4.4). *Let* $f(X)=\sum_{n=0}^{+\infty} a_nX^n$, *and define*

$$\rho=\frac{1}{\limsup\sqrt[n]{|a_n|}.}$$

(i) *If* $\rho=0$, *then* $f(x)$ *converges only when* $x=0$.

(ii) *If* $\rho=+\infty$, *then* $f(x)$ *converges for every* $x\in\mathbb{C}_p$.

(iii) *If* $0<\rho<+\infty$ *and* $|a_n|\rho^n\to 0$ *as* $n\to+\infty$, *then* $f(x)$ *converges if and only if* $|x|\le\rho$.

(iv) *If* $0<\rho<+\infty$ *and* $|a_n|\rho^n$ *does not tend to zero as* $n\to+\infty$, *then* $f(x)$ *converges if and only if* $|x|<\rho$.

We are now going to consider the zeros of the function defined by power series. The next theorem is a fundamental result about the zeros of functions.

Theorem (4.5) (Strassman). *Let*

$$f(X)=\sum_{n=0}^{+\infty} a_n\,X^n$$

be non-zero power series with coefficients in $\mathbb{C}_p$, *and suppose that* $f(x)$ *converges for all* $x\in\mathfrak{O}$. *Let* N *be the positive integer defined by the following two conditions:*

$$|a_N|=\max_n|a_n|\quad\text{and}\quad|a_n|<|a_N|\text{ for }n>N.$$

Then the function $\mathfrak{O}\to\mathbb{C}_p$ *has at most* N *zeros.*

This theorem is usually proved using p-adic Weierstrass Preparation Theorem. For the direct proof without using p-adic Weierstrass Preparation Theorem, see [**?**, p. 63] and [**?**, p. 106]. Strassman's Theorem is only the first of several theorems about zeros of function on $\mathbb{C}_p$ defined by power series. Here are some consequences.

Corollary (4.6). *Let* $f(X)$ *as in Theorem* (4.5), *and let* $\alpha_1,\cdots,\alpha_m$ *be the roots of* $f(x)=0$ *in* $\mathfrak{O}$. *Then there exists a power series* $g(X)$ *which converges on* $\mathfrak{O}$ *but has no zeros in* $\mathfrak{O}$ *such that*

$$f(X)=(X-\alpha_1)\cdots(X-\alpha_m)\,g(X).$$

Corollary (4.7). *Let* $f(X)=\sum a_n\,X^n$ *be a non-zero power series which converges on* $p^m\,\mathfrak{O}$ *for some* $m\in\mathbb{Z}$. *Then* $f(X)$ *has a finite number of zeros in* $p^m\,\mathfrak{O}$.

Corollary (4.8). *Let* $f(X)=\sum a_n\,X^n$ *and* $g(X)=\sum b_n\,X^n$ *be two non-zero p-adic power series which converge in a disk* $p^m\,\mathfrak{O}$. *If there exist infinitely many number* $\alpha\in p^m\,\mathfrak{O}$, *then* $a_n=b_n$ *for all* n.

Corollary (4.9). *Let* $f(X)=\sum a_n\,X^n$ *be a p-adic power series which converge in some disk* $p^m\,\mathfrak{O}$. *If the function* $f:p^m\,\mathfrak{O}\to\mathbb{C}_p$ *is periodic, then* $f(X)$ *is constant.*

Corollary (4.10). *Let* $f(X)=\sum a_n\,X^n$ *be a non-zero p-adic power series which converge on* $\mathbb{C}_p$. *Then* $f(X)$ *has at most denumerable zeros. Furthermore, if the set of zeros is not finite, then the zeros form a sequence* α_n *with* $|\alpha_n|\to+\infty$.

§5 Weierstrass Preparation Theorem for p-adic Power Series.

The purpose of this section is to give a theorem that has become known as the "p-adic Weierstrass Preparation Theorem." This is a p-adic version of a classical theorem due to Weierstrass which dealt with power series in several complex variables and is an important tool in the theory of functions of several complex variables. The p-adic version gives fundamental information on p-adic functions defined by power series. In the proof of Weierstrass Preparation Theorem, we will want to think of a power series as a limit of polynomials, and our results will be proved first for polynomials, then for power series. For details, we refer [**?**, §6.5] and [**?**, §6.2].

Definition (5.1). Let c be an arbitrary positive real number. We define A_c to be the ring of power series $\sum a_nX^n\in\mathbb{C}_p[[X]]$ which satisfy the condition $\lim_{n\to+\infty}|a_n|c^n=0$.

Theorem (5.2). *Let c be an arbitrary positive real number. Define a function* $||\cdot||_c:A_c\to\mathbb{R}^+$ *as follows*: *for each power series*

$$f(X)=a_0+a_1X+a_2X^2+\cdots+a_nX^n+\cdots$$

belonging to A_c*, set*

$$||f(X)||_c = \max_n \ |a_n|\, c^n.$$

Then

(i) $||f(X)||_c = 0$ *if and only if* $f(X)$ *is identically zero.*

(ii) $||f(X) + g(X)||_c \le \max\{||f(X)||_c,\ ||g(X)||_c\}$.

(iii) $||f(X)g(X)||_c = ||f(X)||_c ||g(X)||_c$.

(iv) $||\cdot||_c$ *induce the* p*-adic absolute value on the constant power series.*

(v) *If* $|x| \le c$*, then* $|f(x)| \le ||f(X)||_c$.

We are now ready to give the main result in this subsection, the p-adic Weierstrass Preparation Theorem. This can be viewed as a direct extension of Strassman's Theorem.

Theorem (5.3) (p-adic Weierstrass Preparation Theorem). *Let* c *be an arbitrary positive real number, and let* $f(X) = \sum a_n X^n \in A_c$*. Let* N *be the number defined by the following two conditions*:

$$|a_N| = ||f(X)||_c \quad \text{and} \quad |a_n| c^n < |a_N| c^N \text{ for all } n > N.$$

Then there exists a polynomial

$$g(X) = b_0 + b_1 X + \cdots + b_N X^N$$

of degree N *and with coefficients in* $\mathbb{C}_p$*, and a power series*

$$h(X) = 1 + d_1 X + d_2 X^2 + \cdots$$

with coefficients in $\mathbb{C}_p$*, satisfying*:

(i) $f(X) = g(X)h(X)$,

(ii) $|b_N| c^N = \max_n \ |b_n| c^n$*, so that* $||g(X)||_c = |b_N| c^N$,

(iii) $h(X) \in A_c$,

(iv) $|d_n| c^n < 1$ *for all* $n \ge 1$ *, so that* $||h(X) - 1||_c < 1$*, and*

(v) $||f(X) - g(X)||_c < 1$.

In particular, $h(X)$ *has no zeros in* $\overline{B}(0,\ c)$.

This is closely related to Strassman's Theorem. In fact, let $c = 1$. Since $h(X)$ has no zeros in $\mathfrak{O}$, it is clear that the zeros of $f(X)$ in $\mathfrak{O}$ are exactly the same as the zeros of $g(X)$. Since $g(X)$ is a polynomial of degree N and $\mathbb{C}_p$ is algebraically closed, we know that, counting multiplicities, $g(X)$ has exactly N zeros in $\mathbb{C}_p$, and the conditions on its coefficients means that all of them are in $\mathfrak{O}$. Indeed, suppose that $g(\alpha) = 0$. We may assume that $b_N = 1$. Then it follows from

$$\alpha^N = -(b_{n-1}\alpha^{n-1} + \cdots + b_1\alpha + b_0)$$

that $|\alpha|^N \le \max\{|b_i||\alpha|^i\}$. Since $|b_i| \le 1$, we see $|\alpha|^N \le \max_{1 \le i \le N-1} |\alpha|^i$. This implies $|\alpha| \le 1$. Hence we know that, counting multiplicities, $f(X)$ has exactly N zeros in $\mathfrak{O}$, which gives a stronger form of Strassman's Theorem. In the case of $c \ne 1$, we also have the same result for a power series which converges on $\overline{B}(0,\ c)$ by argument similar to the above one. Furthermore, this fact gives a precise sense to the "multiplicity" of a zero of a p-adic power series: it is just the multiplicity of that zero in the polynomial appearing in the Weierstrass factorization.

§6 Newton Polygons.

One of the best ways to understand the theory of polynomials and power series with coefficients in $\mathbb{C}_p$ is to introduce the concept of the Newton polygon of polynomials and of power series. This gives us a clear geometric picture that encodes much of information we have collected about zeros of polynomials and power series. For the details on the Newton polygon, we refer [**?**, §6.4] and [**?**, §4.3 and §4.4].

We begin by considering polynomials. Let $f(X) \in \mathbb{C}_p[X]$. We may assume that $f(0) = 1$. Thus we take a polynomial

$$f(X) = 1 + a_1 X + \cdots + a_n X^n$$

with $a_i \in \mathbb{C}_p$. Consider the following set of points in the real coordinate plane.

$$(0,\ 0),\ (1,\ v_p(a_1)), \cdots, (i,\ v_p(a_i)), \cdots, (n,\ v_p(a_n)).$$

If $a_i = 0$ for some i, we omit that point, or we think of the point as "infinitely" far above the horizontal axis. The *Newton polygon* of $f(X)$ is defined to be the lower boundary of the "convex hull" of the this set of points.

We now focus on how to extract information about roots of the polynomial from this polygon. The crucial things in which we are interested are as follows:

(i) the "slopes" of the line segments appearing in the polygon;

(ii) the "length" of each slope (by which we mean the length of the projection of the corresponding segment on x-axis);

(iii) the "breaks," that is, the values of i such that the point $(i,\ v_p(a_i))$ is a vertex of the polygon.

Notice that the sum of all the lengths are equal to the degree, and $(0,\ 0)$ and $(n,\ v_p(a_n))$ always are vertices. It is also clear that the slopes form an increasing sequence. We consider a polynomial $f(X) = 1 + a_1X + a_2X^2 + \cdots + a_nX^n$. Let us look at the first segment of the Newton polygon. If this segment has slope m, it connects the point $(0,\ 0)$ to some other point $(i,\ mi)$. First it means that there exists no point below the line $y = mx$; in other words, $v_p(a_j) \geq mj$ for every j. Second, the point $(i,\ mi)$ itself tells us that $v_p(a_i) = mi$. Third, the existence of a break tells us that the subsequent points are really above the line; in other words, $v_p(a_j) > mj$ if $j > i$. Hence we have

- $|a_j|(p^{-m})^j \leq 1$ for all j,
- $|a_i|(p^{-m})^i = 1$, and
- $|a_j|(p^{-m})^j < 1$ if $j > i$.

If we now let $c = p^m$, then we see that $||f(X)||_c = 1$ and i is the largest integer with $||f(X)||_c = |a_i|c^i = 1$. In other words, the fact that the first break is at $(i,\ mi)$ means that if we take $c = p^m$ then $||f(X)||_c = 1$ and i is the distinguished number that appears in Theorem (5.3). Thus we have

Lemma (6.1). *Let $f(X)$ be as above and assume that the Newton polygon of $f(X)$ has its first break at $(i,\ mi)$. Then there exist polynomials $g(X),\ h(X) \in \mathbb{C}_p[X]$ satisfying*

(i) *$f(X) = g(X)h(X)$,*

(ii) *$g(X)$ has the degree i and pure of slope m,*

(iii) *$h(X)$ has no zeros in the closed ball of radius p^m around* 0.

Notice that a polynomial $g(X)$ is said to be *pure of slope m* if its Newton polygon has only one slope m. Furthermore, by considering the slope of the first segment, we have information about zeros as follows:

Proposition (6.2). *Let $f(X) \in \mathbb{C}_p[X]$ and assume that the first break of the Newton polygon of $f(X)$ occurs at the point $(i,\ mi)$. Then $f(X)$ has no roots with absolute value less than p^m and has exactly i roots (counting multiplicities in $\mathbb{C}_p$) with absolute value p^m.*

Now let us move to the second segment. In other words, let us assume that there are breaks at $(i,\ mi)$ and at $(k,\ mi + m'(k - i))$. Then by an argument similar to the above, we get, for $c = p^{m'}$, that $||f(X)||_c = p^{(m'-m)i}$ and that k is the distinguish number in Theorem (5.3). In this case, we can go through a process completely analogous to what we did before to conclude $f(X)$ has exactly k roots in the closed disk of radius $p^{m'}$ around zero, i of which have absolute value p^m, and $k - i$ of which have absolute value $p^{m'}$. We can go through a similar argument at other breaks. In the end, we get all the roots, and we will know exactly what their absolute value.

Theorem (6.3). *Let*

$$f(X) = 1 + a_1X + a_2X^2 + \cdots + a_nX^n \in \mathbb{C}_p[X],$$

and let $m_1, \cdots, m_r$ be the slope of its Newton polygon (in increasing order). Let $i_1, \cdots, i_r$ be the corresponding length. Then, for each $1 \leq k \leq r$, $f(X)$ has exactly i_k roots of absolute value p^{m_k}.

Next we consider the Newton polygon of a power series. The definition is formally identical: given a power series of the form

$$f(X) = 1 + a_1X + a_2X^2 + \cdots a_nX^n + \cdots$$

we plot all of points

$$(i,\ v_p(a_i)) \quad \text{for } i = 0, 1, 2, \cdots,$$

ignoring, as before, any points where $a_i = 0$. The Newton polygon of $f(X)$ is obtained by the "rotating line" procedure: rotate the vertical line through $(0,\ 0)$ until it hits a point $(i,\ v_p(a_i))$, then rotate it about farthest such point it hits, and so on. In this case, however, things are more complicated than in the case of polynomials. Hence we must amend our rules for obtaining the Newton polygon.

Start with the vertical half-line which is the negative part of y-axis. Rotate that line counter-clockwise until one of the following happens:

(i) The line simultaneously hits infinitely many of the points we have plotted. In this case, stop, and the polygon is complete.

(ii) The line reaches a position where it contains only one of our point (the one currently serving as the center of rotation) but can be rotated no further without leaving behind some points. In this case, stop, and the polygon is complete.

(iii) The line hits a finite number of the points. In this case, break the line at the last point that was hit, and begin the whole procedure again. Notice that the segment beginning at the last point hit may find itself immediately in the situation of case (ii), so that there may be no further change.

To handle the case of a polynomial in a unified way, we would have to add one further stopping procedure: if the line reaches the vertical position, we stop. The Newton polygon of a polynomial will then end with an infinite vertical segment. Notice that there are only three ways for the procedure to end:

(i) the last segment contains an infinite number of points,

(ii) the last segment contains a finite number of points, but can be rotated no further.

(iii) there is an infinite sequence of segments of finite length.

The examples of these cases are found in [**?**, §6.4]. We notice here the connection between the slope of the final segment and the radius of convergence. We first have the following (cf. [**?**, p. 223]).

Lemma (6.4). *Let m be the supremum of the slopes appearing in the Newton polygon of a series $f(X) = 1 + a_1X + a_2X^2 + \cdots$. Then the radius of convergence is p^m (which we understand as $+\infty$ if $m = +\infty$).*

The next lemma gives the exact region of convergence (cf. [**?**, p. 224]):

Lemma (6.5). *Let m be the supremum of the slopes appearing in the Newton polygon of a series $f(X) = 1 + a_1X + a_2X^2 + \cdots$.*

(i) *If the polygon ends in an infinite segment of slope m which contains infinitely many of the points $(i,\ v_p(a_i))$, then the region of convergence is the open ball of radius p^m.*

(ii) *If the polygon contains an infinite number of segments of finite length, then the region of convergence is an open ball of radius p^m.*

We now want to go on to obtain power series versions of the results describing how the Newton polygon carries information about zeros of a power series. Since we deal with power series, we need some assumptions. In the case where there exists an infinite line of slope m, we give special assumptions on the definition of the length of that segment. If the Newton polygon of a series ends in an infinite portion of slope m we will say that the length of that portion is ℓ if ℓ is the distance between the x-coordinates of the first and the last of the points $(n,\ v_p(a_n))$ which are on line, provided that the series converges on the closed ball of radius p^m. Otherwise, we say that the length corresponding to slope m is zero. Under these assumptions, the arguments we obtained for a polynomials all works without change for power series. We finally get the following (cf. [**?**, p. 227]):

Theorem (6.6). *Let*

$$f(X) = 1 + a_1X + a_2X^2 + \cdots + a_nX^n + \cdots$$

be a power series. Let $m_1, \cdots, m_k$ be the first k slopes of the Newton polygon of $f(X)$, and assume that $f(X)$ converges on the closed ball of radius $c = p^{m_k}$. Let N be the x-coordinate of the k-th segment of the Newton polygon. Then there exist a polynomial of degree N and a power series $h(X)$ such that

(i) *$f(X) = g(X)h(X)$,*

(ii) *$||f(X) - g(X)||_c < 1$,*

(iii) *$h(X)$ converges on the closed ball of radius c,*

(iv) *$||h(X) - 1||_c < 1$, and*

(v) *the Newton polygon of* $g(X)$ *is equal to the portion of the Newton polygon of* $f(X)$ *contained in the region* $0 \le x \le N$.

Corollary (4.25). *Let*

$$f(X) = 1 + a_1X + a_2X^2 + \cdots + a_nX^n + \cdots$$

be a power series which converges on the closed ball of radius $c = p^m$. *Let* $m_1, \cdots, m_k$ *be the first* k *slopes of the Newton polygon of* $f(X)$ *which are not greater than* m, *and let* $i_1, \cdots, i_k$ *be their lengths. Then, for each* j, $f(X)$ *has* i_j *zeros with absolute value* p^{m_j}, *and there are no zeros in the closed ball of radius* p^m.

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Unicity theorems for meromorphic mappings with deficient divisors

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Introduction.

In this article, we give some unicity theorems for meromorphic mappings from the complex m-space $\mathbb{C}^m$ into compact complex manifolds with the same inverse image of divisors. In particular, we study unicity theorems for meromorphic mappings under the additional conditions on deficiencies. The defect relation for meromorphic mappings implies that the deficient divisors in the sense of Nevanlinna are very few. In fact, the set of these divisors is at most countable. It therefore seems that the existence of deficient divisors imposes a strong restriction on the behavior of meromorphic mappings. In the case where meromorphic functions on $\mathbb{C}$, unicity theorems under the conditions on Nevanlinna's deficiencies were already studied and some interesting results were obtained (cf. [**?**], [**?**] and [**?**]). In this article we deal with the uniqueness problem in the case where dominant meromorphic mappings into projective algebraic manifolds and meromorphic mappings into complex projective spaces with hyperplanes as divisors. Our main results are unicity theorems for meromorphic mappings with deficient divisors in the sense of Nevanlinna. These theorems show that the existence of deficient divisors affects the uniqueness problem of meromorphic mappings. For details, see [**?**] and [**?**].

§1. Preliminaries.

Let $z = (z_1, \cdots, z_m)$ be the natural coordinate system in $\mathbb{C}^m$, and set

$$\|z\|^2 = \sum_{\nu=1}^{m} z_\nu \overline{z}_\nu, \qquad B(r) = \{z \in \mathbb{C}^m;\ \|z\| < r\},$$

$$d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial), \qquad \alpha = dd^c\|z\|^2.$$

For a (1,1)-current φ of order zero on $\mathbb{C}^m$ we set

$$N(r, \varphi) = \int_1^r \langle \varphi \wedge \alpha^{m-1}, \chi_{B(t)} \rangle \frac{dt}{t^{2m-1}},$$

where $\chi_{B(r)}$ denotes the characteristic function of $B(r)$.

Let M be a compact complex manifold and $L \to M$ a line bundle over M. We denote by $|L|$ the complete linear system of L. Let $|\cdot|$ be a hermitian fiber metric in L, and let ω be its Chern form. Let $f : \mathbb{C}^m \to M$ be a meromorphic mapping. We set

$$T_f(r, L) = N(r, f^*\omega),$$

and call it the characteristic function of f with respect to L. In the case where $M = \mathbb{P}_n(\mathbb{C})$ and $L = H$ is the hyperplane bundle, we simply write $T_f(r)$ for $T_f(r, H)$. We now have Nevanlinna's inequality for meromorphic mappings as follows (cf. [**?**, Theorem 2.3]):

Theorem (1.1). *Let $L \to M$ be a line bundle over M and $f : \mathbb{C}^m \to M$ a meromorphic mapping. Then*

$$N(r,\, f^*D) \leq T_f(r,\, L) + O(1)$$

for $D \in |L|$ with $f(\mathbb{C}^m) \not\subseteq \operatorname{Supp} D$, where $O(1)$ stands for a bounded term as $r \to +\infty$.

Let f and D be as above. We define Nevanlinna's deficiency $\delta_f(D)$ by

$$\delta_f(D) = 1 - \limsup_{r\to+\infty} \frac{N(r,\, f^*D)}{T_f(r,\, L)}.$$

It is clear that $0 \leq \delta_f(D) \leq 1$ and $\delta_f(D) = 1$ if $f(\mathbb{C}^m) \cap \operatorname{Supp} D = \emptyset$. If $\delta_f(D) > 0$, then D is called *a deficient divisor in the sense of Nevanlinna.*

Let E be an effective divisor on $\mathbb{C}^m$ such that $E = \sum_j \nu_j E_j$ for distinct irreducible hypersurfaces E_j in $\mathbb{C}^m$ and for nonnegative integers ν_j, and let k be a positive integer. We set

$$N_k(r,\, E) = \sum_j \min\{k,\, \nu_j\}\, N(r,\, E_j).$$

We also define the support of E with order at most k by

$$\operatorname{Supp}_k E = \bigcup_{0<\nu_j\leq k} E_j.$$

§2. Unicity theorems for families of dominant meromorphic mappings.

In this section we give unicity theorems for some families of dominant meromorphic mappings of $\mathbb{C}^m$ into a projective algebraic manifold M. A meromorphic mapping $f : \mathbb{C}^m \to M$ is said to be *dominant* provided that $\dim\ M = \operatorname{rank} f$. In the proofs of the theorems in this section, we essentially use the the following second main theorem for dominant meromorphic mappings (cf. [**?**, Theorem 2] and [**?**, Theorem 3.2]):

Theorem (2.1). *Let $L \to M$ be a big line bundle and $D_1, \cdots, D_q$ divisors in $|L|$ such that $D_1 + \cdots + D_q$ has only simple normal crossings. Let $f : \mathbb{C}^m \to M$ be a dominant meromorphic mapping. Then*

$$q\,T_f(r,\, L) + T_f(r,\, K_M) \leq \sum_{j=1}^{q} N_1(r,\, f^*D_j) + S_f(r),$$

where $S_f(r) = O(\log T_f(r,\, L)) + o(\log r)$ except on a Borel subset $E \subseteq [1, +\infty)$ with finite measure.

Let L and $D_1, \cdots, D_q$ be as in Theorem (2.1). Let $k_1, \cdots, k_q$ be positive integers. Set $k_0 = \max_{1\leq j\leq q} k_j$. Assume that there exists a dominant meromorphic mapping $f_0 : \mathbb{C}^m \to M$. We notice that K_M is not big in our case. Set $E_j = \operatorname{Supp}_{k_j} f_0^* D_j$ for all $1 \leq j \leq q$ and assume that $\dim E_i \cap E_j \leq m - 2$ for any $i \neq j$. Let

$$\mathcal{F} = \mathcal{F}(f_0; \{k_j\}; (\mathbb{C}^m, \{E_j\}), (M, \{D_j\}))$$

be the set of all *dominant* meromorphic mappings $f : \mathbb{C}^m \to M$ such that

$$\operatorname{Supp}_{k_j} f^*D_j = E_j \qquad \text{and} \qquad f = f_0 \text{ on } E_j$$

for all $1 \leq j \leq q$. Let $\mathcal{F}_0$ be the subfamily of $\mathcal{F}$ defined by

$$\mathcal{F}_0 = \{f \in \mathcal{F};\ \delta_{f_0}(D_j) \leq \delta_f(D_j) \text{ for all } 1 \leq j \leq q\}.$$

Let $\mathbb{P}_n(\mathbb{C})$ be the n-dimensional complex projective space and $\Phi : M \to \mathbb{P}_n(\mathbb{C})$ a nonconstant meromorphic mapping. In this paper, we always assume that $\operatorname{rank} \Phi = \dim M$. Set

$$G_0 = M - (\{w \in M - I(\Phi);\ \operatorname{rank} d\Phi(w) < \dim\ M\} \cup I(\Phi)),$$

where $I(\Phi)$ is the locus of indeterminacy of Φ.

Definition (2.2). A set $\{D_j\}_{j=1}^q$ of divisors is said to be *generic with respect to f_0 and Φ* provided that

$$f_0(\mathbb{C}^m - I(f_0)) \cap \operatorname{Supp} D_j \cap G_0 \neq \emptyset$$

for at least one $1 \leq j \leq q$, where $I(f_0)$ denotes the locus of indeterminacy of f_0.

We denote by H the hyperplane bundle over $\mathbb{P}_n(\mathbb{C})$. We define $F_1 \in \mathrm{Pic}(M) \otimes \mathbb{Q}$ by

$$F_1 = \left(\sum_{j=1}^{q} \frac{k_j}{k_j + 1} \right) L \otimes \left(-\frac{2k_0}{k_0 + 1} \right) \varPhi^* H.$$

If F_1 is sufficiently big, we can conclude $\mathcal{F} = \{f_0\}$ as follows.

Theorem (2.3). *Suppose that the set $\{D_j\}_{j=1}^q$ is generic with respect to f_0 and $\varPhi$. If $F_1 \otimes K_M$ is big, then the family $\mathcal{F}$ contains just one mapping f_0.*

Remarks (2.4). (1) In the definition of the family $\mathcal{F}$, we assume that $f = f_0$ on all E_j for every $f \in \mathcal{F}$. However this assumption cannot be simply dropped (cf. [**?**, Remarks (2.8)] and [**?**, p. 357]).

(2) Let $e_0 = \sharp \varPhi^{-1}(\varPhi(w))$ for $w \in G_0$. In the case where $\{D_j\}_{j=1}^q$ is not generic with respect to f_0 and $\varPhi$, we can conclude $\sharp \mathcal{F} \leq e_0$ (cf. [**?**, Remarks (2.8)]).

In the case of $M = \mathbb{P}_1(\mathbb{C})$, we have the unicity theorem due to Gopalakrishna and Bhoosnurmats ([**?**, Theorem 1]). In the case where $M = \mathbb{P}_n(\mathbb{C})$ and $q = 1$, we have the following unicity theorem (cf. [**?**, Theorem 4.1]):

Theorem (2.5). *Let D be a hypersurface in $\mathbb{P}_n(\mathbb{C})$ with simple normal crossings. Suppose that the degree d of D is greater than $n + 3 + (n+1)/k$. Then the family $\mathcal{F}(f_0; \{k\}; (\mathbb{C}^m, \{E\}), (\mathbb{P}_n(\mathbb{C}), \{D\}))$ contains just one mapping f_0.*

Next we consider the case where $[F_1^{-1} \otimes K_M^{-1}/L] = 0$.

Theorem (2.6). *Suppose that the set $\{D_j\}_{j=1}^q$ is generic with respect to f_0 and $\varPhi$, and that*

$$\left[\frac{F_1^{-1} \otimes K_M^{-1}}{L} \right] = 0.$$

If $\delta_{f_0}(D_j) > 0$ for at least one $1 \leq j \leq q$, then the family $\mathcal{F}$ contains just one mapping f_0.

Remark (2.7). In the case where $[F_1^{-1} \otimes K_M^{-1}/L]$ is positive, we can not conclude $\mathcal{F} = \{f_0\}$ under the condition on the existence of deficient divisors. Indeed, let $f_0 : \mathbb{C} \to \mathbb{P}_1(\mathbb{C})$ be a meromorphic function defined by $f_0(z) = \exp z$. Set $D_1 = 0$, $D_2 = \infty$, $D_3 = 1$ and $D_4 = -1$. Let all $k_j = 1$ and put $E_j = \mathrm{Supp}_1 f_0^* D_j$ for $1 \leq j \leq 4$. Let $\varPhi : \mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_1(\mathbb{C})$ be the identity mapping. In this case $L = H$ and $[F_1^{-1} \otimes K_{\mathbb{P}_1(\mathbb{C})}^{-1}/L] = 1$. Now we see $\delta_{f_0}(D_1) = \delta_{f_0}(D_2) = 1$ but $\sharp \mathcal{F} \geq 2$. In fact, $f(z) = \exp(-z)$ is contained in $\mathcal{F}$ and $f_0 \neq f$.

In the case where $M = \mathbb{P}_1(\mathbb{C})$ and $\varPhi : \mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_1(\mathbb{C})$ is the identity mapping, we have H. Ueda's unicity theorem ([**?**, Theorem 1]) by Theorem (2.6) and Remark (2.10) below. In the case where $M = \mathbb{P}_n(\mathbb{C})$ and $q = 1$, we have the following:

Corollary (2.8). *Let D be a hypersurface in $\mathbb{P}_n(\mathbb{C})$ of degree $n + 4$ with simple normal crossings. If $\delta_{f_0}(D) > 0$, then the family $\mathcal{F}(f_0; \{n+1\}; (\mathbb{C}^m, \{E\}), (\mathbb{P}_n(\mathbb{C}), \{D\}))$ contains just one mapping f_0.*

For the family $\mathcal{F}_0$, we have the following:

Theorem (2.9). *Suppose that the set $\{D_j\}_{j=1}^q$ is generic with respect to f_0 and $\varPhi$, and that*

$$\left[\frac{F_1^{-1} \otimes K_M^{-1}}{L} \right] < \frac{1}{k_0 + 1} \sum_{j=1}^{q} \delta_{f_0}(D_j).$$

Then the family $\mathcal{F}_0$ contains just one mapping f_0.

Remark (2.10). We define the subfamily $\mathcal{F}_1$ of $\mathcal{F}$ by

$$\mathcal{F} = \left\{ f \in \mathcal{F};\ \left[\frac{F_1^{-1} \otimes K_M^{-1}}{L} \right] < \frac{1}{k_1 + 1} \sum_{j=1}^{q} \min \{\delta_f(D_j),\, \delta_{f_0}(D_j)\} \right\}.$$

Then we can always have $\sharp \mathcal{F}_1 = 1$ if the generic condition on $\{D_j\}_{j=1}^q$ is satisfied. Notice that $\mathcal{F}_0 \subseteq \mathcal{F}_1$ if the assumption of Theorem (2.9) is satisfied.

§3. Unicity theorems for meromorphic mappings into $\mathbb{P}_n(\mathbb{C})$.

In this section we consider the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}_n(\mathbb{C})$ and give some unicity theorems. We first notice that $\mathrm{Pic}(\mathbb{P}_n(\mathbb{C})) \cong \mathbb{Z}$ and the hyperplane bundle $H \to \mathbb{P}_n(\mathbb{C})$ is the generator of $\mathrm{Pic}(\mathbb{P}_n(\mathbb{C}))$ with $c_1(H) = 1$. We say that $F \in \mathrm{Pic}(\mathbb{P}_n(\mathbb{C})) \otimes \mathbb{Q}$ is *positive* if a line bundle $\nu F \in \mathrm{Pic}(\mathbb{P}_n(\mathbb{C}))$ is positive for some positive integer ν.

In the proofs of the theorems in this section, the following second main theorem plays an essential role (cf. [**?**, p. 501]):

Theorem (3.1). *Let $f : \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ be a meromorphic mapping with rank at least μ. Let l be the dimension of the smallest linear subspace of $\mathbb{P}_n(\mathbb{C})$ containing $f(\mathbb{C}^m)$. Let $H_1, \cdots, H_q$ be hyperplanes located in general position in $\mathbb{P}_n(\mathbb{C})$. Then*

$$(q - 2n + l - 1)\, T_f(r) \leq \sum_{j=1}^{q} N_{l-\mu+1}(r,\ f^* H_j) + S_f(r),$$

where $S_f(r) = O(\log\ T_f(r)) + o(\log\ r)$ except on a Borel subset $E \subseteq [1,\ +\infty)$ with finite measure.

Let $H_1, \cdots, H_q$ be as above, and let $k_1, \cdots, k_q$ be as in §2. Assume that there exists a nonconstant meromorphic mapping $f_0 : \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ such that rank $f_0 \geq \mu$ and the dimension of the linear span of $f_0(\mathbb{C}^m)$ is l. Set $E_j = \mathrm{Supp}_{k_j} f_0^* D_j$ for $1 \leq j \leq q$ and assume that dim $E_i \cap E_j \leq m - 2$ for any $i \neq j$. We denote by

$$\mathcal{G} = \mathcal{G}(f_0; \mu; l; \{k_j\}; (\mathbb{C}^m, \{E_j\}), (\mathbb{P}_n(\mathbb{C}), \{H_j\}))$$

the set of all meromorphic mappings $f : \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ such that satisfy the following conditions:

(1) the linear span of $f(\mathbf{C}^m)$ is of dimension l and rank $f \geq \mu$.

(2) $\mathrm{Supp}_{k_j} f^* H_j = E_j$ $(1 \leq j \leq q)$ and $f = g$ on all E_j.

We also define the subfamily $\mathcal{G}_0$ of $\mathcal{G}$ by

$$\mathcal{G}_0 = \{f \in \mathcal{G};\ \delta_{f_0}(H_j) \leq \delta_f(H_j) \ \text{ for all } 1 \leq j \leq q\}.$$

Set $p = l - \mu + 1$ and let

$$C(\mu; l; \{k_j\}) = q - n + l - \sum_{j=1}^{q} \frac{p}{k_j + 1} - \frac{2pk_0}{k_0 + 1}.$$

We define $F \in \mathrm{Pic}(\mathbb{P}_n(\mathbb{C})) \otimes \mathbb{Q}$ by $F = C(\mu; l; \{k_j\})H$. Then $F \otimes K_{\mathbb{P}_n(\mathbb{C})}$ is positive if and only if $C(\mu; l; \{k_j\}) > n + 1$. If F is sufficiently positive, we can conclude $\sharp\,\mathcal{G} = 1$. Namely, by making use of Theorem (3.1), we have the following unicity theorem for the family $\mathcal{G}$:

Theorem (3.2). *Suppose that $n + 1 < C(\mu; l; \{k_j\})$. Then the family $\mathcal{G}$ contains just one mapping f_0.*

The following are immediate consequences of Theorem (3.2):

(i) Let $\mu = n$. If $q > 2n + 4$ and $k_j = 1$ for all j, then $\mathcal{G}$ contains just one mapping f_0 (cf. [**?**, p. 128]). If $q \geq n + 4$ and $k_j \geq n + 2$ for all j, then $\mathcal{G}$ contains just one mapping f_0 (cf. [**?**, p. 355]).

(ii) Let $\mu = n$. If $q > n + 2p + 1$ and all $k_j \geq p(n + 2p + 1) - 1$, then $\mathcal{G}$ contains just one mapping f_0 (cf. [**?**, p. 153]).

In the case where $F \otimes K_{\mathbb{P}_n(\mathbb{C})}$ is trivial, we have the following:

Theorem (3.3). *Suppose that $C(\mu; l; \{k_j\}) = n + 1$. If $\delta_{f_0}(H_j) > 0$ for at least one $1 \leq j \leq q$, then the family $\mathcal{G}$ contains just one mapping f_0.*

By Theorem (3.3), we can conclude certain consequences. For instance, we have the following:

(iii) Let $\mu = n$ and $q = 2n + 4$. Suppose that $k_j = 1$ for all j. If $\delta_{f_0}(H_j) > 0$ for some j, then $\mathcal{G}$ contains just one mapping f_0.

For the family $\mathcal{G}_0$, we have the following unicity theorem:

Theorem (3.4). *Suppose that*

$$n + 1 - C(\mu; l; \{k_j\}) < \frac{p}{k_0 + 1} \sum_{j=1}^{q} \delta_{f_0}(H_j).$$

Then the family $\mathcal{G}_0$ contains just one mapping f_0.

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Topics in p-Adic Function Theory

William Cherry

1. Picard Theorems

I would like to begin by recalling the Fundamental Theorem of Algebra.

Theorem 1.1. (Fundamental Theorem of Algebra) *A non-constant polynomial of one complex variable takes on every complex value. Moreover, if the polynomial is of degree d, then every complex value is taken on d times, counting multiplicity.*

Because entire functions have power series expansions, they are sort of like polynomials of infinite degree. Picard's well-known theorem is a complex analytic analog of the Fundamental Theorem of Algebra.

Theorem 1.2. (Picard's (Little) Theorem) *A non-constant entire function takes on all but at most one complex value. Moreover, a transcendental entire function must take on all but at most one complex value infinitely often.*

The function e^z shows that a complex entire function can indeed omit one value.

Lately, it has become fashionable to prove p-adic versions of value distribution theorems, of which Picard's Theorem is an example, though not a recent one. More recent examples can be found in the works listed in the references section. Recall that the p-adic absolute value $|\ |_p$ on the rational number field $\mathbf{Q}$ is defined as follows. If $x \in \mathbf{Q}$ is written $p^k a/b$, where p is a prime, k is an integer, and a and b are integers relatively prime to p, then $|x|_p = p^{-k}$. Completing $\mathbf{Q}$ with respect to this absolute value results in the field of p-adic numbers, denoted $\mathbf{Q}_p$. Taking the algebraic closure of $\mathbf{Q}_p$, extending $|\ |_p$ to it, and then completing once more results in a complete algebraically closed field, denoted $\mathbf{C}_p$, and often referred to as the p-adic complex numbers.

Recall that the absolute value $|\ |_p$ satisfies a very strong form of the triangle inequality, namely $|x+y|_p \le \max\{|x|_p, |y|_p\}$. This is referred to as a non-Archimedean triangle inequality, and this non-Archimedean triangle inequality is what accounts for most of the differences between function theory on $\mathbf{C}_p$ and on $\mathbf{C}$.

Recall that an infinite series $\sum a_n$ converges under a non-Archimedean norm if and only if $\lim\limits_{n\to\infty} a_n = 0$. By an entire function on $\mathbf{C}_p$, one means a formal power series $\sum\limits_{n=0}^{\infty} a_n z^n$, where a_n are elements of $\mathbf{C}_p$, and $\lim\limits_{n\to\infty} |a_n|_p r^n = 0$, for every $r > 0$, so that plugging in any element of $\mathbf{C}_p$ for z results in an absolutely convergent series.

Most of what I will discuss here is true over an arbitrary algebraically closed field complete with respect to a non-Archimedean absolute value, but for simplicity's sake, I will stick with the concrete case $\mathbf{C}_p$ here.

If one tries to prove Picard's Theorem for p-adic entire functions, what one gets is the following theorem.

Theorem 1.3. **(p-Adic Case)** *A non-constant p-adic entire function must take on every value in $\mathbf{C}_p$. Moreover, a transcendental p-adic entire function must take on every value in $\mathbf{C}_p$ infinitely often.*

Proof. Let $f(z) = \sum a_n z^n$ be a p-adic entire function, so $\lim_{n\to\infty} |a_n|_p r^n = 0$, for all $r > 0$. Denote by $|f|_r = \sup |a_n|_p r^n$. The graph of

$$\log r \mapsto \log |f|_r = \sup_{n\geq 0}\{\log |a_n|_p + n \log r\}$$

is piecewise linear and closely related to what's known as the Newton polygon. In particular, ther zeros of f occur at the "corners" of the graph of $\log r \mapsto \log |f|_r$ (*c.f.,* [**?**] and [**?**]).

For r close to zero, $|f|_r = |a_0|_p$, provided $a_0 \neq 0$. Moreover, it is clear that if f is not constant, then for all r sufficiently large, $|f|_r \neq |a_0|_p$. Hence, the graph of $\log r \mapsto \log |f|_r$ has a corner, and hence f has a zero.

If f is transcendental, then f has infinitely many non-zero Taylor coefficients, and thus for every n, there exists r_n such that for all $r \geq r_n$, we have $|f|_r > |a_n|_p r^n$. Hence, $\log r \mapsto \log |f|_r$ must have infinitely many corners, and so f has infinitely many zeros. □

Note that Theorem **??** is an even closer analogy to the Fundamental Theorem of Algebra than Picard's Theorem was, since p-adic entire functions, like polynomials, cannot omit any values. Thus, in this respect, the function theory of p-adic entire functions is more closely related to the function theory of polynomials than it is to the function theory of complex holomorphic functions. That will be the theme of this survey.

2. Algebraic Curves

My second illustration that p-adic function theory is more like that of polynomials comes from considering Riemann surfaces. Let X be a projective algebraic curve of genus g. Then, the three analogous theorems we have are:

Theorem 2.1. **(Polynomial Case)** *If $f\colon \mathbf{C} \to X$ is a non-constant polynomial mapping, then $g = 0$.*

Theorem 2.2. **(Complex Case)** *If $f\colon \mathbf{C} \to X$ is a non-constant holomorphic mapping, then $g \leq 1$.*

Theorem 2.3. **(p-Adic Case)** *If $f\colon \mathbf{C}_p \to X$ is a non-constant p-adic analytic mapping, then $g = 0$.*

The polynomial case follows from the Riemann-Hurwitz formula, which says that the genus of the image curve cannot be greater than the genus of the domain.

The complex case was again proved by Picard. Riemann surfaces of genus ≥ 2 have holomorphic universal covering maps from the unit disc, and thus any holomorphic map form $\mathbf{C}$ to a Riemann surface of genus ≥ 2 lifts to a holomorphic map to the unit disc, which must then be constant by Liouville's Theorem.

The p-Adic analog of this theorem was proven only recently, by V. Berkovich [**?**].

One of the major difficulties in p-adic function theory is the fact that the natural p-adic topology is totally disconnected, and therefore analytic continuation in these circumstances is a delicate task. Moreover, geometric techniques that are commonplace in complex analysis cannot be applied in the p-adic case. In order to prove his p-adic analog of Picard's Theorem, Berkovich developed a theory of p-adic analytic spaces that enlarges the natural p-adic spaces so that they become nice topological spaces, and geometric techniques, such as universal covering spaces, can be used to prove theorems.

3. Berkovich Theory

Berkovich's theory is somewhat deep, and I do not have ther required space to go into it in much detail here. However, the reader may find the following brief description of his theory helpful. The interested reader is encouraged to look at: [**?**], [**?**], and [**?**]. The last reference covers the more traditional theory of rigid analytic spaces.

Although one can associate a Berkovich space to any p-adic analytic variety, we will concentrate here on the special case of the unit ball in $\mathbf{C}_p$, which is the local model for smooth p-adic analytic spaces, at least in dimension one.

Consider the closed unit ball $\mathbf{B} = \{z \in \mathbf{C}_p : |z|_p \le 1\}$. The p-adic analytic functions on $\mathbf{B}$ are of the form $\sum a_n z^n$, with $\lim_{n\to\infty} |a_n|_p = 0$. These functions form a Banach algebra $\mathcal{A}$ under the norm $|f|_{0,1} = \sup_n |a_n|_p$.

The Berkovich space associated to $\mathbf{B}$ consists of all bounded multiplicative semi-norms on $\mathcal{A}$. This space is provided with the weakest topology such that all maps of the form $|\ | \mapsto |f|$, $f \in \mathcal{A}$ are continuous maps to the real numbers with their usual topology. Here $|\ |$ denotes one of the bounded multiplicative semi-norms in the Berkovich space.

Berkovich spaces have many nice topological properties, such as local compactness and local arc-connectedness. They also have universal covering spaces, which are again Berkovich spaces.

For $f \in \mathcal{A}$, $z_0 \in \mathbf{B}$, and $0 \le r \le 1$, define $|f|_{z_0,r}$ by $|f|_{z_0,r} = \sup_n |c_n|_p r^n$, where $f = \sum c_n (z - z_0)^n$, or in other words, the c_n are the coefficients of the Taylor expansion of f about z_0. Note that if $r = 0$, then $|f|_{z_0,0} = |f(z_0)|_p$, and note that by the non-Archimedean triangle inequality, if $|z_0 - w_0|_p \le r$, then $|\ |_{z_0,r} = |\ |_{w_0,r}$. There are in fact more bounded multiplicative semi-norms on $\mathbf{B}$ than these, but these are the main ones to think about.

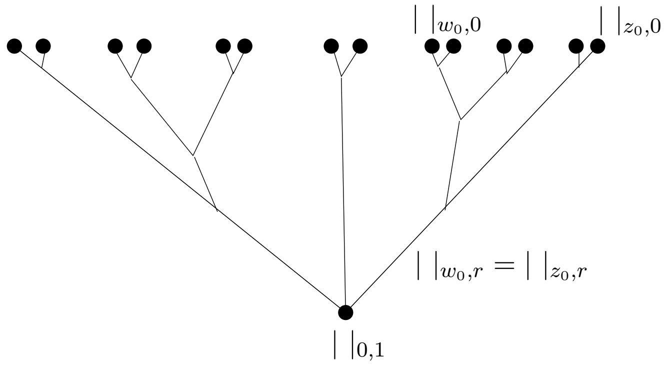


FIGURE 1.

Figure **??** gives a sort of intuitive "tree-like" representation for the Berkovich space associated to $\mathbf{B}$. The dots at the top correspond to the totally disconnected points in $\mathbf{B}$. Of course there are infinitely many of these, and there are points arbitrarily close together, much like a Cantor set. The lines represent the connected continuum of additional multiplicative semi-norms connecting the Berkovich space. There are of course infinitely many places where lines join together, and the junctures are by no means discrete. Finally, the point at the bottom corresponds to the one semi-norm $|\ |_{z_0,1}$ which is the same for all points z_0 in $\mathbf{B}$.

We say that two points z_0 and w_0 in $\mathbf{B}$ are in the same residue class if $|z_0 - w_0|_p < 1$. This leads to a concept called "reduction," whereby the space is "reduced" to the space of residue classes. The reduction of $\mathbf{B}$ can be naturally identified with $\mathbf{A}^1_{\mathbf{F}_p^{\rm alg}}$, the affine line over the algebraic closure of the field of p elements. This process of reduction extends to the Berkovich space associated to $\mathbf{B}$, and there is a reduction mapping π from the Berkovich space $\mathbf{B}$ to $\mathbf{A}^1_{\mathbf{F}_p^{\rm alg}}$. The reduction mapping π has what I would call an anti-continuity property, in that π^{-1} of a Zariski open sets in $\mathbf{A}^1_{\mathbf{F}_p^{\rm alg}}$ will be closed in the Berkovich topology and π^{-1} of a Zariski closed set will be open in the Berkovich topology.

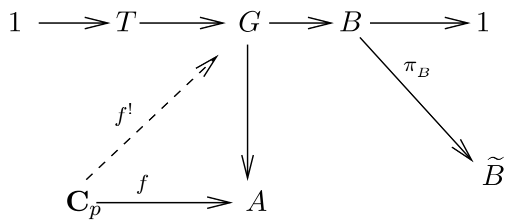
In Figure **??**, two points in the Berkovich tree are in different residue classes if their branches do not join except at the one point $|\ |_{0,1}$, which is kind of like a "generic" point in algebraic geometry, and is in fact the inverse image of the generic point in $\mathbf{A}^1_{\mathbf{F}_p^{\rm alg}}$ under the reduction map. Thus, three residue classes are shown in Figure **??**.

4. Abelian Varieties

In my Ph.D. thesis [**?**], I extended Berkovich's Theorem to Abelian varieties. See also: [**?**] and [**?**].

Theorem 4.1. (Cherry) *If $f \colon \mathbf{C_p} \to A$ is a p-adic analytic map to an Abelian variety, then f must be constant.*

Proof sketch.



T is a product of multiplicative groups (i.e. a multiplicative torus).
G is the universal cover of A in the sense of Berkovich, and a semi-Abelian variety.
B is an Abelian variety with good reduction, meaning it has a reduction mapping π_B to an Abelian variety $\widetilde{B}$ over $\mathbf{F}_p^{\mathrm{alg}}$.

FIGURE 2.

Step 1. First, we use Berkovich theory to lift f to a map $f^!\colon \mathbf{C}_p \to G$ to the universal covering of A.

Step 2. Next we use p-adic uniformization ([?], [?], [?]) to identify G as a semi-Abelian variety, as in Figure **??**.

Step 3. Then, we use reduction techniques. We get a map

$$\mathbf{C}_p \to G \to B \to \widetilde{B}.$$

This map must be constant because if it were not we would induce a non-constant rational map from the projective line over $\mathbf{F}_p^{\mathrm{alg}}$ to the Abelian variety $\widetilde{B}$. Thus, the image in B lies above a single smooth point in $\widetilde{B}$. The inverse image of a smooth point in $\widetilde{B}$ is isomorphic to an open ball in $\mathbf{C}_p^n$, where n is the dimension of B. Thus, the map to B is also constant, by the p-adic version of Liouville's Theorem, for example.

Step 4. Thus, we only need consider mappings from $\mathbf{C}_p$ to T. But,

$$T \cong \mathbf{C}_p \setminus \{0\} \times \cdots \times \mathbf{C}_p \setminus \{0\}.$$

The projection onto each factor is constant by the p-Adic version of Picard's Little Theorem. □

Because p-adic analytic maps to Abelian varieties must be constant, the following conjecture seems plausible.

Conjecture 4.1. *Let X be a smooth projective variety. If there exists a non-constant p-adic analytic map from $\mathbf{C}_p$ to X, then there exists a non-constant rational mapping from $\mathbf{P}^1$ to X.*

5. VALUE SHARING

One of the more striking consequences of Nevanlinna theory is Nevanlinna's theorem that if two non-constant meromorphic functions f and g share five values, then f must equal g, [?]. The polynomial version of this was taken up by Adams and Straus in [?].

Theorem 5.1. (Adams and Straus) *If f and g are two non-constant polynomials over an algebraically closed field of characteristic zero such that $f^{-1}(0) = g^{-1}(0)$ and $f^{-1}(1) = g^{-1}(1)$, then $f \equiv g$.*

Proof. Assume $\deg f \geq \deg g$ and consider $[f'(f-g)]/[f(f-1)]$. This is a polynomial because if $f(z) = 0$ or 1, then $f(z) = g(z)$ by assumption, and hence the zeros in the denominator are canceled by the zeros in the numerator, and the f' in the numerator takes care of multiple zeros. On the other hand, the degree of the numerator is strictly less than the degree of the denominator, so the numerator must be identically zero. In other words f is constant, or f is identically equal to g. □

Theorem 5.2. (Adams and Straus) *If f and g are non-constant p-adic (characteristic zero) analytic functions such that $f^{-1}(0) = g^{-1}(0)$, and $f^{-1}(1) = g^{-1}(1)$, then $f \equiv g$.*

Proof. We may assume without loss of generality that there exist $r_j \to \infty$ such that $|f|_{r_j} \geq |g|_{r_j}$. Let $h = [f'(f-g)]/[f(f-1)]$. Then, h is entire since, as in the polynomial case, zeros in the denominator are always matched by zeros in the numerator. On the other hand, by the non-Archimedean triangle inequality, we have for r_j sufficiently large that

$$|h|_{r_j} = \left|\frac{f'}{f}\right|_{r_j} \cdot \frac{|f-g|_{r_j}}{|f-1|_{r_j}} \leq \left|\frac{f'}{f}\right|_{r_j} \cdot \frac{|f|_{r_j}}{|f|_{r_j}} = \left|\frac{f'}{f}\right|_{r_j}.$$

Now, I claim $|f'/f|_r \leq r^{-1}$, and therefore $|h|_{r_j} \to 0$ as $r_j \to \infty$. Hence, $h \equiv 0$, and again, either f is constant of $f \equiv g$. □

The claim that $|f'/f|_r \leq 1/r$ is the p-adic form of the Logarithmic Derivative Lemma, and note this is much stronger than what is true in the complex case.

Theorem 5.3. (p-Adic Logarithmic Derivative Lemma) *If f is a p-Adic analytic function, then $|f'/f|_r \leq 1/r$.*

Proof. Write $f = \sum a_n z^n$. Then, since $|n|_p \leq 1$, we have

$$|f'|_r = \sup_{n\geq 1}\{|na_n|_p r^{n-1}\} = \frac{1}{r}\sup_{n\geq 1}\{|na_n|_p r^n\} \leq \frac{1}{r}\sup_{n\geq 0}\{|a_n|_p r^n\} = \frac{1}{r}|f|_r \quad \square$$

Notice the similarity in both the proof and the statement of both of Adams and Straus's theorems.

An active topic of current research has to do with so called "unique range sets." Rather than considering functions which share distinct values, one considers finite sets and functions f and g such that $f^{-1}(S) = g^{-1}(S)$. Here, Boutabaa, Escassut, and Haddad [**?**] gave a nice characterization for unique range sets of polynomials, in the counting multiplicity case.

Theorem 5.4. (Boutabaa, Escassut, and Haddad) *If f and g are polynomials over an algebraically closed field F of characteristic zero, and if S is a finite subset of F such that $f^{-1}(S) = g^{-1}(S)$, counting multiplicity, then either $f \equiv g$ or there exist constants A and B, $A \neq 0$, such that $g = Af + B$ and $S = AS + B$.*

Proof. Let $S = \{s_1, \ldots, s_n\}$ and let

$$P(X) = (X - s_1)\cdots(X - s_n).$$

Then, $P(f)$ and $P(g)$ are polynomials with the same zeros, counting multiplicity by the assumption $f^{-1}(S) = g^{-1}(S)$. Thus, $P(f)/P(g)$ is some non-zero constant C, and if we set $F(X,Y) = P(X) - CP(Y)$, we have $F(f,g) = 0$. Thus, $z \mapsto (f(z), g(z))$ is a rational component of the possibly reducible algebraic curve $F(X,Y) = 0$. Because $F(X,Y) = 0$ has n distinct smooth points at infinity in $\mathbf{P}^2$ (characteristic zero!) and because $(f(z), g(z))$ has only one point at infinity, $(f(z), g(z))$ must in fact be a linear component of $F(X,Y) = 0$. □

Boutabaa, Escassut, and Haddad also made a preliminary analysis of the p-adic entire analog of their theorem, and solved the case when the cardinality of S equals three completely. C.-C. Yang and I, [**?**], combined Berkovich's Picard theorem with their argument to complete the p-adic entire case.

Theorem 5.5. (Cherry and Yang) *If f and g are p-adic entire functions and S is a finite subset of $\mathbf{C}_p$ such that $f^{-1}(S) = g^{-1}(S)$, counting multiplicity, then there exist constants A and B, with $A \neq 0$, such that $g = Af + B$, and $S = AS + B$.*

Proof. Again, set

$$P(X) = (X - s_1)\cdots(X - s_n).$$

Again, $P(f)/P(g)$ is a constant $C \neq 0$. Again, set $F(X,Y) = F(X) - CF(Y)$. By Berkovich's p-Adic Picard Theorem, $(f(z), g(z))$ is contained in a rational component of $F(X,Y) = 0$. Thus, there exist rational functions u and v, and a p-adic entire function h, such that $f = u(h)$ and $g = v(h)$. It is then easy to see that u and v must in fact be polynomials, and we are then back to the polynomial case, thinking of h as a variable. □

6. Concluding Remarks

In many respect, it appears that algebraic geometry, rather than complex Nevanlinna theory, is the appropriate model for p-adic value distribution theory. At least, that is what I hope this survey has conveyed to the reader. This leads me to a general principle.

Principle 1. *Appropriately stated theorems about the value distribution of polynomials should also be true for p-adic entire functions. Similarly, theorems for rational functions should also be true for p-adic meromorphic functions.*

Conjecture **??** is a special case of this principle. With some luck, solving a p-adic problem based on the above principle might help us better understand complex Nevanlinna theory. For example, it would be reasonable to make the following conjecture.

Conjecture 6.1. *If $f\colon \mathbf{C}_p \to X$ is a p-adic analytic map to a $K3$ surface X, the the image of f must be contained in a rational curve.*

This conjecture can be thought of as a special case of a p-adic version of the Green-Griffiths conjecture [**?**] that says a holomorphic curve in a smooth projective variety of general type must be algebraically degenerate. One might hope to attack Conjecture **??** since much is known about $K3$ surfaces and they have a close connection to Abelian varieties. It might also be that finding a proof for Conjecture **??** would shed some light on an attack of the general Green-Griffiths conjecture over the complex numbers.

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On uniqueness of meromorphic functions sharing finite sets

Hirotaka Fujimoto

1. Introduction

In this talk, we mean by a meromorphic function a meromorphic function on the complex plane $\mathbb{C}$ and by a divisor a map $\nu : \mathbb{C} \to \mathbb{Z}$ whose support $\overline{\{z; \nu(z) \neq 0\}}$ is discrete. For a divisor ν, we set $\bar{\nu}(z) := \min(\nu(z), 1)$.

Let f be a nonconstant meromorphic function. For a point $a \in \mathbb{C}$, we define the divisor $\nu_f^a : \mathbb{C} \to \mathbb{Z}$ by

$$\nu_f^a(z) := \begin{cases} 0 & \text{if } f(z) \neq a \\ m & \text{if } f - a \text{ has a zero of order } m \text{ at } z, \end{cases}$$

and set $\nu_f^\infty := \nu_{1/f}^0$, $\nu_f := \nu_f^0 - \nu_f^\infty$. With a discrete set S in $\mathbb{C}$, we associate the divisors $\nu_f^S := \sum_{a \in S} \nu_f^a$ and $\bar{\nu}_f^S := \sum_{a \in S} \bar{\nu}_f^a$.

We call a set S a uniqueness range set for meromorphic function if for any nonconstant meromorphic functions f and g the condition $\nu_f^S = \nu_g^S$ implies $f = g$, and a uniqueness range set for entire functions if for any nonconstant entire functions f and g the condition $\nu_f^S = \nu_g^S$ implies $f = g$. We call also a set S a uniqueness set for meromorphic (or entire) function ignoring multiplicities if it has the analogous property for which the condition $\nu_f^S = \nu_g^S$ is replaced by $\bar{\nu}_f^S = \bar{\nu}_g^S$.

In 1982, F. Gross and C. C. Yang proved that the set $S := \{w \in \mathbb{C}; w + e^w = 0\}$ is a uniqueness range set for entire functions([4]). Afterwards, many efforts were made to seek uniqueness range sets which are as small as possible by several authors P. Li, H.X. Yi, B. Shiffman and so on. In fact, in their papers [5], [9], [7] and so on, they showed that the set $\{w; w^q + aw^{q-r} + b = 0\}$ gives small unique range sets for meromorphic functions or entire functions under suitable conditions on constants a, b and positive integers q, r. Recently, Frank and Reinders gave a uniqueness range set for meromorphic functions with 11 elements ([2]), which are given as the set of all zeros of the polynomial

$$P^{FR}(w) = \frac{(q-1)(q-2)}{2} w^q - q(q-2) w^{q-1} + \frac{q(q-1)}{2} w^{q-2} - c$$

for $q = 11$ and a constant $c \neq 0, 1$.

The purpose of this talk is to give some sufficient conditions for a finite subset S of $\mathbb{C}$ to be a uniqueness range set for meromorphic functions or entire functions by applying the arguments used in [2] to more general settings.

2. Some conditions for uniqueness polynomials

For a given finite set S, we consider the monic polynomial $P(w)$ which has simple zeros exactly on S and study the polynomial $P(w)$ instead of the set S. We say that a nonconstant polynomial $P(w)$ is a uniqueness polynomial if the identity $P(f) = cP(g)$ implies $f = g$ for arbitrary nonconstant meromorphic functions f, g and a nonzero constant c.

In their papers [7] and [8], B. Shiffman, C.C. Yang and X. Hua studied polynomials $P(w)$ satisfying the condition that $P(f) = P(g)$ implies $f = g$ for arbitrary nonconstant meromorphic functions f and g. In this talk, we call such a polynomial a uniqueness polynomial in the wider sense.

If a set $S := \{a_1, \cdots, a_q\}$ is a uniqueness range set for meromorphic functions, then the polynomial $P(w) = (w - a_1) \cdots (w - a_q)$ is a uniqueness polynomial Here, the converse is not necessarily valid.

Let $P(w)$ be a nonconstant polynomial of degree q without multiple zeros and let the derivative $P'(w)$ be given by

$$P'(w) = q(w - d_1)^{q_1}(w - d_2)^{q_2} \cdots (w - d_k)^{q_k}, \tag{2.1}$$

where $d_1, \cdots, d_k$ are mutually distinct.

We first give conditions for $P(w)$ to be a uniqueness polynomial in the wider sense.

Theorem 2.1. *Assume that $k \geq 4$. If $P(w)$ satisfies the hypothesis*

(H) *$P(d_\ell) \neq P(d_m)$ for any ℓ, m with $1 \leq \ell < m \leq k$,*

then $P(w)$ is a uniqueness polynomial in the wider sense.

Theorem 2.2. *Assume that $k = 3$ and $\min(q_1, q_2, q_3) \geq 2$. If $P(w)$ satisfies the hypothesis* (H) *in Theorem 1 and any one of three values d_1, d_2 and d_3 cannot be a the arithmetic mean of two others, then $P(w)$ is a uniqueness polynomial in the wider sense.*

These theorems are proved by applying the second main theorem in Nevanlinna theory to meromorphic functions f and g with $P(f) = P(g)$ and the values $d_1, \cdots, d_k, d_{k+1} = \infty$. We omit the details.

Without the hypothesis (H) of Theorem 1 is not valid in general. In fact, for generically chosen constants $a_j (1 \leq j \leq n)$, where $n \geq 1$, a polynomial

$$P(w) := w^{2n} + a_1 w^{2n-2} + \cdots + a_{n-1} w^2 + a_n$$

has not multiple zeros and $P'(w)$ has $2n - 1$ distinct zeros $0, \pm d_1, \cdots, \pm d_{n-1}$. Then, $P(d_\ell) = P(-d_\ell)$ $(1 \leq \ell \leq n - 1)$, and $P(w)$ is not a uniqueness polynomial in the wider sense because $P(f) = P(-f)$ for any meromorphic function f (cf.,[8]).

We can prove also the following theorems on uniqueness polynomials.

Theorem 2.3. *Let $P(w)$ be a polynomial with $k \geq 4$ satisfying the hypothesis* (H) *stated in Theorem 1. If $P(w)$ is not a uniqueness polynomial, then there is some permutation $(t_1, t_2, \cdots, t_k)$ of $(1, 2, \cdots, k)$ such that*

$$\frac{P(d_{t_1})}{P(d_1)} = \frac{P(d_{t_2})}{P(d_2)} = \cdots = \frac{P(d_{t_k})}{P(d_k)} \neq 1.$$

Theorem 2.4. *Let $P(w)$ be a polynomial satisfying the hypothesis* (H) *and assume that $k = 3$ and $\min\{q_1, q_2, q_3\} \geq 2$. If $P(w)$ is not a uniqueness polynomial, then, after suitable changes of indices of d_j's, $d_3 = (d_1 + d_2)/2$ or*

$$\frac{P(d_2)}{P(d_1)} = \frac{P(d_3)}{P(d_2)} = \frac{P(d_1)}{P(d_3)}.$$

The outlines of the proofs of these theorems will be stated in §4.

As a consequece of Theorem 3, we have the following:

Corollary 2.5. *Let $P(w)$ be a polynomial with $k \geq 4$ satisfying the hypothesis* (H). *If*

$$P(d_1) + P(d_2) + \cdots + P(d_k) \neq 0, \tag{2.2}$$

then $P(w)$ is a uniqueness polynomial.

In fact, in the conclusion of Theorem 3, the assumption (2) deduces a contradiction

$$\frac{P(d_{t_1}) + P(d_{t_2}) + \cdots + P(d_{t_k})}{P(d_1) + P(d_2) + \cdots + P(d_k)} = 1.$$

By $\mathcal{P}_q$ we denote the set of all monic polynomials of degree q. With each $P(w) = w^q + A_1 w^{q-1} + \cdots + A_q \in \mathcal{P}_q$ by associating the point $(A_1, \cdots, A_q) \in \mathbb{C}^q$, we can identify $\mathcal{P}_q$ with $\mathbb{C}^q$. We say that some fact holds for generic polynomials of degree q if it holds for all polynomials in $\mathcal{P}_q$ except the zero set of a nonzero polynomial in $A_1, \cdots, A_q$.

By using Corollary 5 we can prove the following improvement of the fact which was shown by B. Shiffman in [7].

Theorem 2.6. *For $q \geq 5$, generic polynomials of degree q are uniqueness polynomials.*

Proof 1. As is easily seen, for generic polynomials, their derivatives have no multiple zeros and so we may consider the only case $k = q - 1 (\geq 4)$. Moreover, it is easily seen that the conditions (H) and (2) hold for generic polynomials of degree $q \geq 5$.

Here, the number five is best-possible. In fact, it is known that every polynomial of degree less than five is not a uniqueness polynomial ([5]).

Consider a polynomial without multiple zero given by $P(w) = w^q + aw^{q-r} + b$ $(a, b \in \mathbb{C} - \{0\})$. It is known that, if $r \geq 2$ and $q > r + 1$ and $(r, q) = 1$, then $P(w)$ is a uniqueness polynomial in the wider sense (cf.,[8]). In this connection, as an application of Corollary 5, we can show the following;

Claim 1. *If $r \geq 3$, $q > r + 1$ and $(r, q) = 1$, then $P(w)$ is a uniqueness polynomial.*

In fact, since $P'(w) = qw^{q-r-1}(w^r + a(q - r)/q)$, $k = r+1$ and $d_\ell = (-(q-r)a/q)^{1/r}\zeta^\ell$ $(1 \leq \ell \leq r)$, $d_{r+1} = 0$ in our notation, where ζ denotes a primitive r-th root of unity. By the assumption $r \geq 3$, we see $k \geq 4$. On the other hand, $P(0) = b$ and $P(d_\ell) = d_\ell^{q-r}(r/q)a + b$. Since $(r, q - r) = 1$, d_ℓ^{q-r} $(1 \leq \ell \leq r)$ are mutually distinct and not equal to 0. This shows that the hypothesis (H) is satisfied. Moreover, (2) is also satisfied because

$$P(d_1) + \cdots + P(d_k) = rb + \frac{r}{q}\left(d_1^{q-r} + \cdots + d_r^{q-r}\right) + b = (r + 1)b \neq 0.$$

3. Some conditions for uniqueness range sets

Take a subset $S := \{a_1, a_2, \cdots, a_q\}$ of $\mathbb{C}$ and consider the polynomial

$$P(w) := (w - a_1)(w - a_2)\cdots(w - a_q).$$

Assume that the derivative is given by (1) and $k \geq 2$.

We can prove the following:

Theorem 3.1. *For a positive integer m_0 or $m_0 = \infty$, assume that $q > 2k + 12$ for the case $m_0 = 1$, $q > 2k + \dfrac{4}{m_0 - 1} + 6$ for the case $m_0 \geq 2$, and $q > 2k + 6$ for the case $m_0 = \infty$. If f and g satisfy the condition*

$(C)_{m_0}$ $\quad \sum_{j=1}^q \min(\nu_f^{a_j}(z), m_0) = \sum_{j=1}^q \min(\nu_g^{a_j}(z), m_0)$,

then there are some constants $c_0 (\neq 0)$ and c_1 such that

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1. \tag{3.1}$$

For the particular case where f and g are holomorphic, (3) remains valid if $q > 2k + 5$ in case $m_0 = 1$, if $q > 2k + \dfrac{2}{m_0 - 1} + 2$ in case $m_0 \geq 2$, or if $q > 2k + 2$ in case $m_0 = \infty$.

The outline of the proof will be stated in §5.

By some improvements of the arguments in [2], we can show the following:

Proposition 3.1. *Let $P(w)$ be a polynomial of degree $q \geq 5$ without multiple zeros whose derivative is given by* (1)*. Assume that $P(w)$ satisfies the hypothesis* (H) *and* (3) *holds for two distinct nonconstant meromorphic functions f and g. If $k \geq 3$, or if $k = 2$ and* $\min\{q_1, q_2\} \geq 2$*, then $c_1 = 0$.*

Combining this with Theorem 7, we have the following theorem:

Theorem 3.2. *Suppose that $k \geq 3$, or $k = 2$ and $\min\{q_1, q_2\} \geq 2$, and that $P(w)$ is a uniqueness polynomial satisfying the hypothesis* (H)*. Take a positive integer m_0 and assume that*

(a) *for the case $m_0 = 1$, $q > 2k + 12$,*

(b) *for the case $m_0 \geq 2$, $q > 2k + \dfrac{4}{m_0 - 1} + 6$ and*

(c) *for the case $m_0 = \infty$, $q > 2k + 6$.*

Particularly, in case where f and g are holomorphic, assume that

(d) *for the case $m_0 = 1$, $q > 2k + 5$,*

(e) *for the case $m_0 \geq 2$, $q > 2k + \dfrac{2}{m_0 - 1} + 2$ and*

(f) *for the case* $m_0 = \infty$, $q > 2k+2$.

Then, if two nonconstant meromorphic functions f *and* g *satisfy the condition* $(\mathrm{C})_{m_0}$, *then* $f = g$.

In [2], Frank and Reinders showed the following:

Theorem 3.3. *Consider the polynomial*

$$P^{FR}(w) = \frac{(q-1)(q-2)}{2} w^q - q(q-2)w^{q-1} + \frac{q(q-1)}{2} w^{q-2} - c,$$

where $c \neq 0, 1$. *If* $q > 7$ *then* $P^{FR}(w)$ *is a uniqueness polynomial and, if* $q \geq 7$, *then it is a uniqueness polynomial for entire functions.*

For the polynomial $P^{FR}(w)$ as in Theorem 8,

$$(P^{FR})'(w) = \frac{q(q-1)(q-2)}{2} w^{q-3}(w-1)^2.$$

This shows that we can apply Theorem 8 to the polynomial $P^{FR}(w)$ for the case $k = 2$.

By these observations, we can conclude that there exists a uniqueness range set for meromorphic functions if $q \geq 11$ ([2]), a uniqueness range set for meromorphic functions ignoring multiplicities if $q \geq 17$ ([1]), a uniqueness range set for entire functions if $q \geq 7$, and a uniqueness range set for entire functions ignoring multiplicitiesif $q \geq 10$, respectively.

4. The outline of the proofs of Theorems 3 and 4

We state the outline of the proofs of Theorems 3 and 4. We use the standard terminology and notations in Nevanlinna theory the order function $T(r, f)$, the proximity function $m(r, f)$, the counting function $N(r, f)$ of a meromorphic function f and so on. In this talk, $S(r, f)$ means the term with the property that $S(r, f) = o(T(r, f))$ holds for all r except a subset E of the interval $[0, +\infty)$ with $\int_E dr < +\infty$.

The counting function of a divisor $\nu : \mathbb{C} \to \mathbb{Z}$ is defined by

$$N(r, \nu) := \int_0^r \left(\sum_{0<|z|\leq t} \nu(z) \right) \frac{dt}{t} + \nu(0) \log r.$$

Using the above notation, we may write $N(r, f) = N(r, = \nu_f^\infty)$.

For convenience' sake, we introduce some notations. Let ν be a divisor and E a discrete subset of $\mathbb{C}$ or some condition prescribing a set. We define

$$\nu|_E(z) := \begin{cases} \nu(z) & \text{if } z \in E \text{ or } z \text{ satisfies the condition } E \\ 0 & \text{otherwise,} \end{cases}$$

$$(\nu - 1)^+(z) := \nu(z) - \bar{\nu}(z) \quad (z \in \mathbb{C}),$$

$$\nu_{f'}^*(z) := \begin{cases} \nu_{f'}^0 & \text{if } f(z) \neq d_\ell \text{ for any } \ell \\ 0 & \text{otherwise.} \end{cases}$$

By assumption, there are a nonzero constant c and two distinct nonconstant meromorphic functions f and g such that $P(f) = cP(g)$, where $c \neq 1$ by Theorem 1. We set

$$\Lambda := \{(\ell, m); P(d_\ell) = cP(d_m)\}$$

and $k_0 := \#\Lambda$. As is easily seen, the hypothesis (H) yields the following:

Claim 2. *For* (ℓ, m) *and* (ℓ', m') *in* Λ, *if* $\ell = \ell'$ *or* $m = m'$, *then* $(\ell, m) = (\ell', m')$.

We choose labels $r(1), \cdots, r(k), s(1), \cdots, s(k)$ *of indices so that*

$$\Lambda = \{(r(1), s(1)), (r(2), s(2)), \cdots, (r(k_0), = s(k_0))\}$$

and

$$\begin{aligned} \{1, 2, \cdots, k\} &= \{r(1), \cdots, r(k_0), r(k_0+1), \cdots, r(k)\} \\ &= \{s(1), \cdots, s(k_0), s(k_0+1), \cdots, s(k)\}. \end{aligned}$$

For our purpose, we need some lemmas.

Lemma 4.1. *Suppose that there are no constants* $c_0(\neq 0)$ *and* c_1 *such that* $g = c_0 f + c_1$*. Then,* $k \leq 3 + k_0 - k^*$*, where* $k^* := \min(k_0, 2)$*. If* $k = 3 + k_0 - k^*$*, then* $N(r, \nu^*_{g'}|_{f=d_r(\ell)}) = S(r)$ *for each* ℓ *with* $q_{r(\ell)} \geq 2$ *and, similarly,* $N(r, \nu^*_{f'}|_{g=d_s(\ell)}) = S(r)$ *for each* ℓ *with* $q_{s(\ell)} \geq 2$.

Lemma 4.2. *In the same situation as in Lemma 1, assume furthermore that* $c \neq 1$ *and* $k = k_0+1 \geq 4$*. Then,*

(1) $N(r, (\nu_f^\alpha - 1)^+) = S(r)$ *for each* α *with* $P(\alpha) = P(d_{r(\ell)})$ $(1 \leq \ell \leq k_0)$*, and* $N(r, (\nu_g^\beta - 1)^+) = S(r)$ *for each* β *with* $P(\beta) = P(d_{s(\ell)})$ $(1 \leq \ell \leq k_0)$,

(2) $q_{r(\ell)} = q_{s(\ell)}$ *for* $1 \leq \ell \leq k$.

These lemmas are proved by comparing the zeros of the both sides of the identity $P'(f)f' = cP'(g)g'$ and by applying the second main theorem to the values $d_1, \cdots, d_k, \infty$. We omit the details.

Using these lemmas, we can prove that

$$N(r, |\nu_f^{d_{r(\ell)}} - \nu_g^{d_{s(\ell)}}|) = S(r) \quad \text{for } \ell = 1, 2, \cdots, k_0(\geq 3).$$

Here, we use the following generalizations of the four value theorem;

Lemma 4.3. *Let* f *and* g *be nonconstant distinct meromorphic functions such that* $T(r, f) = O(T(r, g))$ *and* $T(r, g) = O(T(r, f))$*. Assume that there are values* d_1, d_2, d_3*,* d_4 *and* e_1, e_2, e_3, e_4*, which are mutually distinct respectively such that*

$$N(r, |\nu_f^{d_\ell} - \nu_g^{e_\ell}|) = S(r, f) \quad (\ell = 1, 2, 3, 4).$$

Then, $g = (\varphi_1 f + \varphi_2)/(\varphi_3 f + \varphi_4)$ *for suitable functions* $\varphi_\ell \in \mathcal{S}$ $(\ell = 1, \cdots, 4)$ *with* $\varphi_1\varphi_4 - \varphi_2\varphi_3 \neq 0$*, where* $\mathcal{S}$ *denotes the set of all meromorphic functions* φ *with* $T(r, \varphi) = S(r, f)$.

Lemma 4.4. *Under the same situation as in Lemma 3, if* $d_\ell = e_\ell$ $(\ell = 1, \cdots, 4)$*, then* g *is a Möbius transformation of* f *and* $(d_1, d_2, d_3, d_4) = -1$ *after a suitable change of indices of* d_ℓ*'s.*

By virtue of Lemma 3, we can write $g = (\varphi_1 f + \varphi_2)/(\varphi_3 f + \varphi_4)$ with $\varphi_\ell \in \mathcal{S}$ $(1 \leq \ell \leq 4)$. Then, for $Q(w) := P((\varphi_1 w + \varphi_2)/(\varphi_3 w + \varphi_4))(\varphi_3 w + \varphi_4)^q \in \mathcal{S}[w]$, we have $P(f)(\varphi_3 f + \varphi_4)^q = cQ(f)$. This yields the polynomial identity $P(w)(\varphi_3 w + \varphi_4)^q = cQ(w)$. We easily see $\varphi_3 = 0$. Changing notations, we write $g = \varphi f + \psi$ for $\varphi(\neq 0)$ and ψ in $\mathcal{S}$.

From the identity $P(w) = cP(\varphi w + \psi)$ in $\mathcal{S}[w]$, we obtain

$$(w - d_1)^{q_1} \cdots (w - d_k)^{q_k} = c\varphi^q \left(w - \frac{d_1 - \psi}{\varphi}\right)^{q_1} \cdots \left(w - \frac{d_k - \psi}{\varphi}\right)^{q_k}.$$

By unique factorization theorem, we have $d_{t_\ell} = (d_\ell - \psi)/\varphi$ $(1 \leq \ell \leq k)$ for a permutation $(t_1, t_2, \cdots, t_k)$ of $(1, 2, \cdots, k)$. This gives $P(d_{t_\ell}) = cP(d_\ell)$ $(1 \leq \ell \leq k)$, whence we get $k = k_0$. Thus, we have Theorem 3.

The proof of Theorem 5 is similar. To this end, we use the last part of Lemma 1 and Lemma 4. The details are omitted.

5. The outline of the proof of Theorem 7

We state the outline of the proof of Theorem 7 for the only case where $m_0 = \infty$ and f, g are meromorphic, because the similar arugments are available for the other cases. For meromorphic functions f, g with $\nu_f^S = \nu_g^S$, we set $F := 1/P(f)$, $G := 1/P(g)$ and define

$$H(z) := \frac{F''(z)}{F'(z)} - \frac{G''(z)}{G'(z)}.$$

We can easily show that H has the following properties:

$$\nu_H^\infty(z) \leq \sum_{\ell=1}^{k} (\bar{\nu}_f^{d_\ell}(z) + \bar{\nu}_g^{d_\ell}(z)) + \bar{\nu}_f^\infty(z) + \bar{\nu}_g^\infty(z) \tag{5.1}$$
$$+ \bar{\nu}|_{(f')^{-1}(0)\setminus f^{-1}(S)} + \bar{\nu}|_{(g')^{-1}(0)\setminus g^{-1}(S)},$$

$$m(r, H) = S(r, f) + S(r, g). \tag{5.2}$$

For our purpose, we have only to show that $H = 0$, because this yields $F = c_0G + c_1$ for $c_0(\neq 0), c_1 \in \mathbb{C}$ and hence Theorem 7. So, the proof of Theorem 7 is reduced to show the following:

Proposition 5.1. *If $H \neq 0$, then $q \leq 2k + 6$.*

Proof 2. For brevity, we set

$$N^S(r, \nu^0_{f'}) := N(r, \nu|_{(f')^{-1}(0)\cap f^{-1}(S)}),\ N^{CS}(r, \nu^0_{f'}) := N(r, \nu|_{(f')^{-1}(0)\setminus f^{-1}(S)}).$$

Each $z_0 \in f^{-1}(S)$ is a zero of H for the case $\nu_f^{f(z_0)}(z_0) = \nu_g^{g(z_0)}(z_0) = 1$, and a zero of f' (and g') for the other cases. Therefore,

$$\begin{aligned}(q-1)T(r,f) &\leq N(r,\bar{\nu}_f^{\infty}) + \sum_{i=1}^{q} N(r,\bar{\nu}_f^{a_i}) - N^{CS}(r,\nu^0_{f'}) + S(r)\\ &\leq N(r,\bar{\nu}_f^{\infty}) + N(r,\bar{\nu}_H^0) + N^S(r,\nu^0_{f'}) - N^{CS}(r,\nu^0_{f'}) + S(r).\end{aligned}$$

On the other hand, by the help of the first main theorem, (4) and (5) yield

$$\begin{aligned}N(r,\bar{\nu}_H^0) &\leq T(r,H) + O(1) \leq N(r,\nu_H^{\infty}) + S(r)\\ &\leq \sum_{\ell=3D1}^{k} (N(r,\bar{\nu}_f^{d_\ell}) + N(r,\bar{\nu}_g^{d_\ell})) + N(r,\bar{\nu}_f^{\infty}) + N(r,\bar{\nu}_g^{\infty})\\ &\quad + N^{CS}(r,\nu^0_{f'}) + N^{CS}(r,\nu^0_{g'}) + S(r).\end{aligned}$$

Therefore,

$$\begin{aligned}(q-1)T(r,f) &\leq kT(r) + 2N(r,\bar{\nu}_f^{\infty}) + N(r,\bar{\nu}_g^{\infty})\\ &\quad + N^{CS}(r,\nu^0_{g'}) + N^S(r,\nu^0_{f'}) + S(r).\end{aligned}$$

To this inequality adding a similar inequality for g, we conclude

$$\begin{aligned}(q-1)T(r) &\leq 2kT(r) + 3(N(r,\nu_f^{\infty}) + N(r,\nu_g^{\infty}))\\ &\quad + (N(r,\nu^0_{f'}) + N(r,\nu^0_{g'})) + S(r),\\ &\leq (2k+5)\,T(r) + S(r).\end{aligned}$$

This gives Proposition 2.

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On certain Diophantine Equation

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Introduction

We give here a brief survey on some exponential Diophantine equations. Let x, y, m, q be rational integers with $x > 1, y > 1, m > 2, q \geq 2$. We consider the equation

$$\frac{x^m - 1}{x - 1} = y^q. \tag{1}$$

with x, y, m, q unknowns. Ljunggren [5] proved that equation (1) with $q = 2$ has no solution other than $x = 3, y = 11, m = 5$ and $x = 7, y = 20, m = 4$. Thus, there is no loss of generality in assuming that q is an odd prime number. Saradha and Shorey [7] proved that equation (1) has only finitely many solutions whenever x is a square.Recently, Y. Bugeaud and M. Mignotte announced that there is no such solution. Now we overview an analogous question if x is a cube or a higher power. Namely being μ a fixed rational integer with $\mu \geq 3$ and we consider equation (1) with $x = z^\mu$ where $z > 1$ is a rational integer. There is no loss of generality in assuming that μ is an odd prime number. This turns us to see (1) in unknowns z, x, y, m.

With T.N. Shorey, we proved the following property [3].

Proposition Let $z > 1$ be an integer and $\mu \geq 3$ be a prime number. Assume that

$$q > 2(\mu - 1)(2\mu - 3). \tag{2}$$

Then equation (1) with $x = z^\mu$ implies that max (x, y, m, q) is bounded by an effectively computable number c depending only on μ.

If $\mu = q$, Shorey [9] showed that the assertion of the Proposition is valid with c replaced by an absolute constant. The case $q = 3$ of the preceding result is due to Inkeri [4]. Consequently, we derive from the Proposition that equation (1) with $x = z^3$ and $q \neq 5, 7, 11$ implies that max (x, y, m, q) is bounded by an effectively computable absolute constant.

Diophantine approximations

Let us state the following result of Shorey and Tijdeman [11] on equation (1).

Lemma 1 Equation (1) has only finitely many solutions if either x is fixed or m has a fixed prime divisor. Furthermore, the assertion is effective.

Apllying Lemma 1, we could obtain the following.

Lemma 2 Equation (1) with $x = z^\mu$ implies that either $\max(x, y, m, q)$ is bounded by an effectively computable number depending only on μ or

$$\frac{z^m - 1}{z - 1} = y_1^q, \quad \frac{z^{m(\mu-1)} + z^{m(\mu-2)} + \cdots + 1}{z^{\mu-1} + z^{\mu-2} + \cdots + 1} = y_2^q \tag{3}$$

where $y_1 > 1$ and $y_2 > 1$ are relatively prime integers such that $y_1 y_2 = y$.

The next result is a refinement of Lemma 1 in [10] which is proved by using Padé approximations.

Lemma 3 Let A, B, K and n be positive integers such that $A > B, K < n, n \geq 3$ and $\omega = (B/A)^{1/n}$ is not a rational number. For $0 < \phi < 1$, put

$$\delta = 1 + \frac{2-\phi}{K},\ s = \frac{\delta}{1-\phi},$$

$$u_1 = (3^{2K+1} \cdot 2^{s(4K+2+3n(K+1))+(1+(3n)/2)(K+1)})^{1/(Ks-1)},$$

$$u_2^{-1} = 3^{2K+1}K^2\big(1+\frac{1}{2^{29}}\big)^{K-1}n^{2K}2^{K+s+2+3n(K+1)}.$$

Assume that

$$A(A-B)^{-\delta}u_1^{-1} > 1.$$

Then

$$\mid \omega - \frac{p}{q} \mid > \frac{u_2}{Aq^{K(s+1)}}$$

for all integers p and q with $q > 0$.

By using Lemma 1 and Lemma 3, we obtain Lemma 4:

Lemma 4 Equation (1) with $x = z^\mu$ and (2) implies that max (x, y, m) is bounded by an effectively computable number depending only on q and μ.

In view of the above lemma, it remains to show that equation (1) with $x = z^\mu$ implies that q is bounded. The following statement is a consequence of a result of Philippon and Waldschmidt [6] on linear forms in logarithms.

Lemma 5 Let $n > 1$ be an integer and $\tau_1 \geq 1, \tau_2 \geq 1$ be real numbers. Let $\alpha_1, \cdots, \alpha_{n-1}$ and α_n be positive rational numbers of heights (the height of a non-zero rational number a/b with $\gcd(a,b) = 1$ is defined as max $(\mid a \mid, \mid b \mid)$ not exceeding A_1 and A, respectively, where $A_1 \geq 3, A \geq 3$ and

$$(\log A)(\log A_1)^{-1} \geq \tau_1^{-1}.$$

Further assume that $\mid \log \alpha_i \mid \leq A_1^{-1/\tau_2}$ *for* $1 \leq i \leq n$. Let $b_1, \cdots, b_n$ be rational integers of absolute values not exceeding $B \geq 2$ such that

$$\wedge = b_1 \ \log \alpha_1 + \cdots + b_n \ \log \ \alpha_n \neq 0.$$

There exists an effectively computable number c_1 depending only on n, τ_1 and τ_2 such that

$$\mid \wedge \mid \geq \ \exp \ \ (-c_1(1 + \frac{\log B}{\log A_1}) \log A).$$

From approximations to Diophantine equations

Now we describe the outline of the proof of our proposition. We denote by $c_2, \cdots, c_5$ effectively computable positive numbers depending only on μ. Suppose that equation (1) with $x = z^\mu$ is satisfied. By Lemma 4, it suffices to show that $q \leq c_2$. Now we refer to Lemma 1 to suppose that min $(q, m, z) \geq c_3$ with c_3 sufficiently large. Then we derive (3). Consequently

$$0 < \mid \log \ \alpha_1 + q \ \log \ \alpha_2 \mid < 8\mu z^{-m} \tag{4}$$

where $\alpha_1 = \omega^q, \alpha_2 = y_1^{\mu-1}/y_2$ and ω is given by

$$\omega = \left(\frac{(z-1)^{\mu-1}}{z^{\mu-1} + \cdots + 1} \right)^{1/q}.$$

We observe that

$$\mid \log \ \alpha_1 \mid < \ \log\ (1 + \frac{\mu z^{\mu-2}}{(z-1)^{\mu-1}}) < \frac{2\mu}{z} < z^{-1/2}$$

which, together with (4), implies that

$$0 < \mid \log \ \alpha_2 \mid < z^{-1/2}.$$

Further we observe that the heights of α_1 and α_2 do not exceed $2z^{\mu-1}$ and $2z^{(\mu-1)(m-1)/q}$. Consequently

$$\frac{1}{2z^{(\mu-1)(m-1)/q}} \leq \frac{\mid y_1^{\mu-1} - y_2 \mid}{\max(y_1^{\mu-1}, y_2)} \leq 2z^{-1/2}$$

which implies that

$$q \leq 4(\mu-1)(m-1)$$

and

$$\frac{\log\ (2z^{(\mu-1)(m-1)/q})}{\log(2z^{\mu-1})} \geq \frac{m-1}{2q} \geq \frac{1}{8(\mu-1)}.$$

Now we apply Lemma 5 with $n = 2, A_1 = 2z^{\mu-1}, A = 2z^{(\mu-1)(m-1)/q}, \tau_1 = \tau_2 = 8(\mu-1)$ and $B = q$ to conclude that

$$\mid \log\ \alpha_1 + q\ \log\ \alpha_2 \mid \geq\ \exp\ (-c_4 m q^{-1}\ \log\ (qz)). \tag{5}$$

Finally we combine (4) and (5) to conclude that $q \leq c_5$.

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On p-adic Montel Theorem and its applications

Liang-chung Hsia

1. Introduction

Let $\mathbb{C}_v$ be a non-archimedean field which is algebraically closed and complete with respect to a non-archimedean absolute value $|?|_v$. Let ϕ be a rational function defined over $\mathbb{C}_v$. ϕ can be regarded as continuous self map of the projective line $\mathbb{P}^1(\mathbb{C}_v) := \mathbb{C}_v \cup \{\infty\}$. By non-archimedean dynamics we mean the dynamical system associated with the action of ϕ on $\mathbb{P}^1(\mathbb{C}_v)$ under the iteration of ϕ. The dynamics is partitioned into two parts. One consists of areas where small neighborhood remains small under the iterations of ϕ and the other part reveals the opposite characteristic. Borrowing terminology from complex dynamics, we will call the former Fatou set and the latter Julia set. To have more precise definition of the Fatou set and Julia set, a natural metric which is analogous to the classical chordal metric on Riemann sphere is introduced on $\mathbb{P}^1(\mathbb{C}_v)$.

Definition 1. For any two points $P_1 = [x_1, y_1], P_2 = [x_2, y_2] \in \mathbb{P}^1(\mathbb{C}_v)$, define

$$\| P_1, P_2 \| = \frac{|x_1y_2 - x_2y_1|_v}{\max\{|x_1|_v, |y_1|_v\} \ \max\{|x_2|_v, |y_2|_v\}}.$$

Regarding $\mathbb{P}^1(\mathbb{C}_v)$ as a metric space, the rational function ϕ is a continuous map with respect to the chordal metric. Then, the Fatou set is defined to be the subset F_ϕ of $\mathbb{P}^1(\mathbb{C}_v)$ where the family of iterations $\{\phi^n\}_{n\geq 0}$ of ϕ is equicontinuous. The Julia set J_ϕ is the complement of F_ϕ.

2. Montel Theorem over nonarchimedean field and applications

Put $\bar{D}_r(a) = \{x \in \mathbb{C}_v : |x - a|_v \leq r\}$ to be the closed disc of radius r centered at a. Due to the ultrametricity of non-archimedean absolute value, $\bar{D}_r(a)$ is both open and closed. In complex dynamics, Montel's theorem is a powerful tool in proving global results of complex dynamics. Using complex dynamics as our model, we first show an analogue of Montel's Theorem over nonarchimedean fields. Any function f analytics $\bar{D}_r(a)$ is viewed as a continuous map from $\bar{D}_r(a)$ as a metric space to the metric space $\mathbb{P}^1(\mathbb{C}_v)$. Then, our version of Montel's Theorem over nonarchimedean fields is as follows.

Theorem 2.1. *Let* $\mathfrak{F}$ *be a family of meromorphic functions in* $\bar{D}_r(a)$. *Assume that* $\cup_{f\in\mathfrak{F}} f(\bar{D}_r(a))$ *omit two distinct points of* $\mathbb{P}^1(\mathbb{C}_v) = \mathbb{C}_v \cup \{\infty\}$, *then* $\mathfrak{F}$ *is equicontinuous in* $\bar{D}_r(a)$.

A variant of Montel's Theorem is given by the following Theorem.

Theorem 2.2. *Let* ϕ_1, ϕ_2 *be analytic functions in* $\bar{D}_r(a)$. *Suppose the regions* $\phi_j(\bar{D}_r(a)), j = 1, 2$ *are mutually disjoint. Let* $\mathfrak{F}$ *be a family of meromorphic functions in* $\bar{D}_r(a)$ *such that for every* z *in* $\bar{D}_r(a)$ *and every* f *in* $\mathfrak{F}$, $f(z) \neq \phi_j(z)$, $j = 1, 2$. *Then* $\mathfrak{F}$ *is equicontinuous in* $\bar{D}_r(a)$.

Several consequences can be derived directly from Montel's Theorem.

Corollary 2.1. Let $w \in J_\phi$ and let N be a neighborhood of w. Then the set $E_N = \mathbb{P}^1(\mathbb{C}_v) \setminus \cup_{n>0} \phi^n(N)$ contains at most one point.

For any given $w \in J_\phi$, the set $E_w = \cup_{w\in N} E_N$ contains at most one point with the union ranges over all neighborhood of w.

Theorem 2.3. *Let* $w \in J_\phi$ *and* E_w *be defined as above. If* E_w *is non-empty, then* ϕ *is conjugate to a polynomial map. Moreover, the set* E_w *is independent of* $w \in J_\phi$.

We can characterize nonempty Julia set as the smallest closed, completely invariant set with at least two points.

Corollary 2.4. *If* $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus E_\phi$, *then*

$$J_\phi \subset \{\textit{accumulation points of } \cup_{n\geq 0} \phi^{-n}(z)\}.$$

Consequently, if $z \in J_\phi$, *then*

$$J_\phi = \textit{closure}\left(\cup_{n\geq 0}\phi^{-n}(z)\right).$$

Having established Montel's Theorem and assuming the rational map in question has nonempty Julia set, one of our main result is the following.

Theorem 2.5. *Let* $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ *be a rational map defined over* $\mathbb{C}_v$ *with* $\deg(\phi) \geq 2$ *and non-empty Julia set* J_ϕ. *Then the Julia set* J_ϕ *is contained in the closure of the set of periodic points of* ϕ.

3. Examples

Over nonarchimedean field, many examples also show that the Julia set is the closure of repelling periodic points. The following example is derived from [**?**, Example 4.11], for the details of the assertion we refer to [**?**].

Example 3.1. *Let $\phi(z) = (z^p - z)/p$ over $\mathbb{Q}_p$, the p-adic numbers. Then, we have that :*
(1) $J_\phi = \mathbb{Z}_p$, the p-integers,
(2) Every element $\omega \in J_\phi$ is represented by an infinite sequence $(a_1, a_2, \cdots)$ with each a_i being element of the set of p symbols,
(3) The dynamics on ϕ on J_ϕ is the one sided shift map. Namely,

$$\phi((a_1, a_2, a_3, \cdots)) = (a_2, a_3, \cdots).$$

This map has the property that every periodic point, except the point at infinity, is repelling. (2) shows that J_ϕ is in fact a symbol space in p symbols and (3) describes the dynamics in J_ϕ as the shift map. It is known that in this case the repelling periodic points are dense in J_ϕ. That is, J_ϕ is the closure of the repelling periodic points.

In contrast to the case of complex dynamics, there exist many rational maps with nonempty Julia set and infinitely many non-repelling cycles as the following example shows.

Example 3.2. *Consider the map $\phi(z) = pz^3 + az^2 + b$ over $\mathbb{Q}_p$ where p is a prime number and $a, b \in \mathbb{Z}_p^*$. This map has a fixed point with absolute value $|\alpha_3|_p = p > 1$. In fact, in terms of its p-adic expansion, we can write $\alpha_3 = \frac{a_{-1}}{p} + \sum_{i=0}^{\infty} a_i p^i$ where $a_{-1} \equiv -a \pmod{p}$ and $0 \leq a_i \leq p - 1$. Moreover, its multiplier $|\phi'(\alpha_3)|_p = p > 1$. Hence, J_ϕ is not empty. Applying methods from [**?**], we can show that after some ramified extension of $\mathbb{Q}_p$, there are infinitely many repelling periodic points rational over the extension field.*

*On the other hand, it is easy to see that $\bar{D}_1(0) \subset F_\phi$ and ϕ has infinitely many periodic points in $\bar{D}_1(0)$. Also applying method from [**?**] and by Theorem **??**, we can conclude that the Julia set is contained in the closure of the repelling periodic points and therefore, it is the closure of the periodic points.*

We expect phenomena similar to complex dynamics would also occur over nonarchimedean field. Namely,
Conjecture : The Julia set, if nonempty, is the closure of repelling periodic points of ϕ.

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Value sharing problems in value distribution theory

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Mathematics Subject: Primary 11D88, 11E95, 11Q25. Secondary 30D35.

1. Main notations

Let κ be an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedian absolute value $|\cdot|$. Let K denote the field κ or $\mathfrak{C}$ and set $\bar{K} = K \cup \{\infty\}$. We will denote by $\mathcal{A}(K)$ the ring of entire functions in K, that is, the ring of power series $f(z) = \sum a_n z^n$ $(a_n \in K)$ which satisfy the condition $\lim |a_n| r^n = 0$ for any $r > 0$, and by $\mathcal{M}(K)$ the field of meromorphic functions on K, i. e., the field of fractions of the ring $\mathcal{A}(K)$.

For a subset S of K and $f \in \mathcal{M}(K)$, we denote by $E_f(S)$ the set

$$\bigcup_{a\in S} \{(z,q) \mid z \text{ is a zero of order } q \text{ of } f(z) - a\},$$

and denote the preimage of S under f by $\overline{E}_f(S) = f^{-1}(S)$.

Definition 1.1. Given a family $\mathcal{F}$ of $\mathcal{M}(K)$, a non-empty set S in $\bar{K}$ is called a *urscm* (resp., *ursim*) for $\mathcal{F}$ if for any non-constant functions $f, g \in \mathcal{F}$ satisfying $E_f(S) = E_g(S)$ (resp., $\overline{E}_f(S) = \overline{E}_g(S)$), one has $f = g$.

In the Definition **??**, "urscm" (resp., "ursim") means a *unique range set for counting multiplicity* (resp., *unique range set for ignoring multiplicity*). If two functions $f, g \in \mathcal{F}$ satisfy $E_f(S) = E_g(S)$ (resp., $\overline{E}_f(S) = \overline{E}_g(S)$), we also say that f and g *share the set* S CM (resp., IM). These notations were introduced by Gross-Yang [**?**]. They constructed a urscm for $\mathcal{A}(\mathfrak{C})$, but with infinite cardinality. A finite cardinal urscm for $\mathcal{A}(\mathfrak{C})$ and $\mathcal{M}(\mathfrak{C})$ was first found by H. X. Yi [**?**], and further was studied by Li-Yang [**?**] and [**?**], Mues-Reinders [**?**], Frank-Reinders [**?**], Hu-Yang [**?**], and so on.

Given a family $\mathcal{F}$ of $\mathcal{M}(K)$, we obtain two numbers

$$c(\mathcal{F}) = \min\{\#S \mid S \text{ is a urscm for } \mathcal{F}\ \}$$

and

$$i(\mathcal{F}) = \min\{\#S \mid S \text{ is a ursim for } \mathcal{F}\ \},$$

where $\#S$ is the cardinal number of the set S.

2. Characterization of urscm

Let $\mathrm{Aut}(K)$ be the group of non-constant linear polynomials in $K[z]$, that is, $\sigma \in \mathrm{Aut}(K)$ if and only if $\sigma(z) = az + b$ with $a \neq 0$. Let *id* be the *identical mapping* in K. Note that a set S in K is a ursim for $\mathrm{Aut}(K)$ if and only if S is a urscm for $\mathrm{Aut}(K)$. Take $\sigma,\ \eta \in \mathrm{Aut}(K)$. We see that $\sigma^{-1}(S) = \eta^{-1}(S)$ if and only if $\eta \circ \sigma^{-1}(S) = S$. Thus S is a ursim for $\mathrm{Aut}(K)$ if and only if there exists no $\xi \in \mathrm{Aut}(K) - \{id\}$ such that $\xi(S) = S$. For any $n \in \mathfrak{Z}^+$, we denote the set of zeros of $z^n - 1$ in K by $\Omega_n(K)$.

Example 2.1. Take $\sigma,\ \eta \in \mathrm{Aut}(K)$ defined by

$$\sigma(z) = -z, \quad \eta(z) = wz\ (w \in \Omega_3(K) - \{1\}).$$

Obviously, one has

$$\sigma(\{-1,0,1\}) = \{-1,0,1\}, \quad \eta(\Omega_3(K)) = \Omega_3(K).$$

Hence $\{-1,0,1\}$ and $\Omega_3(K)$ are not ursim for $\mathrm{Aut}(K)$. Note that for any distinct elements $a, b \in K$, the mapping $\xi \in \mathrm{Aut}(K)$ defined by

$$\xi(z) = -z + a + b$$

satisfies $\xi(\{a,b\}) = \{a,b\}$. Hence a ursim for $\mathrm{Aut}(K)$ contains at least 3 distinct elements. Therefore $c(\mathcal{A}(K)) \geq 3$.

Example 2.2. Take $S = \{a_1, a_2, a_3\} \subset K$ and set

$$c = \frac{1}{3}(a_1 + a_2 + a_3), \quad w \in \Omega_3(K) - \{1\}.$$

Then S is not a ursim for $\mathrm{Aut}(K)$ if and only if it has one of the following two forms:

(i) $S = \{c - a, c, c + a\}$;

(ii) $S = \{c + a, c + wa, c + w^2 a\}$,

where $a = c - a_1$ (see [**?**]). Two mappings in $\mathrm{Aut}(K)$ under which S are invariant are respectively

$$\sigma(z) = -z + 2c, \quad \eta(z) = wz + (1 - w)c.$$

Boutabaa, Escassut and Haddad [?] asked whether a finite set S in κ is a urscm for $\mathcal{A}(\kappa)$ if and only if S is a ursim for $\mathrm{Aut}(\kappa)$. They confirmed it if either S has only three different points or $\mathcal{A}(\kappa)$ is replaced by $K[x]$. The general case is proved by Cherry and Yang [?]:

Theorem 2.1. A finite set S in κ is a urscm for $\mathcal{A}(\kappa)$ if and only if S is a ursim for $\mathrm{Aut}(\kappa)$.

W. Cherry proposed the following principle (see [?]):

Cherry's principle *Any theorem which is true for polynomials (resp., rational functions) will also be true for non-Archimedean entire (resp., meromorphic) functions, unless it is obviously false.*

According to the principle, we suggest the following problem:

Conjecture 2.1. A finite set S in κ is a ursim for $\mathcal{A}(\kappa)$ if and only if S is a ursim for $\kappa[z]$.

Let $\mathrm{Aut}(\bar{K})$ be the group of non-constant fractional linear functions in $K(z)$, that is, $\sigma \in \mathrm{Aut}(\bar{K})$ if and only if

$$\sigma(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in K, \quad ad - cb \neq 0.$$

An element of $\mathrm{Aut}(\bar{K})$ also is called a *Möbius transformation* on $\bar{K}$. Similarly, S is a ursim for $\mathrm{Aut}(\bar{K})$ if and only if there exists no $\xi \in \mathrm{Aut}(\bar{K}) - \{id\}$ such that $\xi(S) = S$. Given 3 points $a_1, a_2, a_3 \in \bar{K}$ and choosed $a_4 \in \bar{K}$ satisfying

$$a_3 \neq \frac{1}{2}(a_1 + a_2), \quad (a_1, a_2, a_3, a_4) = \frac{(a_1 - a_3)(a_2 - a_4)}{(a_1 - a_4)(a_2 - a_3)} = -1,$$

then the following transformation

$$\sigma(z) = \frac{(a_3 + a_4)z - 2a_3a_4}{2z - (a_3 + a_4)}$$

satisfies

$$\sigma(a_1) = a_2, \ \sigma(a_2) = a_1, \ \sigma(a_j) = a_j \ (j = 3, 4).$$

In particular, $\sigma(\{a_1, a_2, a_3\}) = \{a_1, a_2, a_3\}$ so that $\{a_1, a_2, a_3\}$ is not a ursim for $\mathrm{Aut}(\bar{K})$. Thus a ursim for $\mathrm{Aut}(\bar{K})$ must have at least 4 points, and hence $c(\mathcal{M}(K)) \geq 4$. Here we suggest the following problem:

Problem 2.1. A finite set S in κ is a urscm (resp., ursim) for $\mathcal{M}(\kappa)$ if and only if S is a urscm (resp., ursim) for $\kappa(z)$.

3. Yi's set and Frank-Reinders' set

In the research of unique range sets of meromorphic functions on $\mathfrak{C}$, H. X. Yi (cf. [?]) first used the following polynomial

$$Y_{n,m}(z) = Y_{n,m}(a, b; z) = z^n - az^m + b \ (n, m \in \mathfrak{Z}^+, \ n > m), \tag{3.1}$$

where we will let $a, b \in K_*$ with

$$\frac{a^n}{b^{n-m}} \neq \frac{n^n}{m^m(n-m)^{n-m}}. \tag{3.2}$$

The condition (**??**) makes $Y_{n,m}$ only has simple zeros. We will denote the set of zeros of $Y_{n,m}$ by $\dot{Y}_{n,m}$, which will be called the *Yi's set*.

Theorem 3.1 (Hu [?]). Take an integer $n \geq 3$. If the constants $a, b \in \kappa_*$ in the polynomial $Y_{n,n-1}$ also satisfy

$$\frac{a^n}{b} \neq 2\frac{n^n}{(n-1)^{n-1}}, \tag{3.3}$$

then the Yi's set $\dot{Y}_{n,n-1}$ is a urscm for $\mathcal{A}(\kappa)$.

For the case $n \geq 4$, see Boutabaa-Escassut-Haddad [?] and Hu-Yang [?]. In value sharing problems, G. Frank and M. Reinders [?] consider the following polynomial

$$F_{n,b}(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} + b \ (b \in K_* - \{-1\}). \tag{3.4}$$

We will denote the set of zeros of $F_{n,b}$ by $\dot{F}_{n,b}$, and will call the set $\dot{F}_{n,b}$ a *Frank-Reinders' set.* Note that $\dot{F}_{n,b}$ contains n distinct elements.

Theorem 3.2. The Frank-Reinders's set $\dot{F}_{n,b}$ is a urscm for $\mathcal{A}(\kappa)$ for $n \geq 6$.

Problem 3.1. Does there exists a $b \in \kappa_*$ such that $\dot{F}_{3,b}$ is a urscm for $\mathcal{A}(\kappa)$?

4. Urscm for $\mathcal{M}(\kappa)$

By using the Frank-Reinders' technique [?], we can prove the following main lemma:

Lemma 4.1. Let F and G be non-constant meromorphic functions on κ sharing ∞ CM and let $a_1, ..., a_q$ be distinct elements of κ with $q \geq 2$. Then one of the following cases must occur

1)

$$\begin{aligned}\left(q - \frac{3}{2}\right)\{T(r,F) + T(r,G)\} \quad \leq \quad & \sum_{j=1}^{q}\left\{N_2\left(r, \frac{1}{F-a_j}\right) + N_2\left(r, \frac{1}{G-a_j}\right)\right\} \\ & - \log r + O(1);\end{aligned}$$

2) $G = AF + B$, where $A, B \in \kappa$ with $A \neq 0$, and

$$\#(\{a_1, ..., a_q\} \cap \{Aa_1 + B, ..., Aa_q + B\}) \geq 2.$$

Frank-Reinders [?] remarked that if $q = 2$, then case (2) means that $F = G$ or $F + G = a_1 + a_2$. Thus Lemma ?? is the non-Archimedian analogue of Theorem 1 in Yi [?]. The main ideas also come from Yi [?].

Lemma 4.2. Let f and g be non-constant meromorphic functions on κ satisfying

$$E_f(\dot{F}_{n,b}) = E_g(\dot{F}_{n,b}),$$

and define

$$F = \frac{1}{F_{n,b}(f)}, \ G = \frac{1}{F_{n,b}(g)}.$$

If $n \geq 10$, then $G = AF + B$, where $A, B \in \kappa$ with $A \neq 0$, and

$$\#(\{a_1, a_2, a_3\} \cap \{Aa_1 + B, Aa_2 + B, Aa_3 + B\}) \geq 2,$$

where $a_1 = 0, \ a_2 = \frac{1}{b}, \ a_3 = \frac{1}{b+1}$.

From Lemma ?? and Lemma ??, we obtain

Theorem 4.1 ([?]). For any integer $n \geq 10$, the Frank-Reinders' set $\dot{F}_{n,b}$ is a urscm for $\mathcal{M}(\kappa)$.

Take $m, n \in \mathfrak{Z}^+$ and let $b \in \kappa_*$, $Q \in \kappa[z]$ with $\deg(Q) = n - m \geq 1$ such that the polynomial

$$P(z) = z^m Q(z) + b$$

has only simple zeros. Let S_P be the set of zeros of P. Similarly we can prove the following lemma:

Lemma 4.3. Let f and g be non-constant meromorphic functions on κ satisfying

$$E_f(S_P) = E_g(S_P),$$

and define

$$F = \frac{1}{P(f)}, \ G = \frac{1}{P(g)}.$$

If $2m \geq n + 8$, then $G = F$ or $G + F = \frac{1}{b}$.

From Lemma ?? and Lemma ??, we obtain

Theorem 4.2 (Hu-Yang [?]). For any integer $n \geq 12$ with $(n, n-2) = 1$, the Yi's set $\dot{Y}_{n,n-2}$ is a urscm for $\mathcal{M}(\kappa)$.

5. Ursim for $\mathcal{M}(\kappa)$

For the complex case, M. Reinders [?] remarked that if F and G share ∞ IM, the analogue of Lemma ?? also is true if the factor $\left(q-\frac{3}{2}\right)$ is replaced by $\left(q-\frac{5}{3}\right)$, however, A. Boutabaa and A. Escassut [?] pointed out that Reinders' remark is false, and prove the following Lemma ??, where they missed the coefficient 2 at front of $N_2(r,F)-\overline{N}(r,F)$ and $N_2(r,G)-\overline{N}(r,G)$.

Lemma 5.1. Let F and G be non-constant meromorphic functions on κ sharing ∞ IM and let $a_1,...,a_q$ be distinct elements of κ with $q\geq 2$. Then one of the following cases must occur

1)

$$\begin{aligned}(3q-5)\{T(r,F)+T(r,G)\} \leq\ & 2\sum_{j=1}^{q}\left\{N_2\left(r,\frac{1}{F-a_j}\right)+N_2\left(r,\frac{1}{G-a_j}\right)\right\}\\ &+\sum_{j=1}^{q}\left\{\overline{N}\left(r,\frac{1}{F-a_j}\right)+\overline{N}\left(r,\frac{1}{G-a_j}\right)\right\}\\ &+2(N_2(r,F)-\overline{N}(r,F))\\ &+2(N_2(r,G)-\overline{N}(r,G))\\ &-N\left(r,\frac{1}{F'};a_1,...,a_q\right)-N\left(r,\frac{1}{G'};a_1,...,a_q\right)\\ &-4\log r+O(1).\end{aligned}$$

2) $G=AF+B$, where $A,B\in\kappa$ with $A\neq 0$, and

$$\#(\{a_1,...,a_q\}\cap\{Aa_1+B,...,Aa_q+B\})\geq 2.$$

Similarly, one has the following fact:

Lemma 5.2. Let f and g be non-constant meromorphic functions on κ satisfying

$$\overline{E}_f(\dot{F}_{n,b})=\overline{E}_g(\dot{F}_{n,b}),$$

and define

$$F=\frac{1}{F_{n,b}(f)},\ G=\frac{1}{F_{n,b}(g)}.$$

If $n\geq 16$, then $G=AF+B$, where $A,B\in\kappa$ with $A\neq 0$, and

$$\#(\{a_1,a_2,a_3\}\cap\{Aa_1+B,Aa_2+B,Aa_3+B\})\geq 2,$$

where $a_1=0,\ a_2=\frac{1}{b},\ a_3=\frac{1}{b+1}$. In particular, we have

$$E_f(\dot{F}_{n,b})=E_g(\dot{F}_{n,b}).$$

Thus Lemma ?? and Theorem ?? yield the following Boutabaa-Escassut's result [?]:

Theorem 5.1. For any integer $n\geq 16$, the Frank-Reinders' set $\dot{F}_{n,b}$ is a ursim for $\mathcal{M}(\kappa)$.

From the proof of Lemma ?? and Theorem ??, we also have

Theorem 5.2 ([?]). For any integer $n\geq 9$, the Frank-Reinders' set $\dot{F}_{n,b}$ is a ursim for $\mathcal{A}(\kappa)$.

Thus Theorem ?? gives estimates $i(\mathcal{M}(\kappa))\leq 16$ and $i(\mathcal{A}(\kappa))\leq 9$.

Lemma 5.3. Let f and g be non-constant meromorphic functions on κ satisfying

$$\overline{E}_f(\dot{Y}_{n,n-2})=\overline{E}_g(\dot{Y}_{n,n-2}),$$

and define

$$F=\frac{1}{Y_{n,n-2}(f)},\ G=\frac{1}{Y_{n,n-2}(g)}.$$

If $n\geq 19$, then $G=F$ or $G+F=\frac{1}{b}$. In particular, we have

$$E_f(\dot{Y}_{n,n-2})=E_g(\dot{Y}_{n,n-2}).$$

Lemma **??** and Theorem **??** yield the following result:

Theorem 5.3. For any integer $n \geq 19$ with $(n, n-2) = 1$, the Yi' set $\dot{Y}_{n,n-2}$ is a ursim for $\mathcal{M}(\kappa)$.

Lemma 5.4. Let f and g be non-constant entire functions on κ satisfying

$$\overline{E}_f(\dot{Y}_{n,n-1}) = \overline{E}_g(\dot{Y}_{n,n-1}),$$

and define

$$F = \frac{1}{Y_{n,n-1}(f)}, \; G = \frac{1}{Y_{n,n-1}(g)}.$$

If $n \geq 9$, then $G = F$ or $G + F = \frac{1}{b}$. In particular, we have

$$E_f(\dot{Y}_{n,n-1}) = E_g(\dot{Y}_{n,n-1}).$$

Lemma **??** and the proof of Theorem **??** yield the following result:

Theorem 5.4. For any integer $n \geq 9$, the Yi' set $\dot{Y}_{n,n-1}$ is a ursim for $\mathcal{A}(\kappa)$.

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Non-Archimedian dynamics

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Mathematics Subject: Primary 11D88, 11E95, 11Q25. Secondary 30D35.

1. Non-Archimedian analogue of Baker's theorems

Let κ be an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedian absolute value $|\cdot|$. Let f be an entire function on κ. Following Hayman's proof [**?**], we can prove the non-Archimedean analogue of Rosenbloom-Baker's theorem (cf. [**?**],[**?**]).

Theorem 1.1 (Hu [**?**]). If f is a transcendental entire function on κ, then f possesses infinitely many fixed points of exact order n, except for at most one value of n.

For the complex case, the function $f(z) = e^z + z$ has no fixed points of exact order 1, but for the non-Archimedean case, each transcendental entire function has infinitely many fixed points of exact order 1. Thus we suggest the following problem:

Conjecture 1.1. If f is a transcendental entire function on κ, then f possesses infinitely many fixed points of exact order n for all $n \geq 1$.

Following Baker [?] or Hayman [?], we also can prove

Theorem 1.2. If f is a polynomial of degree at least 2 on κ, then f has at least one fixed point of exact order n, except for at most one value of n.

Example 1.1 (cf. [?]). Take $f(z) = z^2 - z$. Soving the equations

$$f(z) = z, \quad f^2(z) = z,$$

that is,

$$z^2 - z = z, \quad (z^2 - z)^2 - (z^2 - z) = z,$$

we obtain $\mathrm{Fix}(f) = \mathrm{Fix}(f^2) = \{0, 2\}$. Thus f has no fixed points of exact order 2.

In the complex case, Example **??** is the only example of this type (up to conjugates). In fact, one has

Theorem 1.3 (cf. [?]). Let f be a polynomial of degree at least 2 on $\mathfrak{C}$ and suppose that f has no fixed points of exact order n. Then $n = 2$ and f is conjugate to $z \mapsto z^2 - z$.

Conjecture 1.2. Theorem **??** is true for polynomials of degree at least 2 on κ.

There is also a corresponding result for rational functions on $\mathfrak{C}$, namely

Theorem 1.4. Let f be a rational function of degree d on $\mathfrak{C}$, where $d \geq 2$, and suppose that f has ho fixed points of exact order n. Then (d, n) is one of the pairs

$$(2, 2), \quad (2, 3), \quad (3, 2), \quad (4, 2);$$

moreover, each such pair does arise from some f in this way.

This result also is due to Baker [?], and we conjecture it is true for rational functions on κ.

2. Montel's theorem

Definition 2.1. Suppose that a subset $D \subset \kappa$ is open. A function $f : D \longrightarrow \kappa$ is called analytic at a point $a \in D$ if there exist $\rho \in \mathfrak{R}^+ \cup \{\infty\}$ and $a_n \in \kappa$ such that $\kappa(a; \rho) = \{z \in \kappa \mid |z - a| < \rho\} \subset D$, but $\kappa[a; \rho'] - D \neq \emptyset$ for any $\rho' > \rho$, where $\kappa[a; \rho'] = \{z \in \kappa \mid |z - a| \leq \rho'\}$, and such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n, \quad z \in \kappa(a; \rho).$$

If f is analytic at every point of D, then f is said to be analytic in D. Denote by $\mathcal{H}(D)$ the set of analytic functions on D.

The disc $\kappa(a; \rho)$ in Definition **??** will be called a *maximal analytic disc of* f at a. Analytic functions on D may be referred to have *maximal analyticity* on D. The field of fractions of $\mathcal{H}(D)$ will is denoted by $\mathcal{M}(D)$. An element f in the set $\mathcal{M}(D)$ will be called a *meromorphic function* on D. If f has no poles in D, then f also is called *holomorphic*. By using a type of non-Archimedian integration introduced by Shnirelman in 1938 [?], we can give the non-Archimedian analogue of the Montel's theorem.

Theorem 2.1 (Hu [?]). Let $\mathcal{F}$ be a family of holomorphic functions in an open set D of κ. If $\mathcal{F}$ is locally uniformly bounded in D, that is, each point $z_0 \in D$ has a disc $\kappa[z_0; r] \subset D$ on which it is uniformly bounded, then $\mathcal{F}$ is equicontinuous in D.

Note that non-Archimedian Cauchy integral formula holds for Shnirelman's integral. Thus Theorem **??** yields the following criterion:

Theorem 2.2 (Hu [?]). Let $\mathcal{F}$ be a family of holomorphic functions in an open set D of $\bar{\kappa}$. If each function in $\mathcal{F}$ does not take the value 0, then $\mathcal{F}$ is spherically equicontinuous in D.

Here we used the following ultrametric

$$\chi(z,w) = \begin{cases} \frac{|z-w|}{|z|^\vee |w|^\vee} & : \quad z, w \in \kappa \\ \frac{1}{|z|^\vee} & : \quad w = \infty \end{cases} \tag{2.1}$$

on $\bar{\kappa}$, where, by definition,

$$x^\vee = \max\{1, x\}.$$

If we set

$$\frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0,$$

then we have

$$\chi\left(\frac{1}{z}, \frac{1}{w}\right) = \chi(z,w), \quad z, w \in \bar{\kappa}.$$

Definition 2.2. A family $\mathcal{F}$ of functions of an open set $D \subset \kappa$ into $\bar{\kappa}$ is called *spherically equicontinuous* or a *spherically equicontinuous family* at $z_0 \in D$ if and only if for every positive ε there exists a positive δ such that for all z in D, and for all f in $\mathcal{F}$,

$$|z - z_0| < \delta \Longrightarrow \chi(f(z), f(z_0)) < \varepsilon.$$

The family $\mathcal{F}$ is said spherically equicontinuous on D iff $\mathcal{F}$ is spherically equicontinuous at each point of D.

Corollary 2.1. Let $\mathcal{F}$ be a family of holomorphic functions in an open set D of $\bar{\kappa}$. If each function in $\mathcal{F}$ does not take a value $a \in \kappa$, then $\mathcal{F}$ is spherically equicontinuous in D.

Corollary 2.2. Let $\mathcal{F}$ be a family of meromorphic functions in an open set D of $\bar{\kappa}$. If each function in $\mathcal{F}$ does not take two distinct values a and b in $\bar{\kappa}$, then $\mathcal{F}$ is spherically equicontinuous in D.

By using the ultrametric, rational functions satisfy the following basic property:

Theorem 2.3. A rational mapping f satisfies some Lipschitz condition

$$\chi(f(z), f(w)) \leq \lambda \chi(z,w)$$

on $\bar{\kappa}$, and hence is uniformly spherically continuous on $\bar{\kappa}$.

We also can prove the following variation of Corollary **??**:

Theorem 2.4. Let $\mathcal{F}$ be a family of meromorphic functions in an open set D of $\bar{\kappa}$. Let φ_1 and φ_2 be analytic functions in D such that

$$\inf_{z,w \in D} \chi(\varphi_1(z), \varphi_2(w)) \geq \alpha > 0.$$

If each function $f \in \mathcal{F}$ satisfies

$$f(z) \neq \varphi_i(z), \quad i = 1, 2, \quad z \in D,$$

then $\mathcal{F}$ is spherically equicontinuous in D.

3. Fatou-Julia theory

In this section, we will consider a non-constant holomorphic mapping $f : D \longrightarrow D$. If $D = \kappa$, then f is an entire function (including polynomial). If $D = \bar{\kappa}$, then f is a rational function, and will be called a *rational mapping*.

Definition 3.1. A family $\mathcal{F}$ of local meromorphic functions defined on an open set D is called *normal* at $z_0 \in D$ if there exists a disc $\kappa[z_0; r] \subset D$ such that $\mathcal{F}$ is normal on $\kappa[z_0; r]$.

In the definition, we used the following notations:

Definition 3.2. Suppose that a subset $D \subset \kappa$ has no isolated points. A function $f : D \longrightarrow \kappa \cup \{\infty\}$ is called locally meromorphic if for every $a \in D$ there exist $r \in \mathfrak{R}^+$, $q \in \mathfrak{Z}_+$, and $a_n \in \kappa$ such that

$$f(z) = \sum_{n=-q}^{\infty} a_n (z-a)^n, \quad z \in D \cap \kappa[a; r].$$

In particular, f is said to be locally analytic if, for each $a \in D$, the corresponding number q is zero. Denote by $\mathrm{Mer}(D)$ the set of locally meromorphic functions on D.

Definition 3.3. A family $\mathcal{F}$ of local meromorphic functions defined on an open set D is called *normal* in D if every sequence $\{f_n\}$ in $\mathcal{F}$ contains a subsequence which is *locally spherically uniformly convergent* on D.

Obviously, the family $\mathcal{F}$ is normal on D if and only if it is normal at each point of D. Taking the collection $\{U_\alpha\}$ to be the class of all open subsets of D on which $\mathcal{F}$ is normal, this leads to the following general principle.

Theorem 3.1. Let $\mathcal{F}$ be a family of local meromorphic functions defined on an open set D. Then there is a maximal open subset $F(\mathcal{F})$ of D on which $\mathcal{F}$ is normal. In particular, if $f : D \longrightarrow D$ is a holomorphic mapping, then there is a maximal open subset $F(f)$ of D on which the family of iterates $\{f^n\}$ is normal.

The sets $F(\mathcal{F})$ and $F(f)$ in Theorem **??** is usually called *Fatou sets* of $\mathcal{F}$ and f respectively. *Julia sets* of $\mathcal{F}$ and f are defined respectively by

$$J(\mathcal{F}) = D - F(\mathcal{F}), \quad J(f) = D - F(f),$$

which are closed subsets of D. If $\mathcal{F}$ is finite, we define $J(\mathcal{F}) = \emptyset$.

Similarly, taking the collection $\{V_\beta\}$ to be the class of all open subsets of D on which $\mathcal{F}$ is spherically equicontinuous , this leads to the following general principle.

Theorem 3.2. Let $\mathcal{F}$ be a family of local meromorphic functions defined on an open set D. Then there is a maximal open subset $F_{equ}(\mathcal{F})$ of D on which $\mathcal{F}$ is spherically equicontinuous. In particular, if $f : D \longrightarrow D$ is a holomorphic mapping, then there is a maximal open subset $F_{equ}(f) = F_{equ}(f, \chi)$ of D on which the family of iterates $\{f^n\}$ is spherically equicontinuous.

Define the closed sets

$$J_{equ}(\mathcal{F}) = D - F_{equ}(\mathcal{F}), \quad J_{equ}(f) = J_{equ}(f, \chi) = D - F_{equ}(f, \chi).$$

We can show

$$F(f) \subset F_{equ}(f), \quad J_{equ}(f) \subset J(f).$$

The following result is basic:

Theorem 3.3. The sets $F = F(f)$ and $J = J(f)$ are backward invariant, that is,

$$f^{-1}(F) = F, \quad f^{-1}(J) = J. \tag{3.1}$$

Also it is easy to prove that the sets $F_{equ}(f)$ and $J_{equ}(f)$ are completely invariant.

Theorem 3.4. For each positive integer $m \geq 2$,

$$F(f) \subset F(f^m), \quad J(f^m) \subset J(f). \tag{3.2}$$

Furthermore, if $D = \bar{\kappa}$, then

$$F(f) = F(f^m), \quad J(f^m) = J(f). \tag{3.3}$$

Theorem **??** is true for sets $F_{equ}(f)$ and $J_{equ}(f)$.

Theorem 3.5. The Julia set $J(f)$ contains all repellers.

Take an open set $D \subset \bar{\kappa}$. Suppose that $f : D \longrightarrow D$ is a holomorphic mapping. Then a fixed point z_0 of f is a repeller if and only if there is a neighborhood U of z_0 such that

$$|f(z) - z_0| > |z - z_0|, \quad z \in U - \{z_0\}. \tag{3.4}$$

This is equivalent to $|f'(z_0)| > 1$. Thus if $D = \bar{\kappa}$, Theorem **??** and Theorem **??** shows that the Julia set $J(f)$ contains the closure of its set of repellers since $J(f)$ is closed. We also have the following result:

Theorem 3.6. The Fatou set $F(f)$ contains all attractors.

A fixed point x_0 of f is called an *attractor* of f if there exists a neighborhood U of x_0 such that

$$\lim_{n\to+\infty} f^n(x) = x_0 \quad \text{for all} \quad x \in U.$$

This is equivalent to $|f'(z_0)| < 1$.

Obviously, Theorem **??** holds for sets $F_{equ}(f)$. Theorem **??** also is true for $J_{equ}(f)$. We also have

Theorem 3.7. The set $F_{equ}(f)$ contains all indifferent fixed points.

4. Riemann-Hurwitz relation

Consider any function f that is non-constant and holomorphic near the point z_0 in κ. Then there exists a unique positive integer m such that the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{(z - z_0)^m}$$

is a finite non-zero constant. We denote this integer m by $\mu_f(z_0)$ and call it *mapping degree* or *valency* of f at z_0. We can prove the *Riemann-Hurwitz relation*:

Theorem 4.1. For any non-constant rational mapping f on κ,

$$\sum_{z\in\bar{\kappa}} (\mu_f(z) - 1) = 2\deg(f) - 2. \tag{4.1}$$

Corollary 4.1. A rational mapping of degree $d \geq 2$ has at most $2d-2$ critical points in $\bar{\kappa}$. A polynomial of degree $d \geq 2$ has at most $d - 1$ critical points in κ.

Corollary 4.2. Let f be a rational mapping of degree at least two and suppose that a finite set E is completely invariant under f. Then E has at most two elements.

Definition 4.1. A point z is said to be exceptional for a rational mapping f when $[z]$ is finite, and the set of such points is denoted by $\mathrm{Exc}(f)$.

Theorem 4.2. A rational mapping f of degree at least two has at most two exceptional points. If $\mathrm{Exc}(f) = \{\zeta\}$, then f is conjugate to a polynomial with ζ corresponding to ∞. If $\mathrm{Exc}(f) = \{\zeta_1, \zeta_2\}$ with $\zeta_1 \neq \zeta_2$, then f is conjugate to some mapping $z \mapsto z^d$ such that ζ_1 and ζ_2 correspond to 0 and ∞.

Theorem 4.3. Let f be a rational mapping of degree at least two. Then the set $\mathrm{Exc}(f)$ of exceptional points contains in $F(f)$.

5. Properties of the Julia set

In this section, we exhibit some results which can be easily proved by using Riemann-Hurwitz relation and Montel's theorem.

Theorem 5.1. Let f be a rational mapping with $\deg(f) \geq 2$, and suppose that E is a closed, completely invariant subset of $\bar{\kappa}$. Then either E has at most two elements and $E \subset \mathrm{Exc}(f) \subset F(f)$ or E is infinite and $J_{equ}(f) \subset E$.

By Theorem **??**, we have

Theorem 5.2. Let f be a rational mapping with $\deg(f) \geq 2$. Then either $J(f)$ (resp., $J_{equ}(f)$) is empty or $J(f)$ (resp., $J_{equ}(f)$) is infinite.

Really $J_{equ}(f)$ may be empty. In fact, if $\mathrm{Exc}(f)$ contains two points, say $f(z) = z^d$, by Corollary **??**, the family $\{f^n\}$ is spherically equicontinuous in $\bar{\kappa} - \mathrm{Exc}(f)$ because $\bar{\kappa} - \mathrm{Exc}(f)$ is completely invariant, and thus $F_{equ}(f) = \bar{\kappa}$ by using Theorem **??**. Here we do not know whether the emptiness of the Julia set $J(f)$ can be removed. For the mapping $f(z) = z^d$, we have

$$J(f) \subset \kappa\langle 0; 1\rangle,$$

but we can not confirm whether $J(f) = \kappa\langle 0; 1\rangle$ or $J(f) = \emptyset$.

Theorem 5.3. Let f be a rational mapping with $\deg(f) \geq 2$. Then either $J(f)$ (resp., $J_{equ}(f)$) $= \emptyset$ or $\bar{\kappa}$, or $J(f)$ (resp., $J_{equ}(f)$) has empty interior.

Theorem 5.4. Let f be a rational mapping with $\deg(f) \geq 2$. Then either the derived set of $J_{equ}(f)$ is empty or it is infinite , and is equal to $J_{equ}(f)$.

Theorem 5.5. Let f be a rational mapping with $\deg(f) \geq 2$. Suppose that $J_{equ}(f) \neq \emptyset$ and let D be any non-empty open set which meets $J_{equ}(f)$. Then

$$\bar{\kappa} - \mathrm{Exc}(f) \subset O^{+}(D).$$

Theorem 5.6. Let f be a rational mapping with $\deg(f) \geq 2$. Suppose that $J_{equ}(f) \neq \emptyset$, and has no isolated points. Then $J_{equ}(f)$ is contained in the derived set of the set $\mathrm{Per}(f)$ of periodic points of f. In particular, one has

$$J_{equ}(f) \subset \overline{\mathrm{Per}(f)}.$$

Theorem 5.7. Let f be a rational mapping with $\deg(f) \geq 2$ and suppose that $J_{equ}(f) \neq \emptyset$.

1) If $z \notin \mathrm{Exc}(f)$, then $J_{equ}(f) \subset \overline{O^{-}(z)}$;

2) If $z \in J_{equ}(f)$, then $J_{equ}(f) = \overline{O^{-}(z)}$.

Theorem 5.8. Let f and g be two rational mappings with $\deg(f) \geq 2$ and $\deg(g) \geq 2$. Suppose that

$$f \circ g = g \circ f, \quad J_{equ}(f) \neq \emptyset, \quad J_{equ}(g) \neq \emptyset.$$

Then $J_{equ}(f) = J_{equ}(g)$.

Conjecture 5.1. Let f be a rational mapping with $\deg(f) \geq 2$. Then $J_{equ}(f)$ is the closure of the repelling periodic points of f.

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Nevanlinna theory and algebraic differential equations

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subjclass: 30D35, 34A20, 34C10, 39A10

keywords: Nevanlinna theory, the Malmquist–Yosida theorem, Complex differential equations

In this note we discuss meromorphic solutions of some algebraic differential equations. The global theory of the complex differential equation is now an active area of mathematics. It is likely to be studied in many different fashions. Through out this note "meromorphic" means meromorphic in the complex plane. The matters in this note have been investigated with the aid of the Nevanlinna theory [**?**]. I mention here that we have now an exposition [**?**], in which we can study the Nevanlinna theory and complex differential equations. The celebrated Malmquist theorem was generalized via the Nevanlinna theory by Yosida [**?**]. It can be said that this is the first application of this theory to the non-linear differential equations. A systematic study of the implication of this theory for complex differential equations was undertaken by Wittich in 1950's. During the last two decades the global theory of complex differential equation became more popular. In 1970's several mathematicians contributed to the research of this area and gave directions e.g., Bank, Hille [**?**] and Laine. The precise theorem for the binomial equation by Steinmetz [**?**], and Bank and Kaufman [**?**] would be regarded as the beginning of the next stage.

We mention generalizations of the Malmquist–Yosida–Steinmetz type theorem considering the existence, or nonexistence theorems of admissible solutions. Roughly speaking admissible solutions are transcendental elements over a class of coefficients of differential equations. We now define the notions "small" and "admissible". A meromorphic function $a(z)$ is *small* with respect to $f(z)$ if $T(r,a) = S(r,f)$. Below, $\mathcal{N} = \{a(z)\}$ denotes a given finite collection of meromorphic functions. A transcendental meromorphic function $f(z)$ is *admissible* with respect to $\mathcal{N}$ if $T(r,a) = S(r,f)$ for all $a(z) \in \mathcal{N}$. Let $\Omega(z,w,w',\dots,w^{(n)})$ be a differential polynomial in w with meromorphic coefficients and let $\mathcal{N}$ be the collection of the coefficients of Ω. A meromorphic solution $w(z)$ of the equation

$$\Omega(z,w,w',\dots,w^{(n)}) = 0 \tag{0.1}$$

is an *admissible solution* if $w(z)$ is admissible with respect to $\mathcal{N}$.

The precious result for binomial equation having an admissible solution is obtained due to He and Laine [?]. Other results on an existence of admissible solution of algebraic equation (??) can be found in e.g., Laine [?], Chapters 6–13. However we do not have many articles in which authors treated a relation between admissible solutions. A question should be discussed, for instance, under what conditions $T(r,f) = T(r,g) + O(1)$ occurs for admissible solutions f and g. To pose a question more detail, we define some notations below. Let f and g denote meromorphic functions. We write $f \sim g$ if there exists a set $E \subset \mathfrak{R}^+$ of finite linear measure such that $T(r,f) = O(T(r,g))$ and $T(r,g) = O(T(r,f))$ as $r \to \infty$, $r \in \mathbb{R}^+ \setminus E$. We denote by $\mathcal{M}(\mathbb{C})$ the set of all meromorphic function on $\mathbb{C}$. Define the set of meromorphic solutions of (??)

$$\mathfrak{S}_\Omega = \{f \in \mathcal{M}(\mathbb{C}) \mid \Omega(z,f(z),f'(z),\dots,f^{(n)}(z)) = 0\}$$

and $\kappa_\Omega = \#(\mathfrak{S}_\Omega/\sim)$. Here we do pose a question:

Question 1. *Find the integer κ_Ω for a given differential equation $\Omega(z,f,f',\dots,f^{(n)}) = 0$.*

A simple example is that $\kappa_\Omega = 2$ when $\Omega = w' - (w^2+1)$. In fact, we have that $\mathfrak{S}_\Omega = \{\pm i, \tan(z+c), c \in \mathbb{C}\}$. Clearly $T(r,\tan(z+c_1)) \sim T(r,\tan(z+c_2))$.

We remark that κ_Ω may be ∞. For example, meromorphic functions $\dfrac{e^{z^k}}{k}$, $k \in \mathbb{N}$ satisfy

$$\Omega(z,f,f',f'',f''') = f(f')^3 - z(f')^4 - f^2f'f'' + zf(f')^2f'' + zf^2(f'')^2 - zf^2f'f''' = 0,$$

which implies that $\kappa_\Omega = \infty$.

At the end of this note, we state results for binomial equations in [?], [?]. The equations

$$(w')^2 = A_1(z)(w^2-1), \tag{0.2}$$

$$(w')^2 = A_2(z)(4w^3 - g_2w - g_3), \tag{0.3}$$

where $A_j(z)$, $j = 1, 2$ are rational functions, g_2, g_3, $27g_3^2 - g_2^3 \neq 0$ are constants, have the following properties: If there exist two transcendental meromorphic solutions f and g to (??), then we have $T(r,f) \sim T(r,g)$, and the same assertion holds for the equation (??). This concludes that κ_Ω is at most 3, for the equations (??) and (??).

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P -adic analysis based on Dirichlet space theory

Hiroshi Kaneko

§1. Recurrent and transient criteria for a process of jump type

In this article, we deal with a regular Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$ which has a representation

$$\mathcal{E}(u,v)=\int_{X\times X}(u(x)-u(y))(v(x)-v(y))J(dx,dy)$$

with the symmetric non-negative Radon measure J on $X\times X$ satisfying $J(\{(x,y)|x=y\})=0$

Assumption. *We assume that there exists an exhaustion function ρ such that*

(i): $\inf\rho=0$ *and* $\sup\rho>1$,
(ii): $1_{B(s)}\in\mathcal{F}$ *for* $\forall B(s)=\{\rho<s\}$.

Then, we can set

$$j(r,s)=\mathcal{E}(1_{B(r)},1_{B(s)})=2J(B(r),B(s)^c)<\infty$$

for $s\ge r$. Let Δ denote a division $\Delta:1=\xi_0<\xi_1<\cdots<\xi_N=R$ of interval $[1,R]$ and $\Sigma=\{\sigma_n\}_{n=0}^N$ denote a positive sequence.

Theorem 1. *If for any Δ there exist sequences $\{f_\Delta(n)\}_{n=0}^N$ and $\{g_\Delta(n)\}_{n=0}^N$ such that $j(\xi_n,\xi_m)\le f_\Delta(n)g_\Delta(m)$ $(n,m=0,\cdots N)$ and if*

$$\sup_R\sup_\Delta\sum_{k=0}^{N-1}\frac{1}{g_\Delta(k)}\Big(\frac{1}{f_\Delta(k)}-\frac{1}{f_\Delta(k+1)}\Big)=\infty,$$

then $(\mathcal{E},\mathcal{F})$ is recurrent.

Proof. One can pick out a positive sequence $\{\frac{1}{\sigma_n}\}_{n=0}^N$ defined as

$$\begin{cases}\dfrac{1}{g_\Delta(n)}\Big(\dfrac{1}{f_\Delta(n)}-\dfrac{1}{f_\Delta(n+1)}\Big), & n=0,\cdots,N-1\\ \dfrac{1}{f_\Delta(N)g_\Delta(N)}, & n=N\end{cases}$$

Then, it turns out that

$$\frac{1}{f_\Delta(k)}=\sum_{\ell=k}^{N}\frac{g_\Delta(\ell)}{\sigma_\ell}\qquad (n=0,1,\cdots,N).$$

Set $C = \frac{1}{\sigma_0} + \frac{1}{\sigma_1} + \cdots + \frac{1}{\sigma_N}$, $p_n = \frac{1}{C\sigma_n}$ and $U = \sum_{k=0}^{N} p_k 1_{B(\xi_k)}$. Then, one obtain

$$
\begin{aligned}
\mathcal{E}(U,U) &= \sum_{k=0}^{N} p_k^2 \mathcal{E}(1_{B(\xi_k)}, 1_{B(\xi_k)}) \\
&\quad + 2\sum_{k=0}^{N} p_k \sum_{\ell=k+1}^{N} p_\ell \mathcal{E}(1_{B(\xi_k)}, 1_{B(\xi_\ell)}) \\
&\leq 2\sum_{k=0}^{N} p_k \sum_{\ell=k}^{N} p_\ell j(\xi_k, \xi_\ell) \\
&= \frac{2}{C^2} \sum_{k=0}^{N} \frac{1}{\sigma_k} \sum_{\ell=k}^{N} \frac{1}{\sigma_\ell} f_\Delta(k) g_\Delta(\ell) \\
&\leq \frac{2}{C^2} \sum_{k=0}^{N} \frac{1}{\sigma_k} = \frac{2}{C}.
\end{aligned}
$$

□

In what follows, we suppose that there exists an at most countable ρ-level sets $\{\rho = a_k\}_{k=-\infty}^{\infty}$ whose union $\cup_{k=-\infty}^{\infty}\{\rho = a_k\}$ covers underlying space X.

Theorem 2. *Suppose that there exists a decreasing function f_R such that*

$$\mathbb{C}(\overline{B(1)}; B(R)) = \mathcal{E}(f_R(\rho), f_R(\rho))$$

for any $R > 1$. If for any Δ there exist positive sequences $\{f_\Delta(n)\}_{n=0}^{N}$ and $\{g_\Delta(n)\}_{n=0}^{N}$ such that $j(\xi_n, \xi_m) \geq f_\Delta(n) g_\Delta(m)$ $(n, m = 0, \cdots N)$ and

$$\sup_R \sup_\Delta \sum_{n=0}^{N} \frac{1}{g_\Delta(n) f_\Delta(n)} < \infty,$$

then $(\mathcal{E}, \mathcal{F})$ is transient.

Proof. For $\Sigma = \{\sigma_n\}_{n=0}^{N}$ and $\Delta = \{\xi_n\}_{n=0}^{N}$, set $C = \frac{1}{\sigma_0} + \frac{1}{\sigma_1} + \cdots + \frac{1}{\sigma_N}$, $p_n = \frac{1}{C\sigma_n}$ and $U = \sum_{k=0}^{N} p_k 1_{B(\xi_k)}$. The assumption of the existence of the functions f_R in Theorem 2 implies

$$\mathbb{C}(\overline{B(1)}; B(R)) = \inf_{\Sigma, \Delta \text{ with } \xi_N = R} \mathcal{E}(U,U).$$

Consider the increasing function $F(t) = \mathcal{E}(U \wedge t, U \wedge t)$ in variable $t \in (0,1)$, then it follows from $F(\sum_{k=n}^{N} p_k) = \sum_{k,k'=n}^{N} p_k p_{k'} \mathcal{E}(1_{B(\xi_k)}, 1_{B(\xi_{k'})})$ that

$$F(\sum_{k=n}^{N} p_k) - F(\sum_{k=n+1}^{N} p_k) \geq \sum_{k=n}^{N} p_k p_n j(\xi_k, \xi_n).$$

The inequality (iv) in §1 applied to justify Hellinger integral implies

$$\begin{aligned}\frac{1}{\mathcal{E}(U,U)} &= \frac{1}{F(1)}\\ &\le \sum_{n=0}^{N-1} \frac{p_n^2}{F(\sum_{k=n}^{N} p_k) - F(\sum_{k=n+1}^{N} p_k)}\\ &\le \sum_{n=0}^{N-1} \frac{1}{\sum_{k=n}^{N-1} \frac{p_k}{p_n} j(\xi_n,\xi_k)}\\ &= \sum_{n=0}^{N-1} \frac{1}{\sum_{k=n}^{N-1} \frac{\sigma_n}{\sigma_k} j(\xi_n,\xi_k)}\\ &\le \sum_{n=0}^{N} \frac{1}{j(\xi_n,\xi_n)} \le \sum_{n=0}^{N} \frac{1}{g_\Delta(n) f_\Delta(n)} < \infty.\end{aligned}$$

□

§2. Spatially homogeneous process on $\mathfrak{Q}_p$

Albeverio and Karwowski showed that each sequence $A = \{a(m)\}_{m=-\infty}^{\infty}$ satisfying

(1): $a(m) \ge a(m+1)$

(2): $\lim_{m\to\infty} a(m) = 0$,

(3): $\lim_{m\to-\infty} a(m) > 0$ or $= \infty$

defines the symmetric process on $\mathfrak{Q}_p$ associated with the regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathfrak{Q}_p;\mu)$ uniquely determined by

$$\begin{aligned}&\mathcal{E}(1_{B(x,p^K)}, 1_{B(y,p^L)})\\ &= -2J(B(x,p^K), B(y,p^L))\\ &= -\frac{p^{K+L-m+1}}{(p-1)}(a(m-1)-a(m)),\end{aligned}$$

where $\mathrm{dist}(B(x,p^K), B(y,p^L)) = p^m$.

Theorem (Yasuda). *$(\mathcal{E},\mathcal{F})$ is recurrent, if and only if* $\displaystyle\sum_{n=0}^{\infty} \frac{1}{p^n a(n)} = \infty.$

Proof. Thanks to the equality

$$\begin{aligned}j(p^n,p^m) =& \mathcal{E}(1_{B(0,p^n)}, 1_{B(0,p^m)})\\ =& 2J(B(0,p^n), B(0,p^m)^c)\\ =& p^n a(m) \quad (m \ge n),\end{aligned}$$

the assertion immediately follows from the theorems in the previous section. □

§3. Spatially inhomogeneous process on $\mathfrak{Q}_p$

Since there exists the additive characters χ_p of the field $\mathfrak{Q}_p$,

$$\chi_p(x+y) = \chi_p(x)\chi_p(y) \quad \text{for any } x,y \in \mathfrak{Q}_p,$$

in terms of the Fourier transformation, the α-stable process $\{X_t\}$ is uniquely characterized as the case that

$$E(\chi_p(-xX_t)) = \exp\,(-t\|x\|_p^\alpha) \qquad (\alpha > 0).$$

On the other hand, the α-order derivative $D^{\alpha}u$ of locally constant function u is justifiable through pseudo-differential operators and actually written as

$$\begin{aligned} D^{\alpha}u(x) &= \int_{\mathfrak{Q}_p} \|\xi\|_p^{\alpha}\hat{u}(\xi)\chi_p(-\xi x)\mu(d\xi) \\ &= \frac{p^{\alpha}-1}{1-p^{-\alpha-1}}\int_{\mathfrak{Q}_p} \frac{u(x)-u(y)}{\|x-y\|_p^{\alpha+1}}\mu(dy), \end{aligned}$$

where $\hat{u}(x) = \int_{\mathfrak{Q}_p} \chi_p(-\xi x)u(x)\mu(dx)$.

It is not difficult to see that the pre-Dirichlet form $\int_{\mathfrak{Q}_p} D^{\alpha/2}u(x)D^{\alpha/2}u(x)\mu(dx)$ is closable and its smallest closed extension generates Albeverio and Karwowski's symmetric Hunt process associated with the parameter sequence $a(m) = C_p p^{-\alpha m}$. Therefore, when $\alpha \geq 1$, the α-stable process is recurrent and when $\alpha < 1$, the α-stable process is transient.

Even though we simply pick out a positive bounded function σ on $\mathfrak{Q}_p$, $\int_{\mathfrak{Q}_p} D^{\alpha/2}u(x)D^{\alpha/2}u(x)\sigma(x)\mu(dx)$ is no longer Markovian in general. Nonethless, if σ is a positive function described as $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 \in L^{\infty}(\mathfrak{Q}_p;\mu)$ and $\sigma_2 \in L^2(\mathfrak{Q}_p;\mu)$ and if the α-order derivative is also replaced by $D^{\alpha,\sigma}u(x) = \frac{p^{\alpha}-1}{1-p^{-\alpha-1}}\int_{\mathfrak{Q}_p} \frac{u(x)-u(y)}{\|x-y\|_p^{\alpha+1}}\sigma(y)\mu(dy)$, then the form

$$\mathcal{E}^{(\alpha,\sigma)}(u,v) = \int_{\mathfrak{Q}_p} D^{\alpha/2,\sigma}u(x)D^{\alpha/2,\sigma}u(x)\sigma(x)\mu(dx)$$

defined for locally constant functions u, v is a closable bilinear form in $L^2(\mathfrak{Q}_p;\mu)$.

Theorem 3. *The pair $(\mathcal{E}^{(\alpha,\sigma)}, \mathcal{F}^{(\alpha,\sigma)})$ of the bilinear form $\mathcal{E}^{(\alpha,\sigma)}$ and its domain $\mathcal{F}^{(\alpha,\sigma)}$ is regular Dirichlet form on $L^2(\mathfrak{Q}_p;\mu)$.*

Theorem 4.

(i): *$\alpha \geq 1$ and $\sigma \in L^{\infty}(\mathfrak{Q}_p;\mu) \Rightarrow (\mathcal{E}^{(\alpha,\sigma)}, \mathcal{F}^{(\alpha,\sigma)})$ is recurrent,*

(ii): *$\alpha < 1$, there exists a decreasing function f_{p^N} such that $\mathbb{C}(\overline{B(1)}; B(p^N)) = \mathcal{E}(f_{p^N}(\|x\|_p), f_{p^N}(\|x\|_p))$ and there exists a positive constant δ and a function ρ such that $\sigma(x) = \rho(\|x\|_p) \geq \delta$ and on $\mathfrak{Q}_p$ $\Rightarrow (\mathcal{E}^{(\alpha,\sigma)}, \mathcal{F}^{(\alpha,\sigma)})$ is transient.*

Proof. It is not difficult to see

$$\begin{aligned} &\mathcal{E}^{(\alpha,\sigma)}(1_{B(x,p^K)}, 1_{B(x,p^L)}) \\ &=C_p^2\Big(\sum_{\nu=L+1}^{\infty}\frac{b_{x,\nu}}{p^{(\alpha/2+1)\nu}}\sum_{\nu=L+1}^{\infty}\frac{b_{x,\nu}}{p^{(\alpha/2+1)\nu}} \\ &\quad+\sum_{\nu=L+1}^{\infty}\frac{b_{x,\nu}}{p^{2(\alpha/2+1)\nu}}\,\bar{b}_{x,L}\Big)\,\bar{b}_{x,K}, \end{aligned}$$

where

$$\begin{aligned} \bar{b}_{x,K} &= \int_{\{z\in\mathfrak{Q}_p \mid \operatorname{dist}(z,x)\leq p^K\}} \sigma(z)\mu(dz) \\ \bar{b}_{x,L} &= \int_{\{z\in\mathfrak{Q}_p \mid \operatorname{dist}(z,x)\leq p^L\}} \sigma(z)\mu(dz) \\ b_{x,\nu} &= \int_{\{z\in\mathfrak{Q}_p \mid \operatorname{dist}(z,x)= p^\nu\}} \sigma(z)\mu(dz) \quad (\nu > L). \end{aligned}$$

If σ is bounded, there exists a positive constant $\Lambda(p,\alpha)$ such that $\Lambda(p,\alpha)p^K p^{-\alpha L}$ is a majorant of the right-hand side. The recurrence of the Hunt process follows from Theorem 1 combining with the assumption $\alpha \geq 1$.

On the other hand, if there exists a positive constant δ such that $\sigma(x) \geq \delta$, the right-hand side has a lower bound $\lambda(p,\alpha)p^K p^{-\alpha L}$ with some positive constant $\lambda(p,\alpha)$. Thanks to the assumption on ρ, there exists a decreasing function f_{p^N} such that $\mathbb{C}(\overline{B(1)}; B(p^N)) = \mathcal{E}(f_{p^N}(\|x\|_p), f_{p^N}(\|x\|_p))$. Combining Theorem 2 with $\alpha < 1$, we obtain the transience. □

§4. Other result on a spatially homogenuous Hunt process on $\mathfrak{Q}_p$

Theorem 5. *If u is a real valued locally bounded $\mathcal{E}$-harmonic function, then u is constant.*

Proof. Because of the translation invariance of the random walks on $\mathfrak{Q}_p$, it is sufficient to prove $u(x) = u(y)$ by assuming that $\|x\|_p = \|y\|_p$. Let $\{X_t\}$ be a random walk on $\mathfrak{Q}_p$ characterized by A with starting point x. If we construct a random walk $\{Y_t\}$ on $\mathfrak{Q}_p$ starting y which is also characterized by A and couples with $\{X_t\}$ in a finite time, i.e., $\tau_{X,Y} = \inf\{t > 0; X_t = Y_t\}$ satisfies $P(\tau_{X,Y} < \infty) = 1$, then the well-known coupling method implies that

$$\begin{aligned}|u(x) - u(y)| &\leq E[|u(X_{\tau_{X,Y}\wedge t}) - u(Y_{\tau_{X,Y}\wedge t})|] \\ &\leq 2 \sup_{\|z\|_p \leq \|x\|_p} |u(z)| \times P(\tau_{X,Y} > t) \to 0 \\ &\quad \text{as } \ t \to \infty\end{aligned}$$

and consequently $u(x) = u(y)$.

We may assume that $\|x\|_p = p^{-i_0}$ and x, y have power series expressions

$$x = p^{i_0} + \sum_{i=i_0+1}^{\infty} \gamma_{x,i} p^i$$

$$y = (p-1)p^{i_0} + \sum_{i=i_0+1}^{\infty} \gamma_{x,i} p^i.$$

and define $\{Y_t\}$ as

$$\begin{cases} X_t + (p-2)p^{i_0}, & \text{if } \tau_1 > t \geq 0, \\ X_t - (p-2)p^{i_0}, & \text{if } \tau_2 > t \geq \tau_1 \text{ and} \\ & \|X_t - (p-1)p^{i_0}\|_p \leq p^{-(i_0+1)}, \\ X_t + (p-2)p^{i_0}, & \text{if } \tau_2 > t \geq \tau_1 \text{ and} \\ & \|X_t - p^{i_0}\|_p \leq p^{-(i_0+1)}, \\ X_t, & \text{if } t \geq \tau_2, \end{cases}$$

where

$$\begin{aligned}\tau_1 &= \inf\{t > 0 \mid \|X_t - p^{i_0}\|_p \geq p^{-i_0}\} \text{ and} \\ \tau_2 &= \inf\{t > 0 \mid \|X_t - (p-1)p^{i_0}\|_p \geq p^{-i_0}, \\ &\qquad \text{and } \|X_t - p^{i_0}\|_p \geq p^{-i_0}\}.\end{aligned}$$

Then we can obtain a random walk $\{Y_t\}$ on $\mathfrak{Q}_p$ which enjoys all properties required. □

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Survey on p-adic nevanlinna theory and recent articles

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Keywords: p-adic Nevanlinna theory, hyperbolic surfaces, unique range set
AMS Classification: 32H20, 11G

§1. Introduction

The Nevanlinna theory studies the problem "*how many times does a meromorphic fonction $f(z)$ take the value $a \in \mathbb{P}^1$?*", on the other words, *how to measure the set $f^{-1}(a)$?*.

The first results in this direction belong to Hadamard.

Hadamard's theorem. *Let $f(z)$ be a holomorphic function in $\mathbb{C}$. Then*

$$\text{(the number of zeros of } f \text{ in } \{|z| \le r\}) \le \log \max_{|z|\le r} |f(z)| + O(1)$$

where O(1) depends on f, but not on r

This result is not yet "ideal" because of the following two deficiencies.

a) When f is a meromorphic function, we have the infinity in the right hand side of the inequality, and in this case, the Hadamard theorem does not give an estimation of the number of zeros of f.

b) There are functions, for example, $f(z) = e^z$, which do not have the zeros, and in this case, Hadamard's inequality become trivial.

For eliminating these deficiencies, R. Nevanlinna defines the following functions.

1.1.*Counting function* . Let $a \in \mathbb{C}$. We set

$$n(a, r) = \sharp\{\text{zeros of } f(z) - a \text{ in } \{|z| < r\}, \text{ with multiplicity}\}$$

$$N(a, r) = \int_o^r \frac{n(a, t) - n(a, 0)}{t} dt + n(a, 0) \log r.$$

1.2. *Characteristic function* . Instead of $\log_{|z|\leq r}|f(z)|$ we consider the function

$$T(r)=\frac{1}{2\pi}\int_{o}^{2\pi}\log^{+}|f(re^{i\theta})|d\theta+N(\infty,r)$$

By using these two functions one get the following inequality:

$$N(0,r)\leq T(r)+O(1)$$

This inequality is valid and non-trivial for meromorphic functions.

For eliminating the second deficiency one notices that while the function e^{z} does not have the zeros, it takes many values "approche to zero". Then one can "measure" this set by using

1.3.*Mean proximity function.*

$$m(a,r)=\frac{1}{2\pi}\int_{o}^{2\pi}\log^{+}|\frac{1}{f(re^{i\theta})-a}|d\theta,$$

(where $\log^{+}=\max(0,\log)$).

It is clair that $m(a,r)$ become "biger" when $f(z)$ approches to a.

There are two "Main Theorems" and defect relations which occupy a central place in Nevanlinna theory. 1.4. *First Main theorem of Nevanlinna. There is a function $T(f)$ such that for any $a\in\mathbb{P}^{1}$ we have*

$$m(a,r)+N(a,r)=T(r)+O(1).$$

As $T(r)$ does not depend on a, one can say that a meromorphic function takes every value $a\in\mathbb{P}^{1}$ (and "approche to a" values) with the same frequency. 1.5. *Second Main theorem of Nevanlinna For arbitrary $q\in\mathbb{N}$ and distinct points $a_i\in\mathbb{P}^{1},\ i=1,\ldots,q$,*

$$\sum_{i=1}^{q}m(a_i,r)<2T(r)+O(\log(rT(r)))$$

where the inequality is valid beside a set of finite measure.

And then, if we set

$$\delta(a)=\lim_{r\to\infty}\frac{m(a,r)}{T(r)}$$

we have

$$\sum_{a\in\mathbb{P}^{1}}\delta(a)\leq 2 \tag{1}$$

$\delta(a)$ is called the defect value at the point a and (1) is "defect relation".

Precisely, $\delta(a)=0$ for almost all a (except a countable set).

1.6. *Why study p-adic Nevanlinna Theory.*

In the famous paper "De la métaphysique aux mathématiques" ([W]) A. Weil discussed about the role of analogies in mathematics. For illustrating he analysed a "metaphysics" of Diophantine Geometry: the resemblance between *Algebraic Numbers* and *Algebraic Functions*. However, the striking similarily between Weil's theory of heights and Cartan's Second Main Theorem for the case of hyperplanes is pointed out by P. Vojta only after 50 years! P. Vojta observed the resemblance between *Algebraic Numbers* and *Holomorphic Functions*, and gave a "dictionary" for translating the results of Nevanlinna Theory in the one-dimensional case to Diophantine Approximations. Due to this dictionary one can regard Roth's Theorem as an analog of Nevanlinna Second Main Theorem. P. Vojta has also made quantitative conjectures which generalize Roth's theorem to higher dimensions. One can say that P. Vojta proposed a "new metaphysics" of Diophantine Geometry: Arithmetic Nevanlinna Theory in higher dimensions. On the other hand, in the philosophy of Hasse-Minkowski principle one hopes to have an "arithmetic result" if one had have it in p-adic cases for all prime numbers p, and in the real and complex cases. Hence one would naturally have interest to determine how Nevanlinna Theory would look in the p-adic case.

§2. Two main theorems

Let p be a prime number, Q_p the field of p-adic numbers, and C_p the p-adic completion of the algebraic closure of Q_p. The absolute value in Q_p is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on C_p which extends ord_p.

We define the counting function in exactly the same way as in classical Nevanlinna theory. That is, given a meromorphic function f, we let $n(f, \infty, r)$ denote the number of poles in $\{|z| \leq r\}$, and we let

$$N(f, \infty, r) = \int_o^r [n(f, \infty, t) - n(f, \infty, 0)] \frac{dt}{t} + n(f, \infty, 0) \log r =$$
$$\sum_{|z| \leq r, z \neq 0} \max\{0, -ord_z f\} \log \frac{r}{|z|} + \max\{0, -ord_o f\} \log r,$$

where $ord_z f$ denotes the order of vanishing of f at z, negative numbers indicating poles. Counting functions for other values are defined similarly.

For the mean proximity function, note that the norms $|\ |_r$ are multiplicative on entire functions and they extend to meromorphic functions. Thus we define

$$m(f, \infty, r) = \log |f|_r,$$

and for finite a,

$$m(f, a, r) = \log \frac{1}{|f - a|_r}.$$

Note that there is no need to do any sort of "averaging" over $|z| = r$, since by the strong maximum modulus principle, for suitable generic z with $|z| = r$, we know $|f(z)| = |f|_r$. Finally, just as in classical Nevanlinna theory, the characteristic function is given by

$$T(f, a, r) = m(f, a, r) + N(f, a, r).$$

The properties of the valuation polygon imply that

$$\log |f|_r = \sum_{|z| \leq r, z \neq 0} (ord_z f) \log \frac{r}{|z|} + (ord_o f) \log r + 0(1),$$

where the 0(1) term depends on the size of the first non-zero coefficient in the Laurent axpansion for f at 0. This is of course a non-Archimedean Jensen formula and can be written

$$m(f, \infty, r) + N(f, \infty, r) = m(f, 0, r) + N(f, 0, r) + 0(1),$$

from which the non-Archimedean analog to *Nevanlinna first Main Theorem* easily follows.

The Second Main Theorem. Let f be a non-constant meromorphic function on $\mathbb{C}_p$, and let $a_1, a_2, \ldots, a_q$ be q distinct points on $\mathbb{C}_p \cup \{\infty\}$. Then, for all $r \geq r_o > 0$,

$$(q - 2)T(f, r) - \sum_{j=1}^{q} N(f, a_j, r) - N_{Ram}(f, t) \leq -\log r + 0(1),$$

where

$$N_{Ram}(f, a) = N(f', 0, r) + 2N(f, \infty, r) - N(f', \infty, r)$$

measures the grouth of the ramification of f, and the 0(1) term depends only on the a_j, the function f, and the number r_o.

Corollary. *Let f and $a_1, a_2, \ldots, a_q$ be as in the Theorem. Then for all $r \geq r_o$*

$$(q - 2)T(f, r) \leq \sum_{j=1}^{q} N_1(f, a_j, r) - \log r + 0(1),$$

where $N_1(f, a_j, r)$ denotes a modified counting function in that each point where $f = a$ is counted only with multiplicity 1, and again 0(1) term depends on the a_j, f, r_o.

§3. the height function

In the p-adic case we can use so-called "the height function". Notice that, the Newton polygon gives expression to one of the most basic deffrences between p-adic analytic functions and complex analytic functions. Namely, the modulus of a p-adic analytic function depends only on the modulus of the argument, except at a discrete set of values of the modulus of argument. This fact often makes it easier to prove the p-adic analogs of classical results. Now we give the definition of the height function.

Let $f(z)$ be an analytic function on $\mathbb{C}_p$, it is represented by a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

For each n we draw the graph Γ_n which depicts $v(a_n z^n)$ as a function of $v(z) = t$. This graph is a straight line with slope n. Since we have $\lim_{n\to\infty} \{v(a_n) + nt\} = \infty$ for all t it follows that for every t there exists an n for which $v(a_n) + nt$ is minimal. Let $h(f,t)$ denote the boundary of the intersection of all of the half-planes lying under the lines Γ_n. Then in any finite segment $[r,s]$, there are only finitely many Γ_n which appear in $h(f,t)$. Thus, $h(f,t)$ is a polygonal line. This line is what we call *the height* of the function $f(z)$. The points t at which $h(f,t)$ has vertices are called the critical points of $f(z)$. A finite segment contains only finitely many critical points. If t is a critical point, then $v(a_n) + nt$ attains its minimum at least at two values of n. If $v(z) = t$ is not a critical point, then $|f(z)| = p^{-h(f,t)}$.

The height of a function $f(z)$ gives complete information about the number of zeros of $f(z)$. Namely, f has zeros when $v(z) = t_i$ (a critical point) and the number of zeros of f such that $v(z) = t_i$ is equal to the difference $n_{i+1} - n_i$ between slopes of $h(f,t)$ at t_{i-0} and t_{i+0}

For a meromorphic function $f = \dfrac{\phi}{\psi}$, the height of f is defined by $h(f,t) = h(\phi,t) - h(\psi,t)$. We also use the notation

$$h^+(f,t) = -h(f,t).$$

Then we have:

Theorem 3.1. *Let f be a meromorphic function and let $a_1, a_2, \dots, a_q$ be q distinct points in $\mathbb{C}_p \cup \{\infty\}$. Then for t sufficiently small,*

$$(q-2)h^+(f,t) \le \sum_{j=1}^{q} N_1(f,a_j,t) + t + 0(1),$$

where $N_1(f,a,t)$ denotes a modified function in that each point where $f = a$ is counted only with multiplicity 1, and the 0(1) is a bounded value as $t \longrightarrow -\infty$.

The height function is applied to the interpolation problem (see [K1]). Let $u = \{u_1, u_2, \dots\}$ be a sequence of points in $\mathbb{C}_p$. In that follows we shall only consider sequences u for which the numbers of points u_i satisfying $v(u_i) \ge t$ is finite for every t. We shall always assume that $v(u_i) \ge v(u_{i+1})$, $(i = 1, 2, \dots)$.

Definition 3.2 The sequence $u = \{u_i\}$ is called an *interpolating sequence* of f if the sequence of interpolating polynomials for f on u converges to f.

For every sequence u we define a holomorphic function Φ_u as follows. We set

$$N_u(t) = \#\{u_i | v(u_i) \ge t\}.$$

We can write the sequence u in the form:

$$u = \{u_1, u_2, \dots, u_{n_1}, u_{n_1+1}, \dots, u_{n_2}, \dots\},$$

where

$$v(u_i) = t_k \ \text{ for } n_{k-1} < i \le n_k$$

(we take $u_o = 0$), and

$$\lim_{k\to\infty} t_k = -\infty.$$

We choose a sequence a_k with the property

$$v(a_o) = -n_1 t_1, \ v(a_{k+1}) = v(a_k) + (n_k - n_{k+1})t_{k+1}, \ (k = 1, 2, \dots).$$

We set

$$\Phi_u(z) = 1 + \sum_{k=1}^{\infty} a_k z^{n_k}.$$

Then the series converges for $z \in \mathbb{C}_p$ and determines an analytic function $\Phi_u(z)$ on $\mathbb{C}_p$, for which the number of zeros in each region $\{z|v(z) > t\}$ is equal to $N_u(t)$, and

$$h(\Phi_u, t) = \int_{\infty}^{t} N_u(t)dt.$$

Theorem 3.3. *The sequence $u = \{u_i\}$ is an interpolating sequence of the function $f(z)$ if and only if*

$$\lim_{t\to\infty} \{h(f,t) - h(\Phi_u, t)\} = \infty.$$

Remark 3.4 This is the first interpolation theorem for p-adic analytic functions not necessarily bounded. A similar theorem for analytic functions in the unit disc implies that the p-adic L-functions associated to modular forms are uniquelly defined by the values on Dirichlet characters (see [K2]).

Remark 3.5 We can use the interpolation theorem to recover a p-adic meromorphic function if we know the preimages (with multiplicity) of to points (see [K3]).

For *high dimensions*, as well as in the complex case, instead of the study the preimage of a point, we should consider the preimage of a divisor of codimension one. The reason is that in the p-adic case there exist also Fatou-Bieberbach domains (see [S]).

Now let $f = (f_1, \dots, f_{n+1} : C_p \to P^n(C_p)$ be a p-adic holomorphic curve, where the functions f_j have no common zeros.

Definition 3.6 The height of the holomorphic curve f is defined by

$$h(f,t) = \min_{1\le j\le n+1} h(f_j, t),$$

where $h(f_j, t)$ is the height of p-adic holomorphic function on $\mathbb{C}_p$.

Notices that the height of a curve is well defined up to a bounded value.

The following theorem is a p-adic version of the Second Main Theorem in the case of holomorphic curves.

Theorem 3.7 ([KT]). *Let $H_1, \dots, H_q$ be q hyperplanes in general position, and let f be a non-degenerate holomorphic curve in $P^n(C_p)$. Then we have*

$$(q-n-1)h^+(f,t) \le \sum_{j=1}^{q} N_n(f.H_j, t) + \frac{n(n+1)}{2}.t + 0(1),$$

where 0(1) is bounded when $t \to -\infty$ and $h^+(h,t) = -h(f,t)$.

Cherry and Ye [CY] extend the theorem to several variables. Moreover,they considered the case of degenerate curves by using so called Nochka's weights ([N]). Recenly, Hu and Yang [HY] obtain similar results for moving targets.

For the case of hypersurfaces we have the following

Theorem 3.8 ([KA1]). *Let $H_1, \dots, H_q$ be hypersurfaces of degree d in $\mathbb{P}^n(\mathbb{C}_p)$ in general position. Let f be a non-degenerate holomorphic curve . Then*

$$(q-n)h^+(f,t) \le \sum_{j=1}^{q} \frac{N(f \circ H_j, t}{d} + 0(1),$$

where 0(1) is bounded when $t \to -\infty$.

This is a p-adic version of Eremenko-Sodin's theorem ([ES]).

We conclude this section by the following conjecture.

A holomorphic curve f is called *k-non-degenerate* if the image of f is contained in a linear subspace of dimension k and is not contained in any linear subspace of dimension $k-1$.

Conjecture 3.9. *Let $H_1, \dots, H_q$ be hypersurfaces of degree $d_j,\ j = 1, \dots, q$ in $\mathbb{P}^n(\mathbb{C}_p)$ in general position. Let f be a k- non-degenerate holomorphic curve . Let s be an integer $\geq k$, or $s = \infty$. Then*

$$(q - 2n + k - 1)h^+(f,t) \leq \sum_{j=1}^{q} \frac{N_s(H_j \circ f, t)}{d_j} + 0(1).$$

Remark 3.10 In the complex case the above conjecture corresponds to the following cases:
1. Nevanlinna's Second Main Theorem: $n = 1,\ k = 1,\ d_j = 1,\ s = \infty$.
2. Cartan Theorem: $\forall n,\ k = n,\ d_j = 1,\ s = n$.
3. Nochka Theorem (Cartan's conjecture): $\forall n, \forall k \leq n, s = k,\ d = 1$.
4. Eremenko-Sodin's theorem: $\forall n, k = n, \forall d_j, s = \infty$.

§4. Defect relation and Borel's Lemmas

Let H be a hyperplane of $\mathbb{P}^n(\mathbb{C}_p)$ such that the image of f is not contained in H. We say that f ramifies at least d ($d > 0$) over H if for all $z \in f^{-1}H$ the degree of the pull-back divisor f^*H, $deg_z f^*H \geq d$. In case $f^{-1}H = \emptyset$ we set $d = \infty$.

Theorem 4.1. *Let $H_1, \dots, H_q$ be q hyperplanes in general position. Assume f is linearly non-degenerate and ramifies at least d_j over H_j. Then*

$$\sum_{j=1}^{q}(1 - \frac{n}{d_j}) < n + 1.$$

Remark 4.2 In the complex case we have a similar inequality, but with the sign $\leq$. The reason is that in the p-adic case, the error term in Second Main Theorem is simpler that the complex one. This is important for applications.

From Theorem one can prove the following p-adic version of Borel's Lemma.

Theorem 4.3 (p-adic Borel's Lemma [Q]). *Let $f_1, f_2, \dots, f_n$ ($n \geq 3$) be p-adic holomorphic functions without common zeros on $\mathbb{C}_p$ such that $f_1 + f_2 + ... + f_n = 0$.*

Then the functions $f_1, \dots, f_{n-1}$ are linearly dependent if for $j = 1, \dots, n$ every zero of f_j is of multiplicity at least d_j and the following condition holds:

$$\sum_{j=1}^{n} \frac{1}{d_j} \leq \frac{1}{n-2}.$$

By using the defect relation one can prove some generalizations of Borel's lemma.

Let

$$M_j = z_1^{\alpha_{j,1}} \dots z_{n+1}^{\alpha_{j,n+1}}, \qquad 1 \leq j \leq s,$$

be distinct monomials of degree l with non-negative exponents. Let X be a hypersurface of degree dl of $\mathbb{P}^n(\mathbb{C}_p)$ defined by

$$X: \quad c_1 M_1^d + \dots c_s M_s^d = 0,$$

where $c_j \in \mathbb{C}_p^*$ are non-zero constants.

Theorem 4.4 (p-adic analogue of Masuda-Noguchi's Theorem [M-N]).

Let $f = (f_1, .., f_{n+1}) : C_p \longrightarrow X$ be a non-constant holomorphic curve such that any $f_j \not\equiv 0$. Assume that

$$d \geq s(s-2).$$

Then there is a decomposition of indices, $\{1, 2, ..., s\} = \cup I_\gamma$, such that

i) Every I_γ contains at least 2 indices.

ii) The ratio of $M_j^d \circ f(z)$ *and* $M_j^k \circ f(z)$ *is constant for* $j, k \in I_\gamma$.

iii) $\sum_{j \in I_\gamma} c_j M_j^d \circ f(z) \equiv 0$ *for all* γ .

Corrolary 4.5. *For* $d \geq 3$ *there is no solutions of the following equation in the set of p-adic non-constant holomorphic functions having no common zeros:*

$$x^d + y^d = z^d$$

§5. p-adic hyperbolic spaces

Recall that a complex space is said to be hyperbolic if every holomorphic curve in it is a constant curve. In the complex case, the Borel Lemma is often used to establish the hyperbolicity of a complex space. In what follows we show some applications of p-adic Borel's Lemma in the study of p-adic hyperbolic hypersurfaces.

Although the set of hyperbolic hypersurfaces of degree d large enough with respect to n is conjectured to be Zariski dense [Ko]), it is not easy to construct explicit examples of hyperbolic hypersurfaces.

The first example of smooth hyperbolic surfaces of even degree $d \geq 50$ was given by R. Brody and M. Green ([BG]). Now we show how to use p-adic Borel's Lemmas to construct explicit examples of p-adic hyperbolic hypersurfaces.

Let X be a hypersurface defined as above, and let $d \geq s(s-2)$. Suppose that X is not hyperbolic, and let

$$f = (f_1, ..., f_{n+1}) : \mathbb{C}_p \longrightarrow X$$

be a nonconstant holomorphic curve in X. We are going to show that $\{c_j\}$ belongs to an algebraic subset of $(\mathbb{C}_p^*)^s$. We may assume that any $f_j \not\equiv 0$.

By Theorem 4.4, there is a decomposition of indices $\{1, \ldots, s\} = \cup I_\xi$ such that

i) every I_ξ contains at least 2 indices,

ii) the ratio of $M_j^d \circ f(z)$ and $M_k^d \circ f(z)$ is constant for $j,\, k \in I_\xi$,

iii) $\sum_{j \in I_\xi} c_j\, M_j^d \circ f(z) \equiv 0$ for all ξ.

Now for a decomposition of $\{1, \ldots, s\}$ as above, we set $b_{jk} = M_j^d \circ f(z) \,/\, M_k^d \circ f(z)$. Then the linear system of equations

$$AY = B$$

where A is the matrix $\{\alpha_{j\ell} - \alpha_{k\ell}\}$, $Y = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$, $B = \{\log b_{jk}\}$, has the solution $\{\log f_0, ..., \log f_n\}$. Thus, the matrix A satisfies certain conditions on the rank. On the other hand, by the condition iii) there exist $(A_0, ..., A_n) \in \mathbb{P}^n$ such that $(c_i) \in (\mathbb{C}_p^*)^s$ satisfies the following equations

$$\sum_{i \in I_\xi} c_i\, A_o^{\alpha_{io}} \cdots A_n^{\alpha_{in}} = 0$$

Hence, $(c_i) \in (\mathbb{C}_p^*)^s$ belongs to the projection $\Sigma \subset (\mathbb{C}_p^*)^s$ of an algebraic subset in $(\mathbb{C}_p^*)^s \times \mathbb{P}^n$. If we take $(c_i) \notin \Sigma$, we have a hyperbolic hypersurface.

5.1.**Examples**.

Let $N = 4n-3, k = N(N-2) = 16(n-1)^2$. Then for generic linear functions $H_j(z_0, \ldots, z_n) \in \mathbb{C}_p^{n+1}$ $(1 \leq j \leq n)$ the hypersurface

$$X : \sum_{j=1}^{N} H_j^k = 0$$

is hyperbolic. This is p-adic version of a recent result of Siu and Yeung ([SY], 1997).

Proof. By p-adic Borel's lemma, if $f : \mathbb{C}_p \longrightarrow X$ is a non-constant holomorphic curve, then $\mathrm{Im} f \subset \cap_\xi X_\xi$, where

$$X_\xi : \sum_{j \in I_\xi} H_j^k = 0.$$

The genericity of $\{H_j\}$ implies $\cap X_\xi = \emptyset$.

For the case of surfaces in $\mathbb{P}^3$ we can use the following method. Take at first a surface $X \subset \mathbb{P}^3$ such that every holomorphic curve in X is degenerate. This means that the image of a holomorphic map from $\mathbb{C}_p$ in to X, $f : \mathbb{C}_p \longrightarrow X$, is contained in a proper algebraic subset of X. If one could prove that the image $f(\mathbb{C}_p)$ is contained in a curve of genus at least 1, then f is a constant map (Bercovich's theorem).

5.2. **Example** Let X be a surface in $\mathbb{P}^3(\mathbb{C}_p)$ defined by the equation

$$X:\ z_1^d + z_2^d + z_3^d + z_4^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} = 0,$$

where $c \neq 0$, $\sum_{i=1}^{4} \alpha_i = d$, and if there is an exponent $\alpha_i = 0$, the others must be $\neq 1$. Then X is hyperbolic if $d \geq 24$.

5.3. **Example** Let X be a curve in $\mathbb{P}^2(\mathbb{C}_p)$ defined by the following equation:

$$X:\ z_1^d + z_2^d + z_3^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0,$$

where $d \geq 24$, $d > \alpha_i \geq 0$, $\sum \alpha_i = d$. Then the complement of X is p-adic hyperbolic in $\mathbb{P}^2(\mathbb{C}_p)$.

§6. unique range set of meromorphic functions

For a non-constant meromorphic function f on C and a set $S \subset C \cup \{\infty\}$ define

$$E_f(S) = \cup_{a \in S}\{(m, z) | f(z) = a \text{ with multiplicity } m\},$$

and

$$\bar{E}_f(S) = \cup_{a \in S}\{z | f(z) = a \text{ ignoring multiplicities }\}.$$

A set $S \subset C \cup \{\infty\}$ is call *an unique range set for meromorphic functions* (URSM) if for any pair of non-constant meromorphic functions f and g on C, the condition $E_f(S) = E_g(S)$ implies $f = g$. A set $S \subset C \cup \{\infty\}$ is called *an unique range set for entire functions* (URSE) if for any pair of non-constant entire functions f and g on C, the condition $E_f(S) = E_g(S)$ implies $f = g$. Classical theorems of Nevanlinna show that $f = g$ if $\bar{E}_f(a_j) = \bar{E}_g(a_j)$ for distinct values $a_1, \ldots, a_5$, and that f is a Möbius transformation of g if $E_f(a_j) = E_g(a_j)$ for distinct values $a_1, \ldots, a_4$. Gross and Yang show that the set

$$S = \{z \in C | z + e^z = 0\}$$

is an URSE. Recently, URSE and also URSM with finitely many elements have been found by Yi ([Y1], [Y2]), Li and Yang ([LY1], [LY2]), Mues and Reinders [MR], Frank and Reinders [FR]. Li and Yang introduced the notation

$$\lambda_M = \inf\{\#S | S \text{ is a URSM}\},$$
$$\lambda_E = \inf\{\#S | S \text{ is a URSE}\},$$

where $\#S$ is the cardinality of the set S.The best lower and upper bounds known so far are

$$5 \leq \lambda_E \leq 7,\ 6 \leq \lambda_M \leq 11.$$

For p-adic meromorphic or entire function f on C_p, we can similarly define $E_f(S)$ and $\bar{E}_f(S)$ for a set $S \subset C_p \cup \{\infty\}$ and introduce the notation λ_M and λ_E. By using p-adic Nevanlinna theory and theory of singularities we can prove the following theorems:

Theorem 6.1. *Let P be a generic polynomial of degree at least 5. Let f and g be p-adic meromorphic functions such that $P(f) = CP(g)$ with a constant C. Then $f \equiv g$.*

Theorem 6.2. *Let $S = \{a_1, a_2, a_3, a_4\}$ be a generic set of 4 points in $\mathbb{C}_p$. Then for p-adic meromorphic functions f and g, the conditions $E_f(S) = E_g(S)$ and $E_f(\infty) = E_g(\infty)$ imply $f \equiv g$.*

For the proof, see [K7].

Conjecture 6.3. *A generic set of 5 points in $\mathbb{C}_p \cup \{\infty\}$ is a URS for p-adic meromorphic functions.*

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A property of uniformization of Baker domains

Masashi Kisaka

1. Introduction

Let f be a transcendental entire function, and let $F_f \subset \mathbb{C}$ and $J_f \subset \mathbb{C}$ be the Fatou set and Julia set of f respectively. A connected component U of F_f is called a *Fatou component.* Then U is either a *wandering domain* (that is, $f^m(U) \cap f^n(U) = \emptyset$ for all $m, n \in \mathbb{N}$ $(m \neq n)$) or *eventually periodic* (that is, $f^m(U)$ is periodic for an $m \in \mathbb{N}$). If it is periodic, it is well known that there are four possibilities; U is either an attractive basin, a parabolic basin, a Siegel disk, or a Baker domain. Note that U cannot be a Herman ring. This fact follows easily from the maximum principle.

In this note we consider an unbounded periodic (that is, $f^n(U) \subseteq U$ for some $n \in \mathbb{N}$) Fatou component U. It is known that U is simply connected ([**B**], [**EL**]) and so let $\varphi : \mathbb{D} \to U$ be a uniformization (Riemann map) of U, where $\mathbb{D}$ is a unit disk. The boundary ∂U of U can be very complicated as the following example shows:

Example. Let us consider the exponential family $E_\lambda(z) := \lambda e^z$. If the parameter λ satisfies $\lambda = te^{-t}$ $(|t| < 1)$, then there exists a unique unbounded completely invariant attractive basin U which is equal to the Fatou set F_{E_λ} and ∂U is equal to the Julia set J_{E_λ} which is so called a Cantor bouquet. Moreover,

$$\Theta_\infty := \left\{ e^{i\theta} \;\middle|\; \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty \right\} \subset \partial\mathbb{D}$$

is dense in $\partial\mathbb{D}$ ([?]). This implies that φ is highly discontinuous on $\partial\mathbb{D}$ and hence ∂U $(= J_{E_\lambda})$ has a very complicated structure. In fact the Hausdorff dimension of J_{E_λ} is equal to 2 ([?]).

Later, Baker and Weinreich investigated the boundary behavior of φ generally in the case of attractive basins, parabolic basins and Siegel disks and showed the following:

Theorem (Baker-Weinrech, [BW]). Let U be an unbounded invariant Fatou component, then either
(i) $f^n \to \infty$ in U (that is, U is a Baker domain) or
(ii) the point ∞ belongs to the impression of every prime end of U. □

From the classical theory of prime end by Carathéodory it is well known that there is a 1 to 1 correspondence between $\partial\mathbb{D}$ and the set of all the prime ends of U. Let us denote $P(e^{i\theta})$ the prime end corresponding to the point $e^{i\theta} \in \partial\mathbb{D}$. The impression $\mathrm{Im}(P(e^{i\theta}))$ of a prime end $P(e^{i\theta})$ is a subset of ∂U which is known to be written as follows:

$$\mathrm{Im}(P(e^{i\theta})) = \left\{ p \in \partial U \;\middle|\; \begin{array}{l} \text{there exists a sequence } \{z_n\}_{n=1}^{\infty} \subset \mathbb{D} \\ \text{whcih satisfies } \lim\limits_{n\to\infty} z_n = e^{i\theta},\ \lim\limits_{n\to\infty} \varphi(z_n) = p \end{array} \right\}$$

For the details of the theory of prime end, see for example, [?]. Define the set $I_\infty \subset \partial\mathbb{D}$ by

$$I_\infty := \{e^{i\theta} \in \partial\mathbb{D} \mid \infty \in \mathrm{Im}(P(e^{i\theta}))\},$$

then the above result asserts that $I_\infty = \partial\mathbb{D}$ in the case of unbounded attractive basins, parabolic basins and Siegel disks. This shows that ∂U is extremely complicated.

On the other hand, ∂U can be very "simple" in the case when U is a Baker domain. For example, the function

$$f(z) := 2 - \log 2 + 2z - e^z$$

has a Baker domain U on which f is univalent and whose boundary ∂U is a Jordan curve (that is, $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is a Jordan curve and $\partial U \subset \mathbb{C}$ is a Jordan arc, [?, Theorem 2]). In this case I_∞ consists of only a single point.

Then what can we say about the set I_∞ in general when U is a Baker domain? For this problem we obtain the following:

Main Theorem. Let f be a transcendental entire function and suppose that f has an invariant Baker domain U. Let $\varphi : \mathbb{D} \to U$ be a uniformization of U and the set I_∞ as above. Assume that $f|U : U \to U$ is not univalent.
(1) If $f|U$ is semi-conjugate to a hyperbolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$, then I_∞ contains a perfect set $K \subset \partial\mathbb{D}$.
(2) If $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$, then I_∞ contains a perfect set $K \subset \partial\mathbb{D}$.
(3) If $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{C} \to \mathbb{C}\quad z \mapsto z+1$, then $I_\infty = \partial\mathbb{D}$.
If $f|U$ is univalent, then ${}^{\#}I_\infty = 1, 2$ or ∞.

Remark In the Main Theorem we assume that U is an invariant Baker domain for simplicity. Of course, we can obtain the same result when U is a periodic Baker domain of period $p \geq 2$.

This result is based on the classification of Baker domains and an arbitrary Baker domain falls into one of the above three cases. We explain the details in §2. In §3 we show the outline of the proof of the Main Theorem.

2. Classification of Baker domains

In this section we classify Baker domains from the dynamical point of view. Now let U be an invariant Baker domain. By definition $f^n|U \to \infty$ $(n \to \infty)$ locally uniformly, so put

$$g := \varphi^{-1} \circ f \circ \varphi : \mathbb{D} \to \mathbb{D},$$

then g is conjugate to $f|U : U \to U$ and from the dynamics of $f|U$, g has no fixed point in $\mathbb{D}$. By the theorem of Denjoy and Wolff, there exists a unique point $p_0 \in \partial\mathbb{D}$ (which is called Denjoy-Wolff point) and $g^n \to p_0$ locally uniformly. It is known that there exists a radial limit

$$c := \lim_{r \nearrow 1} g'(rp_0) \quad \text{with} \quad 0 < c \leq 1,$$

which means that p_0 is either an attracting or a parabolic fixed point of the boundary map of g. Next put

$$z_n := g^n(0) \quad \text{and} \quad q_n := \frac{z_{n+1} - z_n}{1 - \overline{z_n} z_{n+1}},$$

then by the Schwarz-Pick's lemma $\{|q_n|\}_{n=1}^{\infty}$ turns out to be a decreasing sequence and hence there exists a limit $\lim_{n\to\infty} |q_n|$ ([**P**]). By using this limit and the value c, the dynamics of g on $\mathbb{D}$ can be classified for three different classes as follows. This result is essentially due to Baker and Pommerenke ([**?**], [**?**]). They treated anlytic functions in the halfplane $\mathbb{H}$ and obtained some results. The following is the translation of their results into the case of analytic functions in $\mathbb{D}$ which is conformally equivalent to $\mathbb{H}$.

Theorem (1) If $c < 1$, then g is semi-conjugate to a hyperbolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(z) = \dfrac{(1+c)z + 1 - c}{(1-c)z + 1 + c}$.
(2) If $c = 1$ and $\lim_{n\to\infty} |q_n| > 0$, then g is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(z) = \dfrac{(1 \pm 2i)z - 1}{z - 1 \pm 2i}$.
(3) If $c = 1$ and $\lim_{n\to\infty} |q_n| = 0$, then g is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(z) = z + 1$. □

On the other hand, König investigated the relation between the above classification and the dynamics of $f|U : U \to U$ and obtained the following result:

Theorem (König, [?]) For an arbitrary point $w_0 \in U$ define

$$w_n := f^n(w_0) \quad \text{and} \quad d_n := \text{dist}\ (w_n, \partial U),$$

where "dist" is a Euclidean distance. Then
(1) $f|U$ is semi-conjugate to a hyperbolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ if and only if there exists a constant $\beta = \beta(f) > 0$ such that

$$\frac{|w_{n+1} - w_n|}{d_n} \geq \beta \quad (n \in \mathbb{N})$$

holds for any $w_0 \in U$.
(2) $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ if and only if

$$\liminf_{n\to\infty} \frac{|w_{n+1} - w_n|}{d_n} > 0$$

holds for any $w_0 \in U$ but

$$\inf_{w_0 \in U} \limsup_{n\to\infty} \frac{|w_{n+1} - w_n|}{d_n} = 0.$$

(3) $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(z) = z + 1$ if and only if

$$\lim_{n\to\infty} \frac{w_{n+1} - w_n}{d_n} = 0$$

holds for any $w_0 \in U$. □

For each cases König also gave concrete examples satisfying the above conditions:
(1) $f(z) = 3z + e^{-z}$,
(2) $f(z) = z + 2\pi i\alpha + e^z$, where $\alpha \in (0, 1)$ satisfies the Diophantine condition,
(3) $f(z) = e^{\frac{2\pi i}{p}} \left(z + \displaystyle\int_0^z e^{-\zeta^p} d\zeta \right)$, where $p \in \mathbb{N},\ p \geq 2$.

Note that in the case (3), the function f above has a Baker domain of period $p \geq 2$, not an invariant one. Of course, if we consider f^p instead of f, f^p has an invariant Baker domain.

3. Outline of the proof of Main Theorem

With the above classification, we show the outline of the proof of Main Theorem.

Since $U \subset \mathbb{C}$ is unbounded, we have $I_\infty \neq \emptyset$ and it is easy to see that I_∞ is a closed subset of $\partial\mathbb{D}$. Then $\partial\mathbb{D} \setminus I_\infty$ is open in $\partial\mathbb{D}$ and it can be shown that g can be analytically continued over $\partial\mathbb{D} \setminus I_\infty$. So in particular g is analytic on $\partial\mathbb{D} \setminus I_\infty$ and we have

$$g(\partial\mathbb{D} \setminus I_\infty) \subseteq \partial\mathbb{D} \setminus I_\infty.$$

If g is a d to 1 map $(2 \leq d < \infty)$, then g is a finite Blaschke product of degree d and its Julia set J_g is either $\partial\mathbb{D}$ or a Cantor set (in particular, it is a perfect set) in $\partial\mathbb{D}$. Assume that $J_g \cap (\partial\mathbb{D} \setminus I_\infty) \neq \emptyset$, then from the general property of the dynamics of rational maps and the g-invariance of $\partial\mathbb{D} \setminus I_\infty$ we have

$$\partial\mathbb{D} \subset \partial\mathbb{D} \setminus I_\infty,$$

that is, $I_\infty = \emptyset$, which is a contradiction. Therefore we have $J_g \subset I_\infty$. This proves the case (1) and (2) with a further assumption that g is a finite to one map.

If g is an ∞ to 1 map, we can show that

$$\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D} \subset I_\infty$$

holds for every $z_0 \in \mathbb{D}$ (there may be some exception) and the set $\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D}$ is either equal to $\partial\mathbb{D}$ or at least contains a certain perfect set $K \subset \partial\mathbb{D}$. This result comes from a property of g as a boundary map $g : \partial\mathbb{D} \to \partial\mathbb{D}$. This completes the proof for the case (1) and (2).

For the case (3), since we have $\lim_{n\to\infty} |q_n| = 0$, we can obtain that

$$\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D} = \partial\mathbb{D} \subset I_\infty,$$

and hence $I_\infty = \partial\mathbb{D}$. This fact comes from the ergodic property of g as an inner function. This completes the proof for the case (3).

If g is univalent, then g is either hyperbolic or parabolic Möbius transformation. g has either one or two fixed points and the every orbit of a point other than the fixed points has infinitely many points. On the other hand, we have

$$g(\partial\mathbb{D} \setminus I_\infty) \subseteq \partial\mathbb{D} \setminus I_\infty,$$

so we can conclude that ${}^{\#}I_\infty = 1, 2$ or ∞. □

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Elimination of defects of holomorphic curves into $\mathbf{P}^n(\mathbf{C})$ for rational moving targets

Seiki Mori

Nevanlinna defect relations ware established for various cases, for example, holomorphic (meromorphic) mappings of $\mathbf{C}^m$ into a conplex projective space $\mathbf{P}^n(\mathbf{C})$ for constant targets of hyperplanes or moving targets of hyperplanes (arbitrary $m \geq 1$ and $n \geq 1$), or holomorphic mappings of an affine variety A of dimension m into a projective algebraic variety V of dimension n for divisors on V ($m \geq n \geq 1$), and so on. On the other hand, the size of a set of (Valiron) deficient hyperplanes or deficient divisors are investigated. (e.g. Sadullaev [7], Molzon-Shiffman-Sibony [6]. Mori[4]) Nevanlinna theory asserts that for each holomorphic (or meromorphic) mapping, Nevanlinna defects or Valiron defects of the mapping are very few. Until now, there are few results on defects of a family of mappings. Recently the author [4], [5] proved that for a transcendental horomorphic mapping f of $\mathbf{C}^{\mathrm{m}}$ into $\mathbf{P}^n(\mathbf{C})$, we can eliminate all deficient hyperplanes (or deficient hypersurfaces of degree at most a given integer d) in $\mathbf{P}^n(\mathbf{C})$ by a small deformation of the mapping.

We shall now discuss about an elimination theorem of defects of rational moving targets which consist of systems of polynomials of an arbitrary degree for a holomorphic curve by its small deformation. Here a samll deformation $\tilde{f}$ of f means that their order functions $T_f(r)$ and $T_{\tilde{f}}(r)$ satisfy $|T_f(r) - T_{\tilde{f}}(r)| \leq o(T_f(r))$, as r tends to infinity.

1. Preliminary

Notation and Terminology.

Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ and $\phi : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})^*$ be holomorphic curves, and $(f_0, ..., f_n)$ and $(\phi_0, ..., \phi_n)$ be their reduced representations, respectively. Hence, each coordinate functions $f_j(z)$, or $\phi_l(z)$ (j=0,...,n) are entire functions without common zeros. Then the proximity function $m_f(r, \phi)$ and the counting function $N_f(r, \phi)$ of a moving target ϕ into $\mathbf{P}^n(\mathbf{C})^*$ are given by:

$$m_f(r,\phi) := \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|f\|\|\phi\|}{|\langle f,\phi\rangle|}(re^{i\theta})d\theta$$

and

$$N_f(r,\phi) := \int_{r_0}^{r} n(A,t)\frac{dt}{t},$$

where $n(A,t)$ denotes the number of zeros of $A := \langle f, \phi\rangle$ in $\{|z| < t\}$ counting with multiplicities. The order function $T_f(r)$ of f is given by :

$$\begin{aligned} T_f(r) &:= \frac{1}{2\pi}\int_0^{2\pi} \log(\sum_{j=0}^{n}|f_j(re^{i\theta})|^2)^{1/2}d\theta - \log(\sum_{j=0}^{n}|f_j(0)|^2)^{1/2} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log\sum_{j=0}^{n}|f_j(re^{i\theta})|d\theta + O(1). \end{aligned}$$

The Nevanlinna deficiency $\delta_f(\phi)$ and the Valiron deficiency $\Delta_f(\phi)$ of a moving target ϕ for f are given by:

$$\delta_f(\phi) := \liminf_{r\to+\infty}\frac{m_f(r,\phi)}{T_f(r)} = 1 - \limsup_{r\to+\infty}\frac{N_f(r,\phi)}{T_f(r)}$$

and

$$\Delta_f(\phi) := \limsup_{r\to+\infty}\frac{m_f(r,\phi)}{T_f(r)} = 1 - \liminf_{r\to+\infty}\frac{N_f(r,\phi)}{T_f(r)}.$$

We now define the projective logarithmic capacity of a set in the projective space $\mathbf{P}^n(\mathbf{C})$. (See, Molzon-Shiffman-Sibony [6, p. 46]).
Let E be a compact subset of $\mathbf{P}^n(\mathbf{C})$, and $\mathcal{P}(E)$ denotes the set of probability measures supported on E. We define

$$V_\mu(x) := \int_{w\in\mathbf{P}^n(\mathbf{C})} \log \frac{\|x\|\|w\|}{|\langle x, w\rangle|} d\mu(w), \quad (\mu \in \mathcal{P}(E))$$

and

$$V(E) := \inf_{\mu\in\mathcal{P}(E)} \sup_{x\in\mathbf{P}^n(\mathbf{C})} V_\mu(x).$$

Define the projective logarithmic capacity of E by

$$C(E) := \frac{1}{V(E)}.$$

If $V(E) = +\infty$, we say that the set E is of projective logarithmic capacity zero. For an arbitrary subset K of $\mathbf{P}^n(\mathbf{C})$, we put

$$C(K) = \sup_{E\subset K} C(E),$$

where the supremum is taken over all compact subset E of K.

Some Results.

Theorem A. ([6]) Let $\varphi : [0,1] \to \mathbf{P}^n(\mathbf{C})$ be a real smooth nondegenerate arc in $\mathbf{P}^n(\mathbf{C})$, K a compact subset of [0, 1]. Then the projective logarithmic capacity $C(\varphi(K))$ is positive if and only if K has a positive logarithmic capacity in $\mathbf{C}$. Here "smooth nondegenerate arc φ" means that there exists a lift $\tilde{\varphi} : [0,1] \to \mathbf{C}^{n+1} \setminus \{0\}$ such that the k-th derivatives $\{\tilde{\varphi}^{(k)}(t)\}_{k\geq 1}$ of $\tilde{\varphi}(t)$ spans $\mathbf{C}^{n+1}$ for every $t \in [0,1]$.

Theorem B. [1] If

$$\Lambda(r) := \int_{r_0}^{r} \frac{\psi(t)}{t} dt,$$

where $\psi(r)$ is non-negative, non-decreasing and unbounded, and if $\Lambda(r) < rK$ for some $K > 0$ and all sufficiently large r, then there exists an entire function $f(z)$ of finite order, such that $T_f(r) \sim \Lambda(r), (r \to \infty)$.

Theorem C. [4] Let f be a holomorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ such that $\lim_{r\to+\infty} T_f(r) = +\infty$. Then there exist $r_1 < r_2 < \ldots < r_n \to +\infty$ and sets $E_n : E_{n+1} \subset E_n (n = 1, 2, ...)$ in $\mathbf{P}^n(\mathbf{C})^*$ with

$$V(E_n) \geq 2 \log T_f(r_n)$$

such that, if H does not belong to E_n, then

$$m_f(r, H) \leq 4\sqrt{T_f(r)} \log T_f(r),$$

for $r > r_n$. Hence

$$\liminf_{r\to+\infty} \frac{m_f(r, H)}{T_f(r)} = 0,$$

outside a set $E \subset \mathbf{P}^n(\mathbf{C})^*$ of projective logarithmic capacity zero. Here $\mathbf{P}^n(\mathbf{C})^*$ denotes the dual projective space of $\mathbf{P}^n(\mathbf{C})$.

2. Elimination of defects of holomorphic curves for rational moving targets.

For a transcendental holomorphic curve f of $\mathbf{C}$ into $\mathbf{P}^n(\mathbf{C})$, we can eliminate all defects of rational moving targets by a small deformation of f. We say that holomorphic curve f is transcendental if

$$\lim_{r\to+\infty} \frac{T_f(r)}{\log\ r} = +\infty, \quad \text{as} \quad r \to +\infty.$$

A holomorphic curve f is rational if and only if $T_f(r) = O(\log\ r), \quad (r \to +\infty)$.

THEOREM. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a given transcendental holomorphic curve. Then there exists a regular matrix

$$L = \left(l_{ij}\right)_{0\leq i,j\leq n} \text{ of the form } \ l_{i,j} = c_{ij}g_j + d_{ij}, \ \ (c_{ij}, d_{ij} \in \mathbf{C} : 0 \leq i, j \leq n),$$

such that $\det L \neq 0$ and $\tilde{f} = L \cdot f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ is a holomorphic curve without Nevanlinna defects of rational moving targets and satisfies

$$|T_f(r) - T_{\tilde{f}}(r)| = o(T_f(r)), \qquad (r \to \infty),$$

where g_j (j=1,...,n) are some transcendental entire functions satisfying $T_{g_j}(r) = o(T_{g_{j+1}}(r))$, (j=1,...,n-1) and $T_{g_n} = o(T_f(r)), \ \ (r \to \infty)$.

Here $L \cdot f$ means that $(l_{ij}) \cdot^t (f_0, ..., f_n)$ for a reduced representation $(f_0, ..., f_n)$ of f.

Note that we cannot replace transcendental eitire functions g_j by any rational functions.

Outline of the proof of Theorem.

First Step.

Let h be a transcendental holomorphic curve and $(h_0, ..., h_n)$ be a reduced representation of h. Then there are indices i, j such that h_j/h_i is transcendental, say $i = 0, j = n$. By Theorem B, there are n transcendental entire functions $g_1, ..., g_n$ on $\mathbf{C}$ such that $T_{g_j}(r) = o(T_{g_{j+1}})$, (j=1,...,n-1) and $T_{g_n}(r) = o(T_f(r))$, as $r \to \infty$. Then $g_1, ..., g_n$ are linearly independent over $\mathbf{C}$. Now we expand g_k as the Taylor series at the origin:

$$g_k = \sum_{j=0}^{\infty} \alpha_j^k z^j,$$

and we write

$$P_{kl} := \sum_{j=0}^{l} \alpha_j^k z^j.$$

Then there are n polynonials $P_{1l_0}, ..., P_{nl_0}$ which are linearly independent over $\mathbf{C}$. We put $\mathbf{a}^k := (\alpha_0^k, ..., \alpha_{l_0}^k)$. Then rank ${}^t(\mathbf{a}^1, \mathbf{a}^2, ..., \mathbf{a}^n) = n$. We put $\bar{h}_k = h_k + a_k g_k h_0$, (k=1,...,n) and $\bar{h}_0 = h_0$, and fix arbitrary $m \in \mathbf{N}$. The system of vectors which consist of coefficients of each trems z^k of

$$1, z, ..., z^m, g_1, zg_1, ..., z^m g_1, ..., g_n, zg_n, ..., z^m g_n$$

are of rank $(m + 1)(n + 1)$. Thus there exist coefficients of some terms

$$z^{p_0}, ..., z^{p_m}, z^{q_0}, ..., z^{q_m}, ..., z^{s_0}, ..., z^{s_m},$$

of these functions

$$g_1, zg_1, ..., z^m g_1, ..., g_n, zg_n, ..., z^m g_n$$

which coefficients determinant are not equal to zero, that is,

$$L := \Gamma \begin{vmatrix} \alpha_{p_0}^1 & \alpha_{p_0-1}^1 & \dots & \alpha_{p_0-m}^1 & \alpha_{p_0}^2 & \dots & \alpha_{p_0-m}^2 & \dots & \alpha_{p_0}^n & \dots & \alpha_{p_0-m}^n \\ \alpha_{q_0}^1 & \alpha_{q_0-1}^1 & \dots & \alpha_{q_0-m}^1 & \alpha_{q_0}^2 & \dots & \alpha_{q_0-m}^2 & \dots & \alpha_{q_0}^n & \dots & \alpha_{q_0-m}^n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{s_0}^1 & \alpha_{s_0-1}^1 & \dots & \alpha_{s_0-m}^1 & \alpha_{s_0}^2 & \dots & \alpha_{s_0-m}^2 & \dots & \alpha_{s_0}^n & \dots & \alpha_{s_0-m}^n \end{vmatrix} \neq 0.$$

Then we can show the following lemma.

Lemma 1. Let $h = (h_0 : h_1 : \cdots : h_n)$ be a nonconstant holomorphic curve. Then there exists $(\tilde{a}_1, ..., \tilde{a}_n)$ such that $\tilde{a}_j = \alpha^{k_j}$ (j=1,...,n) with $k_1 = 1, \ k_l = (m+1)\sum_{j=1}^{l-1} k_j + 1$ $(l = 2, 3, ..., n)$ $(\alpha \in \mathbf{C})$), and

$$\begin{aligned} \tilde{h} \ &= \ (h_0 : zh_0 : \cdots : z^m h_0 : h_1 + \tilde{a}_1 g_1 h_0 : z(h_1 + \tilde{a}_1 g_1 h_0) : \cdots : z^m(h_1 + \tilde{a}_1 g_1 h_0) : \\ &\quad\ h_n + \tilde{a}_n g_n h_0 : z(h_n + \tilde{a}_n g_n h_0) : \cdots : z^m(h_n + \tilde{a}_n g_n h_0)), \end{aligned}$$

is linearly nondegenerate.

Second Step.

There is a regular linear change L_1 such that

$$h = L_1 \cdot f = (h_0 : \cdots : h_n) : \mathbf{C} \longrightarrow \mathbf{P}^n(\mathbf{C})$$

a reduced representation of the holomorphic curve h and

$$N(r,0,h_j) \sim T_h(r) \quad \text{as} \quad r \to +\infty \quad (j = 0,1,...,n).$$

We put $\bar{h}_k = h_k + a_k g_k h_0$, (k = 1,..., n) and $\bar{h}_0 = h_0$. Consider a holomorphic curve

$$\bar{h} := (\bar{h}_0 : \bar{h}_1 : \cdots : \bar{h}_n) : \mathbf{C} \longrightarrow \mathbf{P}^n(\mathbf{C}).$$

Then for any $(a_1,...,a_n)$ corresponding to $\alpha \in \mathbf{C}\setminus$ {some countable set } as in Lemma 1,

$$\{\bar{h}_0, z\bar{h}_0, ..., z^m\bar{h}_0, \bar{h}_1, ..., z^m\bar{h}_1, ..., \bar{h}_n, ..., z^m\bar{h}_n\},$$

is linearly independent over $\mathbf{C}$. Consider the Wronskian

$$\mathbf{W} := W(\bar{h}_0, z\bar{h}_0, ..., z^m\bar{h}_0, \bar{h}_1, ..., z^m\bar{h}_1, ..., \bar{h}_n, ..., z^m\bar{h}_n).$$

where $W(\varphi,...,\varphi_n)$ denotes the Wronskian determinant of $\varphi_0,...,\varphi_n$. We write it as

$$\begin{aligned}\mathbf{W} \;:=\; & W_0(h_0, zh_0, ..., z^m h_0, h_1, ..., z^m h_1, ..., h_n, ..., z^m h_n) + a_1(W_{11} + \cdots + W_{1s_1}) \\ & + \cdots + a_n(W_{n1} + \cdots + W_{ns_n}) + a_1^2(W_{1^21} + \cdots + W_{1^2 s_1^2}) + \cdots \\ & + a_1^{m+1}(W_{1^{m+1}1} + \cdots + W_{1^{m+1}s_1^{m+1}}) + a_1a_2(W_{1^121} + \cdots + W_{1^12s_{12}}) + \cdots \\ & + \prod_{j=1}^n a_j^{m+1} W_N(1, ..., z^m, g_1, ..., z^m g_1, ..., z^m g_n) \cdot h_0^{(m+1)(n+1)}.\end{aligned}$$

We rewrite it in the inhomogeneous form as

$$\mathbf{W} = h_0^{(m+1)(n+1)}\{\mathbf{W}_0 + a_1\mathbf{W}_1 + \cdots + \prod_{j=1}^n a_j^{m+1}\mathbf{W}_N\}.$$

where $\mathbf{W}_k$, (k=0,...,N) are sum of some Wronskian determinants, and $N = (m+2)^n - 1$. Consider for any fixed m $\in \mathbf{N}$, a holomorphic curve of the form

$$F_m := (\mathbf{W}_0/d : \mathbf{W}_1/d : \cdots : \mathbf{W}_N/d) : \mathbf{C} \longrightarrow \mathbf{P}^N(\mathbf{C}),$$

where $d = d(z)$ is a meromorphic function whose zeros and poles consist of common factors among $\mathbf{W}_0, ..., \mathbf{W}_N$. Then F_m is a reduced representation of non-constant holomorphic curve in $\mathbf{P}^N(\mathbf{C})$.

Third Step.

Lemma 2 [cf.4]. Consider the set

$$\mathcal{A} := \{(1, a_1, ..., a_1^{m+1}, a_2, ..., a_1^{i_1} \cdots a_n^{i_n}, ..., \prod_{j=1}^n a_j^{m+1} \mid a_j \in \mathbf{C},\ 0 \le i_1, ..., i_n \le m+1\}$$

in $\mathbf{P}^N(\mathbf{C})$, where $N = (m+2)^n - 1$. Then it contains a set $\mathcal{X} := \{(1, \alpha, \alpha^2, ..., \alpha^N) | \alpha \in \mathbf{C}\}$, after some rearrangement. Since the set $\mathcal{X}$ has a positive projective logarithmic capacity, the set $\mathcal{A}$ has a positive projective logarithmic capacity.

Then for a vector $\in \mathcal{X}\setminus$ { countable set}, $\mathbf{W} \not\equiv 0$ by second step, and by Theorem C, for each fixed m $\in \mathbf{N}$ and a vector $\mathbf{a} = (1, a_1, ..., \prod_{j=1}^n a_j^{m+1}) \in \mathcal{X}\setminus$ { projective logarithmic capacity zero},

$$\limsup_{r\to\infty} \frac{m_{F_m}(r, H_{\mathbf{a}})}{T_{F_m}(r)} = 0. \tag{1}$$

A countable union of sets of projective logarithmic capacity zero is of projective logarithmic capacity zero. Hence there is a vector $\mathbf{a} = (1, a_1, ..., \prod_{j=1}^n a_j^{m+1})$ such that (1) holds for any integer m, that is, $N_{F_m}(r, 0, \langle F_m, \mathbf{a}_0\rangle) := N_F(r, H_{\mathbf{a}}) \sim T_{F_m}(r)$, $(r \to +\infty)$. Here

$$H_{\mathbf{a}} = \{\zeta | \langle \zeta, \mathbf{a}\rangle = 0\} \ \text{ and } \ \langle F, \mathbf{a}\rangle = \{\mathbf{W}_0 + a_1\mathbf{W}_1 + \cdots + \prod_{j=1}^n a_j^{m+1}\mathbf{W}_N\}/d,$$

where $\zeta = (\zeta_0, ..., \zeta_N)$.

Lemma 3. Let $h = (h_0 : \cdots : h_n)$ and $\bar{h} = (\bar{h}_0 : \cdots : \bar{h}_n)$ as above. Then we have

$$\int_{\partial B} \log |\bar{h}_k| \sigma \leq (1 + o(1)) \int_{\partial B} \log |h_0| \sigma + o(T_h(r)), \qquad (r \to +\infty).$$

Lemma 4. Let F_m and h be as above. Then there exists a positive number K such that

$$T_{F_m}(r) \leq K T_h(r).$$

Final Step.

For this $(a_1, .., a_n)$, we consider the holomorphic mapping given by a following reduced representation:

$$\tilde{f} := L_2 \cdot h \equiv (\tilde{f}_0, ..., \tilde{f}_n) : \mathbf{C} \longrightarrow \mathbf{P}^n(\mathbf{C}),$$

where

$$L_2 := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 g_1 & 1 & \cdots & 0 \\ a_2 g_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_n g_n & 0 & \cdots & 1 \end{pmatrix}, \qquad (\det L_2 = 1 \neq 0),$$

hence $\tilde{f}_0 = h_0$ and $\tilde{f}_k = h_k + a_k g_k h_0$, (k = 1,..., n). Then we see

$$T_{\tilde{f}}(r) = T_f(r) + o(T_{\bar{h}}(r)) = (1 + o(1)) T_f(r), \qquad (r \to +\infty).$$

Now we take a given integer m and arbitrary rational target ϕ of degree m :

$$\phi = (\phi_0(z), ..., \phi_n(z)) : \mathbf{C} \longrightarrow \mathbf{P}^n(\mathbf{C})^*.$$

Put $A_m :=< \tilde{f}, \phi >= \sum_{k=0}^n \phi_k \tilde{f}_k$. We may assume $\phi_n = b_0^n + b_1^n z + \cdots + b_m^n z^m \not\equiv 0$.
We note that

$$\begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & 0 \\ \phi_1 & \phi_2 & \ldots & \phi_{n-1} & \phi_n \end{vmatrix} = \phi_n \neq 0,$$

hence we see $\tilde{f}_0, \tilde{f}_1, ..., \tilde{f}_{n-1}, A_m$ are linearly independent over $\mathbf{C}$. Thus we have

$$\begin{aligned}
m_{\tilde{f}}(r, \phi) &= \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\tilde{f}\|}{|A_m|} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|W(\tilde{f}_0, ..., z^m \tilde{f}_0, \tilde{f}_1, ..., z^m \tilde{f}_1, ..., \tilde{f}_n, ..., z^m \tilde{f}_n)|}{z^s |A_m| (|\tilde{f}_0| \cdots |\tilde{f}_{n-1}|)^{m+1} |\tilde{f}_n|^m} d\theta \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\tilde{f}\| (|\tilde{f}_0| \cdots |\tilde{f}_{n-1}|)^{m+1} z^s |\tilde{f}_n|^m}{|W(\tilde{f}_0, ..., z^m \tilde{f}_0, \tilde{f}_1, ..., z^m \tilde{f}_1, ... \tilde{f}_n, ..., z^m \tilde{f}_n)|} d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|b_m^n|^{-1} |W(\tilde{f}_0, ..., z^m \tilde{f}_0, \tilde{f}_1, ..., z^m \tilde{f}_1, ... \tilde{f}_n, ..., z^{m-1} \tilde{f}_n, A_m)|}{z^s |A_m| (|\tilde{f}_0| \cdots |\tilde{f}_{n-1}|)^{m+1} |\tilde{f}_n|^m} d\theta \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{(\|\tilde{f}\| |\tilde{f}_0| \cdots |\tilde{f}_{n-1}|)^{m+1}}{|\tilde{f}_0|^{(n+1)(m+1)}} d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|\tilde{f}_0|^{(n+1)(m+1)}}{|\mathbf{W}|} d\theta + O(\log r), \\
&\leq o(T_{\tilde{f}}(r)) + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\tilde{f}\|^{(n+1)(m+1)}}{|\tilde{f}_0|^{(n+1)(m+1)}} d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|\mathbf{W}_0| + |\mathbf{W}_1| + \cdots + |\mathbf{W}_N|}{|\mathbf{W}|} d\theta + O(\log r) \\
&\leq o(T_{\tilde{f}}(r)) + (n+1)(m+1) m_{\tilde{f}}(r, H_{(1,0,...,0)}) \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{(|\mathbf{W}_0| + |\mathbf{W}_1| + \cdots + |\mathbf{W}_N|)(1/|d|)}{|\mathbf{W}|(1/|d|)} d\theta + O(\log r) \\
&= o(T_{\tilde{f}}(r)) + o(T_{\tilde{f}}(r)) + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|F_m\|}{|\langle F_m, \mathbf{a}_0 \rangle|} d\theta = o(T_{\tilde{f}}(r)) + o(T_{F_m}(r)) = o(T_{\tilde{f}}(r)), \quad //
\end{aligned}$$

since $T_{F_m}(r) \leq KT_{\tilde{f}}(r)$ for some $K > 0$, by Lemma 4. Here $s = m(m+1)(n+1)/2$. Therefore we obtain

$$\delta_{\tilde{f}}(\phi) = \liminf_{r \to +\infty} \frac{m_{\tilde{f}}(r, \phi)}{T_{\tilde{f}}(r)} = 0.$$

(q.e.d.)

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Bounded Fatou components of transcendental entire functions

Shunsuke Morosawa

1. Introduction

Let f be a transcendental entire function. We denote the n-th iteration of f by f^n. The maximal open subset of the complex plain $\mathbb{C}$ where $\{f^n\}$ is normal is called the Fatou set of f and is denoted by $F(f)$. The complement of $F(f)$ in $\widehat{\mathbb{C}}$ is called the Julia set of f and is denoted by $J(f)$. Note that we regard ∞ is a point in the Julia set. Since $J(f)$ is completely invariant, $J(f) \setminus \{\infty\}$ is unbounded in $\mathbb{C}$. Kisaka considered a topological property of the boundary of unbounded Fatou components ([**?**]). In this paper, we consider bounded Fatou components. Stallard ([**?**]) showed that if a transcendental entire function has small growth order, then every Fatou component is bounded. Another way to find bounded Fatou components is to find a polynomial-like mapping. If its filled-in Julia set has interior points, then the Fatou set contains a bounded component.

We call $\zeta \in \mathbb{C}$ a singular value if f is not a smooth covering map over any neighborhood of ζ. It plays an important role in studying its dynamics. We denote the set of them by $\mathrm{sing}(f^{-1})$ and set

$$\mathcal{S} = \{f \mid f \text{ is transcendental entire and } \mathrm{sing}(f^{-1}) \text{ is a finite set.}\}.$$

For $f \in \mathcal{S}$, every point of $\mathrm{sing}(f^{-1})$ is either an asymptotic value of f or a critical value of f. It is known that if $f \in \mathcal{S}$, then $F(f)$ does not contain neither a wandering domain nor a Baker domain ([**?**]). Hence if $f \in \mathcal{S}$, then we have

$$\{z \mid f^n(z) \to \infty\} \subset J(f).$$

In § **??**, we give an example of a transcendental entire function whose Fatou set contains bounded and unbounded components.

In § **??**, we consider the family $\{f_\lambda(z) = \lambda z e^z \mid \lambda \in \mathbb{C}\}$, which is contained in $\mathcal{S}$. We see that some functions in the family have bounded Fatou components.

Finally, in § **??**, we give an example of a transcendental entire function whose Julia set is a Sierpinski carpet.

Proofs of theorems in this note are given in [**?**].

Some pictures of Fatou set stated in this note can be seen in the author's home page[1].

[1] http://www.math.kochi-u.ac.jp/morosawa/indexe.html

2. An example obtained by a polynomial-like mapping

We give an example of a transcendental entire function whose Fatou set contains a bounded component by using a polynomial-like mapping.

Example 1. Let

$$f(z) = z^2 \exp \frac{1}{2}(1 - z^4).$$

Then the attracting component containing a super-attracting fixed point 0 is bounded.

Indeed, set

$$U = \left\{ z \,\middle|\, |z| < \frac{1}{2} \right\}$$

and $V = f(U)$. Elementary calculation shows that $f(\partial U) = \partial V$ and $U \subset \overline{U} \subset V$. Hence $(f,\ U,\ V)$ is a polynomial-like mapping of degree two and the filled-in Julia set of $(f,\ U,\ V)$ contains the attracting component containing 0, which we denote by D_0. Note that 0 is both a critical point and an asymptotic value of f. Hence, the components of $f^{-1}(D_0)$ except D_0 are unbounded.

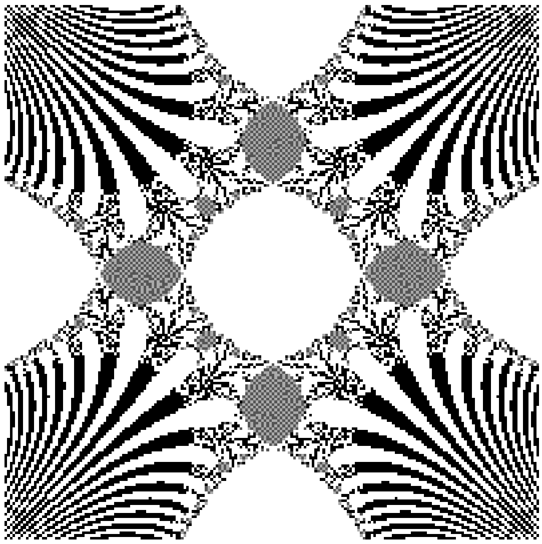

Figure 3. $f(z) = z^2 \exp\,(1 - z^4)/2$

3. Julia sets of $f_\lambda(z) = \lambda z e^z$

We set

$$\mathcal{D} = \{ f_\lambda(z) = \lambda z e^z \mid \lambda \in \mathbb{C} \}.$$

Iteration of elements of $\mathcal{D}$ are studied by some mathematicians (e.g. [?], [?], [?] and [?]). In the case that $\lambda = 0$, we regard that f_0 has an attracting fixed point 0 for convenience. For $f_\lambda(z) \in \mathcal{D}$, we see

$$\operatorname{sing}(f_\lambda^{-1}) = \left\{ f_\lambda(-1) = \frac{\lambda}{e},\ 0 \right\}$$

and the set of the fixed points of f_λ is

$$\{0\} \cup \{-\operatorname{Log} \lambda + 2n\pi i\}_{n \in \mathbb{Z}}.$$

We define

$$B = \{ \lambda \in \mathbb{C} \mid f_\lambda^n(-1) \not\to \infty \}$$

and further

$$B_{-1} = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \quad \text{and}$$
$$B_1 = \{\lambda \in \mathbb{C} \mid |1 - \log \lambda| < 1\}.$$

Since fixed points of f_λ are 0 and $-\mathrm{Log}\ \lambda + 2n\pi i$, it is clear that f_λ has an attracting fixed point if and only if $\lambda \in B_{-1} \cup B_1$ and if $\lambda \in B_{-1}$, then 0 is the attracting fixed point and if $\lambda \in B_1$, then $-\mathrm{Log}\ \lambda$ is the attracting fixed point.

In the case that $\lambda \in B_{-1}$, the attracting component containing the origin is completely invariant, since f_λ is not univalent on it. Hence it contains the critical point of f, which is -1.

If $\lambda \notin \overline{B_{-1}}$, then the asymptotic value 0 is a repelling fixed point. Since f_λ has only two singular values $f_\lambda(-1)$ and 0, we have $J(f_\lambda) = \widehat{\mathbb{C}}$ for $\lambda \notin B$ by the relations between cyclic Fatou components and singular values. Hence we have the following proposition.

Proposition 3.1. *If $\lambda \notin B$, then $J(f_\lambda) = \widehat{\mathbb{C}}$. The function $f_\lambda(z)$ has an attracting fixed point if and only if $\lambda \in B_{-1} \cup B_1$. Further, if $\lambda \in B_{-1}$, then 0 is the attracting fixed point and $f_\lambda^n(-1)$ tends to 0 as n tends to infinity and if $\lambda \in B_1$, then $-\mathrm{Log}\ \lambda$ is the attracting fixed point. Further, if f_λ has an attracting cycle, then $F(f_\lambda)$ is equal to the basin of the attracting cycle.*

We define the following set

$$\Lambda = \{\lambda \in \mathbb{C} \mid |\mathrm{Im}\ \lambda| \geq e\mathrm{Arg}\ \lambda\}.$$

Theorem 1. If $\lambda \in \Lambda$ and $f_\lambda(z) = \lambda z e^z$ has an attracting cycle whose period is greater than one, then each component of $F(f_\lambda)$ is bounded.

To prove the theorem above, we need the following theorem proved by Fagella ([**?**]).

Theorem 2. If $\lambda \in \Lambda$, then there exists a curve $L(t)$ $(t \in [0,\ \infty))$ satisfying the following conditions.
(1) $f_\lambda(L(t)) = L(t)$.
(2) For sufficiently large h

$$L(t) \cap H_h \subset \{z \mid -\pi - \theta \leq \mathrm{Im}\ z \leq \pi - \theta\},$$

where $H_h = \{z \mid \mathrm{Re}\ z \geq h\}$ and $\theta = \mathrm{Arg}\ \lambda$.
(3) $L(0) = 0$.
(4) For $t \neq 0$

$$\lim_{n\to\infty} f_\lambda^n(L(t)) = \infty.$$

Curves in Julia sets of exponential maps were first studied by Devaney and Krych ([**?**]) as Cantor bouquets. In [**?**], Fagella called this $L(t)$ a fixed hair.

Outline of proof of Theorem **??**. Assume that the period of the attracting cycle is $p(> 1)$. Let D_0 be the attracting component of this cycle containing -1. From Theorem **??**, there exists an invariant curve whose end point is 0, which we denote by L. Hence ∂D_0 does not contain the fixed point 0, because the period of attracting cycle is greater than 1. We choose a neighborhood $U_\epsilon = \{z \mid |z| < \epsilon\}$ so that U_ϵ does not intersect D_0 and that $f_\lambda^{-1}(U_\epsilon)$ consists of two simply connected components. We denote the unbounded component of $f_\lambda^{-1}(U_\epsilon)$ by H. It is clear that $H \cap D_0 = \emptyset$ and that H contains some left half plane $\{z \mid \mathrm{Re}\ z < a\}$. Choose l_1 and l_2 from the components of the inverse image of L such that the domain bounded by them contains -1 and does not contain any component of it. Note that each l_i $(i = 1,\ 2)$ intersects both H and H_h in Theorem **??**. Since l_1 and l_2 are in $J(f_\lambda)$, D_0 is contained in the domain bounded by l_1, l_2 and ∂H. Assume that D_0 is unbounded. Then it is in some strip in the right half plane by Theorem **??**. Hence we see that $\{f^{np}(z)\}$ tends to infinity for every z in D_0 whose real part is sufficiently large. This is a contradiction. Hence D_0 is bounded. Since asymptotic value is not on the boundary of any Fatou component, every component of the basin of the attracting cycle is bounded. □

Remark 1 Further, we see that $J(f_\lambda)$ is locally connected if f_λ satisfies the condition in Theorem **??**.
Remark 2 The curve L in the proof does not intersect the boundary of any Fatou component. We call the subset of the Julia set whose points do not lie on the boundary of any component of the Fatou set the residual Julia set. Residual Julia sets for rational functions are considered in [**?**] and those for entire functions are considered in [**?**].

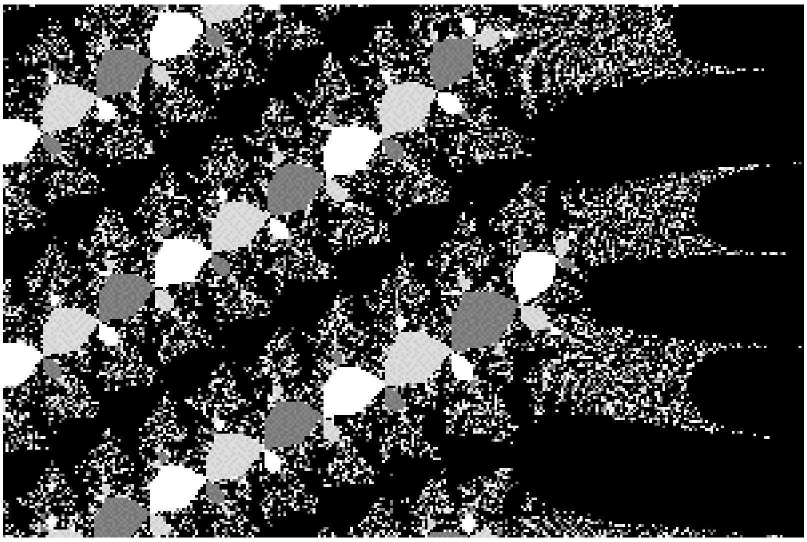

FIGURE 4. $f_\lambda(z) = \lambda z e^z$; $\lambda = 2.68699 + 3.87109i$

4. Another example

We say that a closed subset in $\widehat{\mathbb{C}}$ is a Sierpinski carpet if it is the complement of a countable dense family of open topological discs whose diameters tend to zero and whose closure are pairwise disjoint closed topological discs. Note that any two Sierpinski carpets are homeomorphic. Some rational functions whose Julia sets are Sierpinski carpets are known (see [**?**] [**?**]). We give an example of a transcendental entire function whose Julia set is a Sierpinski carpet by the argument similar to that in the previous section.

Theorem 3. Let

$$g_a(z) = ae^a\{z - (1-a)\}e^z$$

for $a > 1$. Then $J(g_a)$ is a Sierpinski carpet.

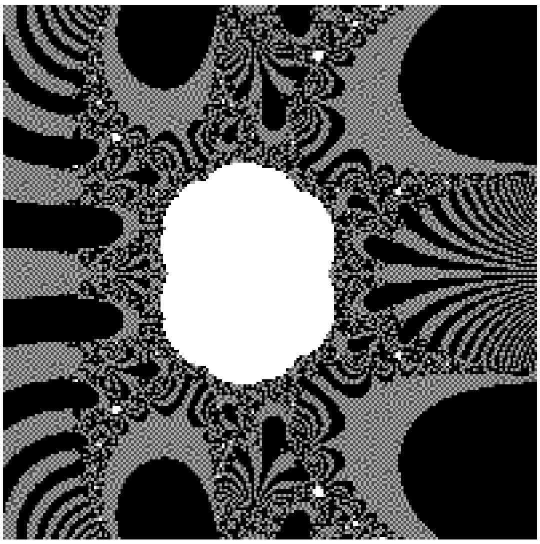

FIGURE 5. $g_a(z) = ae^a\{z - (1-a)\}e^z$; $a = 2$

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On holomorphic curves omitting divisors and related topics

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Partially supported by Grant-in-Aid for Scientific Research (A)(1) 09304014

The above title was slightly modified from the original one. The purpose of this talk is to report and discuss some new results on the subjects of the title. Some new results are joint works with J. Winkelmann.

§1. Tensors and holomorphic mappings

We are interested in the following properties:

[Little Picard] *Let $f : \mathbf{C}^k \to M$ be a holomorphic mapping from the k-dimensional complex affine space into a compact complex manifold M. Assume that* $\operatorname{rank} df = k$*, then the image of f is not Zariski dense (algebraically degenerate).*

[Big Picard] *Let $f : \Delta^* \times \Delta^{k-1} \to M$ be a holomorphic mapping. Assume that* $\operatorname{rank} df = k$*, and the image of f is Zariski dense in M. Then, f extends meromorphically over $\Delta \times \Delta^{k-1}$.*

For instance, if $k = 1$, Schwarz' Lemma implies that

if the holomorphic tangent bundle $\mathbf{T}(M)$ carries a hermitian metric with negative curvature, then M is Kobayashi hyperbolic; $f : \mathbb{C} \to M$ is constant.

[Kobayashi, 75] *If the cotangent bundle $\mathbf{T}^*(M)$ is ample, then M is Kobayashi hyperbolic.*

Hence, we have little and big Picard's theorems for such M.

For $k = m = \dim M$, it follows that

[Kobayashi-Ochiai, 71, 75] *if the canonical bundle $\mathbf{K}_M$ is big, little and big Picard's theorems hold.*

For general $1 \leqq k \leqq m$, we have

[Carlson, 72] *If $\bigwedge^k \mathbf{T}^*(M)$ carries a hermitian metric with positive curvature in the sense of Griffiths, then little and big Picard's theorems hold.*

Let L be a line bundle over M, and let E be a vector bundle over M. Let

$$\pi : \mathbf{P}(E^*) = (E^* \setminus \{O\})/\mathbb{C}^* \to M$$

be the projective bundle. Let $H \to \mathbf{P}(E^*)$ be the hyperplane bundle such that the sections of H correspond to those of E. Let $\tilde{B}(E, L)$ be the stable base locus for $lH - \pi^{-1}L, l = 1, 2, \dots$. Set

$$B(E, L) = \pi(\tilde{B}(E, L)).$$

Then it follows that the condition

(C1) $$B(E, L) \neq M \text{ for big } L$$
is independent of the choice of L, and bimeromorphically invariant.
[Nog., 77] *Assume (C1) for $E = \bigwedge^k \mathbf{T}^*(M)$. Then little and big Picard's theorems hold.*

It is better to obtain a condition described only in terms of $\bigwedge^k \mathbf{T}^*(M)$, which is equivalent to (C1). For that purpose we need to look into a bit more detailed structure than the stable base locus.

In general, let $\lambda : F \to M$ be a line bundle, and set

$$\Phi_l : M \to \mathbf{P}(H^0(N, lF)^*).$$

Let B_l be the base locus of lF, and set

$$C_l = B_l \bigcup \{y \in F \setminus \lambda^{-1} B_l; \; y \in \Phi_l^{-1}(\Phi(y)) \text{ is not an isolated point}\}.$$

Then C_l is an analytic subset. We call C_l the *critical locus* of lF. We set

$$C(F) = \bigcap_{l=1}^{\infty} C_l,$$

and call it the *stable critical locus* of F.

Let $\pi : E \to M$ be a vector bundle, and $H \to \mathbf{P}(E^*)$ be the dual of the tautological line bundle.

Definition. We say that E is *fairly big* if $\pi(C(H)) \neq M$.

Lemma 1.1. *E is fairly big if and only if E satisfies (C1).*

Theorem 1.2. *If $\bigwedge^k \mathbf{T}^*(M)$ is fairly big, then little and big Picard's theorems hold.*

In the proof we use the Stein factorization, and the fact that the pull-back of an ample line bundle by a finite holomorphic mapping is again ample.

Remark. Let M be a surface of general type.
(1) [Bogomolov, 77] If $c_1^2(M) > c_2(M)$, then $\mathbf{T}^*(M)$ is big.
(2) [Lu-Yau, 90] If $c_1^2(M) > 2c_2(M)$, then $\mathbf{T}^*(M)$ is fairly big.

Lu and Yau proved (C1) for such M, and deduced little and big Picard's theorems for $f : \mathbb{C}$ or $\Delta^* \to M$. Write the ratio in order:

$$1 < \frac{c_1^2}{c_2} < 2 < \frac{c_1^2}{c_2} \leqq 3.$$

The last is Miyaoka's inequality. These are all about 1-jets.

§2. Jet differentials

In this section we deal with the case of $k = 1$; $f : \mathbb{C} \to M$. Let $\pi_k : J_k(M) \to M$ be the k-jet bundle over M. A holomorphic functional on $J_k(M)$ which is a polynomial on every fiber is called a (global) k-jet differential. Let $\mathcal{JD}_{k,d}$ denote the sheaf of k-jet differentials which are polynomials of weighted degree d on fibers. Note that
[Nog., 86] *if $\mathcal{JD}_{k,d}$ is "fairly big" in a sense, then little and big Picard's theorems hold for holomorphic curves in M.*

There is its logarithmic version. It is expected that $\mathcal{JD}_{k,d}$ carries more detailed information than holomorphic tensors which are of jet-level 1.

[Basic Idea]: It is the basic idea originally due to Bloch that if there are enough many jet differentials $\phi_j, 1 \leqq j \leqq N$, so that the transcendental basis of the function field of M (here M is assumed to be algebraic) is reproduced by them, then for a non-degenerate (in a sense) holomorphic curve $f : \mathbb{C} \to M$, we have estimates of $\phi_j \circ J_k f$, where $J_k f : \mathbb{C} \to J_k(M)$ denotes the k-jet lifting, so that the order function $T_f(r)$ is bounded as

$$T_f(r) \leqq O(\log r T_f(r)) \quad ||.$$

This implies the degeneracy of f. The problem is reduced to find enough good jet differentials

$$\Phi = (\phi_1, \ldots, \phi_N) : J_k(M) \to \mathbb{C}^N.$$

Definition. A jet differential $\phi : J_k(M) \to \mathbb{C}$ is said to be "*invariant*" or "*conformal*" if for a holomorphic mapping $g : z \in \Delta \to g(z) \in M$ and a change of variable, $z = z(\zeta)$,

$$\phi(J_k(g \circ z)(\zeta)) = \left(\frac{dz(\zeta)}{d\zeta}\right)^d \phi(J_k g(z)).$$

By taking a subspace of the k-jet bundle $J_k(M)$ and and its projectivization $\pi_k : \mathbf{P}_k(M) \to M$, which is called the Semple jet bundle, we have a line bundle $L_k \to \mathbf{P}_k(M)$ such that a global section of L_k is equivalent to a conformal jet differential on $J_k(M)$ ([Demailly, 97]). Let $C_k \subset \mathbf{P}_k(M)$ be the stable critical locus of L_k. Assume that

(C2) $$C_\infty = \bigcap_{k=1}^{\infty} \pi_k(C_k) \neq M.$$

Then we can apply the Basic Idea to a holomorphic curve $f : \mathbb{C} \to M$ to conclude that it has an image included in C_∞; hence it is algebraically degenerate.

The following is an application of this Basic Idea.

[Demailly-Goul, preprint 98] *Let M be a surface of general type. Assume the following:*

1. $\mathrm{Pic}(M) = \mathbf{Z}$*;*
2. $c_1^2 - \frac{9}{10}c_2 > 0$*;*
3. $H^0(S^l\mathbf{T}^*(M)) = \{O\}, \forall l$*;*
4. $H^0(\mathcal{E}_{k,d} \otimes (-tK_M)) = 0$ *for all* $t > 3/4$ *such that* tK_M *is an integral divisor.*

Then every holomorphic $f : \mathbb{C} \to M$ is algebraically degenerate at the level of 2-jet.

[Demailly-Goul, preprint 98] *Let M be a generic hypersurface of $\mathbf{P}^3$ of degree d. Then we have*

1. $\mathrm{Pic}(M) = \mathbf{Z}$*;*
2. $10c_1^2 - 9c_2 = d(d^2 - 44d + 104) > 0$ *for* $d \geqq 42$*;*
3. $H^0(S^l\mathbf{T}^*(M) \otimes \mathcal{O}(k)) = \{O\}, \forall l > 0, k \leqq l$*;*
4. $H^0(\mathcal{E}_{k,d} \otimes (-tK_M)) = 0$ *for* $d \geqq 11$ *and* $t > 1/2$ *such that* tK_M *is an integral divisor.*

Then every holomorphic $f : \mathbb{C} \to M$ is algebraically degenerate at the level of 2-jet.

This combined with McQuillan's work [preprint, 1997] and G. Xu [X94] would imply

Theorem [Demailly-Goul, preprint 98]. *A generic hypersurface of $\mathbf{P}^3$ of degree $\geqq 42$ is Kobayashi hyperbolic.*

Remarks. (1) $c_1^2 = d(d-4)^2 < c_2 = d(d^2 - 4d + 6)$.

(2) Bogomolov's result, the above 2 and 3 imply that

$$\frac{9}{10} < \frac{c_1^2}{c_2} \leqq 1.$$

Thus, the hyperbolicity problem of hypersurfaces of $\mathbf{P}^3$ may be difficult.

Recall another application of the Basic Idea, which is older than the above.

Logarithmic Bloch-Ochiai's Theorem [Nog., 77∼81; cf., Dethloff-Lu, 98]. *Let M be a complex projective algebraic manifold, and let D be a hypersurface. Assume that $q(M \backslash D) = \dim H^0(\Omega^1(M \log D)) > \dim M$. Then every entire holomorphic curve $f : \mathbb{C} \to M \setminus D$ has a non Zariski dense image.*

In the case where $D = \emptyset$, it is easy to show that the same holds for a compact Kähler manifold M. Thus it is natural to ask the case of Kähler M with $D \neq \emptyset$.

Theorem [Nog.-Winkel., 99]. *Let M be a compact Kähler manifold and let D be a hypersurface of M. If the logarithmic irregularity $q(M \setminus D) > dimM$, then the image of an entire holomorphic curve $f : \mathbf{C} \to M \setminus D$ is contained in a proper analytic subset of M.*

For the proof, we first take the quasi-Albanese mapping $\alpha : M \setminus D \to \mathcal{T}$. Then $\mathcal{T}$ is a quasi-torus:

$$0 \to (\mathbb{C})^t \to \mathcal{T} \to T_0 \to 0,$$

where T_0 is the Albanese torus of M. Let B be the maximal closed subgroup which leaves the Zariski closure of $\alpha(M \setminus D)$ invariant. It is a point to show that

the quotient $\mathcal{T}/B$ is again a quasi-torus.

Then one may reduce it to the algebraic case.

In the Diophantine approximation, Vojta generalized Faltings' theorem to

Theorem [Vojta, 96]. *Let K be a number field and S be a finite set of a proper set of inequivalent places (valuations) of K with product formula such that S contains all archimedean places. Let V*

be an algebraic smooth variety defined over a number field K, and let D be a hypersurface of V. If $q(V \setminus D) > \dim V$, then any (D,S)-integral point set A is not Zariski dense in V.

This is an analogue of logarithmic Bloch-Ochiai's theorem. Here the counter objects are

$$\begin{aligned} &\text{a non-constant holomorphic curve } f : \mathbf{C} \to V \setminus D \\ \Longleftrightarrow\ &\text{an infinite } (D,S)\text{-integral point set of } V. \end{aligned}$$

To explain what is a (D,S)-integral point set, we take $K = \mathbf{Q}$. Then S consists of the ordinary absolute value $|\bullet|$ and finitely many primes, $p_i, 1 \leqq i \leqq q < \infty$. Then a rational number of type

$$a = \frac{b}{p_1^{e_1} p_2^{e_2} \cdots p_q^{e_q}}, \quad b, e_i \in \mathbb{Z}$$

is called an S-integer. For the sake of simplicity, assume that D is very ample. Taking a basis $\{\sigma_j\}_{j=0}^N$ of $H^0(V,[D])$ with $(\sigma_0) = D$, we have an affine embedding

$$\Psi = \left(\frac{\sigma_1}{\sigma_0}, \dots, \frac{\sigma_N}{\sigma_0}\right) : V \setminus D \to \mathbf{A}^N.$$

A subset A of the set $V(K)$ of all K-rational points of V is called a (D,S)-integral point set if there is such Ψ that all points of $\Psi(A)$ are S-integral points; that is, its coordinates are S-integers. Cf. S. Lang [L83, L87, L91].

Note that any finite set A is a (D,S)-integral point set, after multiplying large integers to the coordinates; in particular, one point set is always a (D,S)-integral point set. Thus the definition makes sense only for infinite A.

Let M be a compact Kähler manifold of dimension m and let $\{D_i\}_{i=1}^l$ be a family of hypersurfaces of M.

Definition. We say that $\{D_i\}_{i=1}^l$ is *in general position* if for any distinct indices $1 \leqq i_1, \dots, i_k \leqq l$, the codimension of every irreducible component of the intersection $\bigcap_{j=1}^k D_{i_j}$ is k for $k \leqq m$, and $\bigcap_{j=1}^k D_{i_j} = \emptyset$ for $k > m$.

This notion is defined for singular M as well.

Let $\mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^l$ denote the $\mathbf{Z}$-rank of the subgroup of $H^2(M,\mathbf{R})$ generated by $\{c_1(D_i)\}_{i=1}^l$. Let $\mathrm{NS}(M)$ denote the Neron-Severi group of M; i.e., $\mathrm{NS}(M) = \mathrm{Pic}(M)/\mathrm{Pic}^0(M)$. We know that

$$\mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^l \leqq \mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(M).$$

Theorem [Nog.-Winkel., 98]. *Let $\{D_i\}_{i=1}^l$ be a family of hypersurfaces of M in general position. Let $W \subset M$ be a subvariety such that there is a non-constant holomorphic curve $f : \mathbf{C} \to W \setminus \bigcup_{D_i \not\supset W} D_i$ with Zariski dense image. Then we have that*

1. $\#\{W \cap D_i \neq W\} + q(W) \leqq \dim W + \mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^l$;
2. *Assume that all D_i are ample. Then we have*

$$(l-m)\dim W \leqq m\left(\mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^l - q(W)\right)^+.$$

Here $(\cdot)^+$ stands for the maximum of 0 and the number. We have the following corollary which provides also examples.

Corollary [Nog.-Winkel., 98]. *Let the notation be as above.*

1. *Assume that all D_i are ample and that $l > m(\mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(M)+1)$. Then $M \setminus \bigcup_{i=1}^l D_i$ is complete hyperbolic and hyperbolically imbedded into M.*
2. *Let $X \subset \mathbf{P}^m(\mathbf{C})$ be an irreducible subvariety, and let $D_i, 1 \leqq i \leqq l$, be distinct hypersurface cuts of X that are in general position as hypersurfaces of X. If $l > 2\dim X$, then $X \setminus \bigcup_{i=1}^l D_i$ is complete hyperbolic and hyperbolically imbedded into X.*
3. *Let $\{D_i\}_{i=1}^l$ be a family of ample hypersurfaces of M in general position. Let $f : \mathbf{C} \to M$ be a holomorphic curve such that for every D_i, either $f(\mathbf{C}) \subset D_i$, or $f(\mathbf{C}) \cap D_i = \emptyset$. Assume that $l > m$. Then $f(\mathbf{C})$ is contained in an algebraic subspace W of M such that*

$$\dim W \leqq \frac{m}{l-m}\mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(M).$$

In special, if $M = \mathbf{P}^m(\mathbf{C})$, then we have

$$\dim W \leqq \frac{m}{l-m}.$$

Remark. The above Corollary, (ii) for $X = \mathbf{P}^m(\mathbf{C})$ was given by Babets [B84], but his proof seems to carry some incompleteness and confusion. In the case of $\mathbf{P}^m(\mathbf{C})$ and hyperplanes D_i, the above (ii) with $X = \mathbf{P}^m(\mathbf{C})$ and (iii) for $f : \mathbf{C} \to \mathbf{P}^m(\mathbf{C}) \setminus \bigcup_{i=1}^{l} D_i$ were first proved by Fujimoto [F72] and Green [G72], where the linearity of W was also proved, and by their examples the dimension estimate is best possible in general.

In the Diophantine approximation we have the following analogues.

Theorem [Nog.-Winkel., 98]. *Assume that everything is defined over a number field K, and S is a finite subset of a proper set $M(K)$ of inequivalent places of K with product formula such that S contains all infinite places. Let V be a projective smooth variety of dimension m. Let $\{D_i\}_{i=1}^{l}$ be a family of ample hypersurfaces of V in general position. Let $W \subset V$ be a subvariety of V. Assume that there exists a Zariski dense $(\sum_{D_i \not\supset W} D_i \cap W, S)$-integral point set of $W(K)$. Then we have*

$$(l-m)\dim W \leqq m\left(\mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^{l} - q(W)\right)^{+}.$$

Corollary [Nog.-Winkel., 98]. *Let the notation be as above.*

1. *Assume that all D_i are ample and that $l > m(\mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(V)+1)$. Then any $(\sum_{i=1}^{l} D_i, S)$-integral point set of $V(K)$ is finite.*
2. *Let $X \subset \mathbf{P}_K^m$ be an irreducible subvariety, and let $D_i, 1 \leqq i \leqq l$, be distinct hypersurface cuts of X that are in general position as hypersurfaces of X. If $l > 2\dim X$, then any $(\sum_{i=1}^{l} D_i, S)$-integral point set of $X(K)$ is finite.*
3. *Let $D_i, 1 \leqq i \leqq l$, be ample divisors of V in general position. Let A be a subset of $V(K)$ such that for every D_i, either $A \subset D_i$, or A is a $(\sum_{D_i \not\supset A} D_i, S)$-integral point set. Assume that $l > m$. Then A is contained in an algebraic subvariety W of V such that*

$$\dim W \leqq \frac{m}{l-m}\mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(V).$$

In special, if $V = \mathbf{P}_K^m$, then we have

$$\dim W \leqq \frac{m}{l-m}.$$

The dimension estimates obtained above are optimal. These generalize and improve the result of M. Ru and P.-M. Wong [RW91], where they dealt with the case of $V = \mathbf{P}_K^m$ and hyperplanes D_i. In fact, they proved that if A is a $(\sum_{i=1}^{l} D_i, S)$-integral point set, then A is contained in a finite union W of linear subspaces such that

$$\dim W \leqq (2m+1-l)^{+}.$$

Cf. $\dim W \leqq m/(l-m)$ of Corollary, 3.

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Analytic properties of Painlevé transcendents

Shun Shimomura

1. Introduction

Consider an algebraic differential equation of the form

$$F(z, w, w') = 0, \tag{1}$$

where $F(z, w_0, w_1)$ is a polynomial in (w_0, w_1) with coefficients analytic in z. We say that equation (1) has the Painlevé property, if and only if, for every solution, all the movable singularities (namely the singularities depending on initial conditions) are at most poles. For example any solution of the equation

$$(w')^2 = 4w^3 - g_2 w - g_3 \tag{2}$$

is expressible in the form $\wp(z - z_0)$ and hence this equation has the Painlevé property. Equation (1) with the Painlevé property is reduced to equation (2) or to an equation of Riccati type, or is solvable by quadrature.

The problem to find 2-nd order nonlinear equations of the form

$$w'' = R(z, w, w') \tag{3}$$

is far more difficult, where $R(z, w_0, w_1)$ is a rational function of (w_0, w_1) with coefficients analytic in z. The following result is known by P. Painlevé and B. Gambier ([12], [2]).

Theorem 1. *Equation* (3) *with the Painlevé property other than a linear equation, equation* (2), *or an equation solvable by quadrature, is reduced to one of the following*:

$$\text{(I)} \qquad w'' = 6w^2 + z,$$

$$\text{(II)} \qquad w'' = 2w^3 + zw + \alpha,$$

$$\text{(III)} \qquad w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w},$$

$$\text{(IV)} \qquad w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w},$$

$$\text{(V)} \qquad w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \gamma\frac{w}{z} + \delta\frac{w(w+1)}{w-1},$$

$$\text{(VI)} \qquad w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right),$$

where $\alpha, \beta, \gamma, \delta$ *are complex constants.*

The nonlinear equations given above are called Painlevé equations and their transcendental solutions are called Painlevé transcendents.

Equation (VI) is also derived from an isomonodromic deformation of a linear equation of Fuchsian type. R. Fuchs [1] treated an equation of the form

$$\text{(4)} \qquad \frac{d^2u}{dx^2} = \left(\frac{a_0}{x^2} + \frac{a_1}{(x-1)^2} + \frac{a_\infty}{x(x-1)} + \frac{b_1}{(x-z)^2} + \frac{3}{4(x-w)^2} + \frac{hz(z-1)}{x(x-1)(x-z)} - \frac{w(w-1)\nu}{x(x-1)(x-w)}\right)u$$

such that

(a) $x = w$ is a non-logarithmic singular point, namely it is a regular singular point at which the fundamental system of solutions does not contain logarithmic terms,

(b) at each regular singular point, except $z = w$, the characteristic exponents do not differ by an integer.

Here the coefficients a_0, a_1, a_∞, b_1 are determined by the characteristic exponents, and h and ν are accessary parameters. The non-logarithmic condition (a) implies that $h = h(z, w, \nu)$ is written as a certain rational function of z, w, ν. He proved that equation (4) has a fundamental system of solutions whose monodromy representation is independent of z if and only if $w = w(z)$ and $\nu = \nu(z)$ are certain analytic functions. In particular $w = w(z)$ satisfies the equation (VI). Furthermore R. Garnier [3] showed that the other five equations are derived from isomonodromic deformations of linear differential equations with irregular singular points. For example the isomonodromic deformation of

$$\text{(5)} \qquad \frac{d^2u}{dx^2} = \left(4x^3 + 2zx + 2h + \frac{3}{4(x-w)^2} - \frac{\nu}{x-w}\right)u$$

yields equation (I). These linear equations follow from (4) by confluences of singularities, and the corresponding Painlevé equations are also obtained from (VI) by the use of a process of step-by-step degeneration ([11]).

Theorem 2. *The Painlevé equations have the Painlevé property.*

Each linear equation stated above follows from a Schlesinger system of the form

$$\frac{dU}{dx} = A(x, z, w)U, \qquad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{6}$$

where $A(x, z, w)$ is a 2 by 2 matrix function whose components are rational functions of (x, z, w), and the corresponding Painlevé equation is derived from an isomonodromic deformation of this system ([4], [5]). By virtue of [8], [9], from this fact the Painlevé property follows. A rigorous proof due to the method of the original proof of Painlevé is given by [20]. By the Painlevé property, all the solutions of (I), (II), (IV) are meromorphic on the whole complex plane $\mathbb{C}$, those of (III), (V) are meromorphic on the universal covering $\mathcal{R}(\mathbb{C} - \{0\})$, and those of (VI) are meromorphic on $\mathcal{R}(\mathbb{C} - \{0, 1\})$.

2. Analytic expressions of Painlevé transcendents

The fixed singular points $z = 0, 1, \infty$ of (VI), $z = 0$ of (V) (or of (III)) are of regular type. Near each fixed singularity of this type there exists a two-parameter family of solutions expressible by convergent series ([16]).

Theorem 3. *Let κ be an arbitrary complex constant and let ω be an arbitrary constant such that $\omega \in \mathbb{C} - (-\infty, 0] - [1, +\infty)$. Then*

(i) *equation* (III) *admits a solution expressed as*

$$w_{III}(\omega, \kappa, z) = e^{-\kappa} z^{2\omega-1}\big(1 + O(|z|^2 + |e^{-\kappa} z^{2\omega}| + |e^{\kappa} z^{2(1-\omega)}|)\big)$$

in the domain $\{z \in \mathcal{R}_0 \mid z^2 \in \mathcal{D}(r_0)\}$;

(ii) *equation* (V) *admits a solution expressed as*

$$w_V(\omega, \kappa, z) = 1 - e^{-\kappa} z^{\omega}\big(1 + O(|z| + |e^{-\kappa} z^{\omega}| + |e^{\kappa} z^{1-\omega}|)\big)$$

in the domain $\mathcal{D}(r_0)$;

(iii) *equation* (VI) *admits a solution expressed as*

$$w_{VI}(\omega, \kappa, z) = e^{-\kappa} z^{\omega}\big(1 + O(|z| + |e^{-\kappa} z^{\omega}| + |e^{\kappa} z^{1-\omega}|)\big)$$

in the domain $\mathcal{D}(r_0)$.
Here r_0 is a sufficiently small positive constant depending on ω, $\mathcal{R}_0$ is the universal covering of $\mathbb{C} - \{0\}$, and $\mathcal{D}(r_0) = \{z \in \mathcal{R}_0 \mid |z| < r_0,\ |e^{-\kappa} z^{\omega}| < r_0,\ |e^{\kappa} z^{1-\omega}| < r_0\}$.

In special cases, the distribution of poles or zeros along the real axis is clarified.

Theorem 4. (i) *Assume that $\alpha = \gamma = 0$ in* (VI)*. Then, for any real constants c and c', equation* (VI) *admits a solution expressed as*

$$W_{VI}(c, c', z) = \cos^{-2}(c \log z + c' + o(1))$$

as $z \to 0$ through the sector $|\arg z| < \varepsilon$, ε being a sufficiently small positive constant.

(ii) *Assume that $\alpha = \beta = 0$ in* (V)*. Then, for any real constants c and c', equation* (V) *admits a solution expressed as*

$$W_V(c, c', z) = -\tan^2(c \log z + c' + o(1))$$

as $z \to 0$ through the sector $|\arg z| < \varepsilon$.

The fixed singular point $z = \infty$ of (J) (J=I, ..., V) is of irregular type. Near $z = \infty$, there exist solutions expressed asymptotically. For example equation (V) admits solutions given below ([15]).

Theorem 5. *Assume that $\beta = 0$, $\delta > 0$, $\alpha, \gamma \in \mathbb{R}$ in* (V)*. Let $\phi(z)$ be an arbitrary solution of* (V) *such that $0 < \phi(1) < 1$, $\phi'(1) \in \mathbb{R}$. Then*

$$\phi(z) = R_0(1 + o(1))z^{-1} \cos^2(\sqrt{\delta/2}z - C(R_0) \log z + \theta_0 + o(1)),$$
$$C(R_0) = (\gamma/4)\sqrt{2/\delta} - \sqrt{\delta/2}R_0,$$

as $z \to +\infty$ along the positive real axis, where R_0 (> 0) and θ_0 ($\in \mathbb{R}$) are constants depending on the initial values $\phi(1)$, $\phi'(1)$.

3. Value distribution of Painlevé transcendents

Substituting the Laurent series expansion near $z = \infty$ into (I), and comparing coefficients, we have the following.

Theorem 6. *Any solution of* (I) *is a transcendental meromorphic function.*

For equation (II), not every solution is transcendental. For example, when $\alpha = -1$, there exists a rational solution $w = 1/z$. Value distribution properties of transcendental meromorphic solutions of (I) and (II) are studied by [13], [14], [21].

Theorem 7. *Let $\phi(z)$ be a solution of* (I)*. Then, for every $a \in \mathbb{C} \cup \{\infty\}$, $\delta(a, \phi) = 0$, and for every $a \in \mathbb{C}$, $\vartheta(a, \phi) \le 1/6$.*

Theorem 8. *Let $\phi(z)$ be a transcendental meromorphic solution of* (II)*. If $\alpha \neq 0$, then, for every $a \in \mathbb{C} \cup \{\infty\}$, $\delta(a, \phi) = 0$. If $\alpha = 0$, then, for every $a \in \mathbb{C} \cup \{\infty\} - \{0\}$, $\delta(a, \phi) = 0$, and $\delta(0, \phi) \le 1/2$.*

For solutions of (II), the following estimates of $\vartheta(a, \phi)$ is known by [6].

Theorem 9. *Let $\phi(z)$ be a transcendental meromorphic solution of* (II)*. If $\alpha \neq 0$, then $\vartheta(0, \phi) \le 1/5$, and if $\alpha = 0$, then $\vartheta(0, \phi) = 0$. For $a \in \mathbb{C} - \{0\}$, $\vartheta(a, \phi) \le 1/4$.*

In the proofs of these results the following lemma is useful ([10]; Theorem 6).

Lemma 10. *Let $F(z, u)$ be a polynomial in u and its derivatives with meromorphic coefficients $a_\nu(z)$, $\nu \in N$. Assume that $u = f$ is a transcendental meromorphic solution of the differential equation $F(z, u) = 0$, and that c_0 is a complex constant satisfying $F(z, c_0) \not\equiv 0$. Then*

$$m(r, 1/(f - c_0)) = O\Big(\sum_{\nu \in N} T(r, a_\nu) + \log T(r, f)\Big)$$

as $t \to \infty$ outside a possible exceptional set of finite linear measure.

Let $g(z)$ be an arbitrary rational function, and $\phi(z)$ be an arbitrary meromorphic solution of (I). The function $W = \phi(z) - g(z)$ satisfies

$$W'' = 6W^2 + 12g(z)W + p(z), \tag{7}$$

where $p(z) = 6g(z)^2 + z - g''(z) \not\equiv 0$ (cf. Theorem 6). By Lemma 10, any meromorphic solution of (I) is complete with respect to every rational function.

Theorem 11. *An arbitrary meromorphic solution $\phi(z)$ of* (I) *satisfies $\delta(g(z), \phi) = 0$ for every rational function $g(z)$.*

Concerning equation (IV), the following results are known ([19]).

Theorem 12. *Let $\phi(z)$ be an arbitrary transcendental meromorphic solution of* (IV)*.*

(i) *If $\beta \neq 0$, then for every $a \in \mathbb{C} \cup \{\infty\}$, $\delta(a, \phi) = 0$. If $\beta = 0$, then for every $a \in \mathbb{C} \cup \{\infty\} - \{0\}$, $\delta(a, \phi) = 0$.*

(ii) *Suppose that $\beta = 0$. If $\phi(z)$ does not satisfy an equation of Riccati type*

$$w' = \mp(2zw + w^2), \tag{8}$$

then $\delta(0, \phi) \le 1/2$.

Remark There exists a solution satisfying (IV) and (8) simultaneously, if and only if $\alpha = \pm 1$. Every solution of (8) is expressed as $\phi_C(z) = e^{\pm z^2}\Big(C \pm \int e^{\mp z^2} dz\Big)^{-1}$, which satisfies $\delta(0, \phi_C) = 1$.

Theorem 13. *An arbitrary transcendental meromorphic solution $\phi(z)$ of* (IV) *satisfies $\theta(a, \phi) \leq 1/4$ for every $a \in \mathbb{C} - \{0\}$.*

Equations (III) and (V) are treated on the universal covering $\mathcal{R}(\mathbb{C} - \{0\})$. By $z = e^s$, equation (III) and (V) are changed into

$$(\mathrm{III}_0) \qquad \frac{d^2w}{ds^2} = \frac{1}{w}\left(\frac{dw}{ds}\right)^2 + e^s(\alpha w^2 + \beta) + e^{2s}\left(\gamma w^3 + \frac{\delta}{w}\right)$$

and

$$(\mathrm{V}_0) \qquad \frac{d^2w}{ds^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{ds}\right)^2 + (w-1)^2\left(\alpha w + \frac{\beta}{w}\right) + \gamma e^s w + \delta e^{2s}\frac{w(w+1)}{w-1},$$

respectively. Furthermore by $W = e^{s/2}w$, $z = e^{s/2}$, equation (III) is changed into

$$(\mathrm{III}'_0) \qquad \frac{d^2W}{ds^2} = \frac{1}{W}\left(\frac{dW}{ds}\right)^2 + \alpha W + \gamma W^3 + \beta e^s + \frac{\delta e^{2s}}{W}.$$

Value distribution of meromorphic solutions of these equations is studied in [6], [17], [18]. The same problem concerning (VI) is also treated on $\mathcal{R}(\mathbb{C} - \{0, 1\})$ ([6]).

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Holomorphic Solutions of Some Functional Equations II

Mami Suzuki

We will consider here a simultaneous system of difference equations

$$\begin{cases} x(s+1) = X(x(s), y(s)) \\ y(s+1) = Y(x(s), y(s)) \end{cases} \tag{1}$$

with

$$\begin{cases} X = X(x,y) = \lambda x + \displaystyle\sum_{m+n\geq 2, m\geq 1} p_{mn}x^m y^n \\ Y = Y(x,y) = \mu y + \displaystyle\sum_{m+n\geq 2, n\geq 1} q_{mn}x^m y^n \end{cases} \tag{2}$$

where the series on the right hand side are supposed to converge in $\{(x,y)\,;\,|x| < t, |y| < t\}$ for a t, $0 < t \leq \infty$.

Let $x(t)$, $y(t)$ be a pair of (1) with the property

$$x(s+s') \to 0\,, \quad y(s+s') \to 0 \quad \text{as} \quad s' \in \mathfrak{R},\, s' \to -\infty,$$

uniformly for $s \in \mathfrak{K}$, in which $\mathfrak{K}$ is an arbitrarity compact subset of the $s-$plane.

Suppose there is a function $\psi(x)$:

$$\psi(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots\,,\ |x| < \delta \tag{3}$$

for a δ such that $y(s) = \psi(x(s))$, $s \in G$, in which G is a domain on the s-plane with the property that $s \in G$ implies $s - 1 \in G$. Then the system (1) is reduced to a single equation

$$x(s+1) = X(x(s), \psi(x(s))), \quad s+1 \in G, \tag{4}$$

which is important for the study of asymptotic behaviors of solutions of (1).

By (4) we have

$$\psi(X(x(s), \psi(x(s)))) = \psi(x(s+1)) = y(s+1) = Y(x(s), \psi(x(s))),$$

which means

$$\psi(X(x, \psi(x))) = Y(x, \psi(x)), \tag{5}$$

for $x \in G$, in which G is a domain on the x-plane. Thus, if the functional equation (5) admits a solution $\psi(x)$, then the system (1) is reduced to a single equation (4). Therefore it is of significance to investigate equation (5).

The equation (5) has been studied previously for the case $|\lambda| \neq 1, 0$. Now we consider the case $|\lambda| = 1$. In this note we will consider the case $\lambda = \mu = 1$.

By the fact that characteristic values λ, μ are both equal to 1, we are confronted with difficulties in determining a formal solution of (5) in the form (3). We suppose that $X(x,y)$ and $Y(x,y)$ in (2) are of the form

$$\begin{cases} X(x,y) = x + X_1(x,y), \qquad X_1(x,y) = \displaystyle\sum_{m+n\geq 2, m\geq 1} p_{mn}x^m y^n \\ Y(x,y) = y + Y_1(x,y), \qquad Y_1(x,y) = \displaystyle\sum_{m+n\geq 2, n\geq 1} q_{mn}x^m y^n, \end{cases} \tag{6}$$

that is

$$p_{0n} = 0\,(n \geq 2) \quad \text{and} \quad q_{m0} = 0\,(m \geq 2).$$

Further, as shown in the next section, we are forced to lay down the condition

$$kp_{20} = q_{11} \neq 0 \quad \text{for some} \quad k \in \mathfrak{N},\ k \geq 2. \tag{8}$$

Suppose the condition (8) is satisfied.

Let $p_{20} = |p_{20}|e^{i\alpha}$. Put $x = e^{(\pi-\alpha)i}u$, $y = v$ in (6), then we obtain

$$\begin{cases} U = U(u,v) \\ V = V(u,v) \end{cases} \tag{9}$$

with

$$\begin{cases} U(u,v) = u + \sum_{m+n\geq 2, m\geq 1} P_{mn}u^m v^n \\ V(u,v) = v + \sum_{m+n\geq 2, n\geq 1} Q_{mn}u^m v^n \end{cases} \tag{10}$$

where $P_{mn} = p_{mn}e^{(m-1)(\pi-\alpha)i}$, $Q_{mn} = q_{mn}e^{m(\pi-\alpha)i}$. Then $P_{20} = -|p_{20}| < 0$ and $Q_{11} = q_{11}e^{(\pi-\alpha)i} = kp_{20}e^{(\pi-\alpha)i} = kP_{20} < 0$. So we may suppose in (6) that

$$kp_{20} = q_{11} < 0 \quad \text{for some } k \in \mathfrak{N},\ k \geq 2. \tag{11}$$

Put, for some κ and δ,

$$D = D(\kappa, \delta) = \{x; |\arg[x]| < \kappa,\ 0 < |x| < \delta\}. \tag{12}$$

Now we will state our result in this note:

Theorem 0.1. *Suppose $X(x,y)$ and $Y(x,y)$ be of the form (6). Then*
(1) If $kp_{20} \neq q_{11}$ for any $k \in \mathfrak{N}$, $k \geq 2$, then the formal solution $\psi(x)$ of the form (3) are identical to 0, i.e., $a_2 = a_3 = \cdots = 0$.
(2) If $kp_{20} = q_{11}$ for some $k \in \mathfrak{N}$, $k \geq 2$, then we have a formal solution $\psi(x)$ of the form (3) with

$$\psi(x) = \sum_{j=k}^{\infty} a_j x^j, \tag{12}$$

i.e., $a_j = 0$ if $2 \leq j < k$.
(3) Suppose (11). For any κ, $0 < \kappa \leq \frac{\pi}{2}$, there are a $\delta > 0$ and a solution $\psi(x)$ of (5), which is holomorphic and is expanded asymptotically in $D(\kappa, \delta)$ as

$$\psi(x) \sim \sum_{j=k}^{\infty} a_j x^j.$$

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Uniqueness theorems for meromorphic functions sharing five small meromorphic functions

Nobushige TODA

1. Introduction

Let $f(z)$ be a transcendental meromorphic function in the complex plane and let $\boldsymbol{S}(f)$ be the set of meromorphic functions $a(\neq\infty)$ in the complex plane which satisfy

$$T(r,a)=S(r,f),$$

where $S(r,f)$ is any quantity satisfying

$$S(r,f)=o(T(r,f))$$

for $r\to\infty$ except possibly a set of r of finite linear measure. Such a meromorphic function $a(z)$ is said to be *small* for f. The set $\boldsymbol{S}(f)$ is a field over $\boldsymbol{C}$.

We put for $a\in\boldsymbol{S}(f)\cup\{\infty\}$

$$E(f=a)=\{z:f(z)-a(z)=0\},$$

where $f(z)-\infty$ means $1/f(z)$.

Throughout the paper we shall use the standard notation of the Nevanlinna theory of meromorphic functions([2],[4]).

For a positive integer k let

$$n_{k)}(r,1/f)\ \text{(resp. } n_{(k}(r,1/f)\ )$$

be the number of zeros of f with order $\leq k$ (resp. $\geq k$) counting multiplicities in $|z|\leq r$ and put for $r>0$

$$N_{k)}(r,\frac{1}{f})=\int_0^r\frac{n_{k)}(t,\frac{1}{f})-n_{k)}(0,\frac{1}{f})}{t}dt+n_{k)}(0,\frac{1}{f})\log r$$

(resp.

$$N_{(k}(r,\frac{1}{f})=\int_0^r\frac{n_{(k}(t,\frac{1}{f})-n_{(k}(0,\frac{1}{f})}{t}dt+n_{(k}(0,\frac{1}{f})\log r).$$

Similarly for a positive integer k let

$$\overline{n}_{k)}(r,1/f)\ \text{(resp. } \overline{n}_{(k}(r,1/f)\ )$$

be the number of zeros of f with order $\leq k$ (resp. $\geq k$) ignoring multiplicities in $|z|\leq r$ and put for $r>0$

$$\overline{N}_{k)}(r,\frac{1}{f})=\int_0^r\frac{\overline{n}_{k)}(t,\frac{1}{f})-\overline{n}_{k)}(0,\frac{1}{f})}{t}dt+\overline{n}_{k)}(0,\frac{1}{f})\log r$$

(resp.

$$\overline{N}_{(k}(r,\frac{1}{f})=\int_0^r\frac{\overline{n}_{(k}(t,\frac{1}{f})-\overline{n}_{(k}(0,\frac{1}{f})}{t}dt+\overline{n}_{(k}(0,\frac{1}{f})\log r).$$

The truncated counting function $N_k(r,1/f)$ is defined by

$$N_k(r,\frac{1}{f})=N_{k)}(r,\frac{1}{f})+k\overline{N}_{(k+1}(r,\frac{1}{f}).$$

Then, $\overline{N}(r,1/f)=N_1(r,1/f)$. We put for $a\in\boldsymbol{S}(f)$

$$\delta(a,f)=1-\limsup_{r\to\infty}\frac{N(r,1/(f-a))}{T(r,f)}$$

Then, $1\geq\delta(a,f)\geq 0$.

Let f_1 and f_2 be two transcendental meromorphic functions in the complex plane. The following "Unicity Theorem" is well-known:

Unicity Theorem of Nevanlinna. If for five distinct elements $a_1, \cdots, a_5 \in \overline{\boldsymbol{C}}$

$$E(f_1 = a_j) = E(f_2 = a_j) \quad (j = 1, \cdots, 5),$$

then $f_1 = f_2$ ([4],p.109; see also [2], p.48).

This theorem is sharp([4], p.111). It is an open problem to extend this theorem to the case when $a_1, \cdots, a_5$ belong to $\{\boldsymbol{S}(f_1) \cap \boldsymbol{S}(f_2)\} \cup \{\infty\}$.

For $a, b \in \boldsymbol{S}(f)$ such that $0, 1, a, b$ are distinct and one of a and b is not constant, Zhang Qing De([5]) used the determinant

$$\begin{vmatrix} ff' & f' & f^2 - f \\ aa' & a' & a^2 - a \\ bb' & b' & b^2 - b \end{vmatrix}$$

to prove the following

Theorem A. Let f_1 and f_2 be two transcendental meromorphic functions in the complex plane. If for six distinct elements $a_1, \cdots, a_6 \in \{\boldsymbol{S}(f_1) \cap \boldsymbol{S}(f_2)\} \cup \{\infty\}$

$$E(f_1 = a_j) = E(f_2 = a_j) \quad (j = 1, \cdots, 6),$$

then, $f_1 = f_2$ (Theorem 2 in [5]).

Further, for $a, b \in \boldsymbol{S}(f)$ as above we introduced the determinant

$$\Delta_f = \begin{vmatrix} f(f-1) & (f')^2 & ff' \\ a(a-1) & (a')^2 & aa' \\ b(b-1) & (b')^2 & bb' \end{vmatrix}.$$

By using some properties of this determinant and Lemma 4 in [3] we proved the following

Theorem B. Let f_1 and f_2 be two transcendental meromorphic functions in the complex plane. Suppose that there exist five distinct elements $a_1, \cdots, a_5 \in \{\boldsymbol{S}(f_1) \cap \boldsymbol{S}(f_2)\} \cup \{\infty\}$ satisfying

$$E(f_1 = a_j) = E(f_2 = a_j) \quad (j = 1, \cdots, 5)$$

and

$$\overline{N}(r, 1/(f_1 - a_5)) \leq uT(r, f_1) + S(r, f_1)$$

for some $u \in [0, 1/19)$, then $f_1 = f_2$ (Corollary 4 in [3]).

The purpose of this paper is to give some uniqueness theorems containing an improvement of Theorem B for meromorphic functions sharing five small meromorphic functions.

2. Preliminary and lemma

We shall give some lemmas first.

Lemma 1([4]). Let $f(z)$ be a transcendental meromorphic function in the complex plane. Then we have the following inequalities:

(I) For q distinct elements $a_1, \cdots, a_q \in \overline{\boldsymbol{C}}$ $(q < \infty)$,

$$(q-2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}(r, 1/(f - a_j)) + S(r, f).$$

(II) For three distinct elements a_1, a_2 and $a_3 \in \boldsymbol{S}(f) \cup \{\infty\}$

$$T(r, f) \leq \sum_{j=1}^{3} \overline{N}(r, 1/(f - a_j)) + S(r, f).$$

Lemma 2([5]). Let $f(z)$ be a transcendental meromorphic function in the complex plane and let $a_1, \cdots, a_5$ be distinct elements of $\boldsymbol{S}(f) \cup \{\infty\}$. Then

$$2T(r, f) \leq \sum_{j=1}^{5} \overline{N}(r, 1/(f - a_j)) + S(r, f).$$

Let $f(z)$ be a transcendental meromorphic function in the complex plane, a, b be distinct meromorphic functions contained in $\boldsymbol{S}(f)$. Then, we have the following two lemmas.

Lemma 3. Suppose that $0, 1, a, b$ are distinct and both a and b are not constant. If

$$\overline{N}(r, f) \neq S(r, f), \tag{2.1}$$

then, $\Delta_f(z) \not\equiv 0$ (see Lemma 6 in [3]).

Lemma 4. (I) Suppose that $0, 1, a, b$ are distinct and both a and b are not constant. If $\Delta_f \neq 0$ (for example, under the condition (1)), then we have the inequality

$$2T(r, f) < N_{1)}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-1}) + \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f-b}) + 2\overline{N}(r, f) + S(r, f)$$

(see Lemma 6 in [3]).

(II) Let $a_1, \cdots, a_5$ be distinct elements in $\boldsymbol{S}(f) \cup \{\infty\}$ such that $a_j \neq \infty$ $(j = 1, 2, 3, 4)$ and

$$\overline{N}(r, 1/(f - a_5)) \neq S(r, f).$$

Then, we have the inequality

$$2T(r, f) \leq N_{1)}(r, \frac{1}{f - a_j}) + \sum_{k=1, k\neq j}^{5} \overline{N}(r, \frac{1}{f - a_k}) + \overline{N}(r, \frac{1}{f - a_5}) + S(r, f),$$

where $1 \leq j \leq 4$.

Proof. (I) is Lemma 5 in [3].

(II) By a suitable linear transformation of f, we easily obtain the inequality.

Lemma 5([1], se also Lemma 2 in [3]). Let $f(z)$ be a transcendental meromorphic function in the complex plane. Then for any q distinct elements $a_1, \cdots, a_q \in \boldsymbol{S}(f)$ $(2 \leq q < \infty)$ the following inequality holds:

$$(q - 1)T(r, f) \leq \sum_{j=1}^{q} N_m(r, \frac{1}{f - a_j}) + m\overline{N}(r, f) + S(r, f),$$

where m is the number of elements of a maximal linearly independent subset of $\{a_1, \cdots, a_q\}$.

Further we shall give some lemmas for later use.

Let f_1 and f_2 be two transcendental meromorphic functions in the complex plane. We suppose that there are five distinct elements a_1, a_2, a_3, a_4 and a_5 in $\{\boldsymbol{S}(f_1) \cap \boldsymbol{S}(f_2)\} \cup \{\infty\}$ satisfying the following conditions:

(i) $a_j \neq \infty$ $(j = 1, 2, 3, 4)$;

(ii) $E(f_1 = a_j) = E(f_2 = a_j)$ $(j = 1, \cdots, 5)$.

Lemma 6. Under these circumstances, the following relations hold.

(I) $T(r, f_2) = T(r, f_1) + S(r, f_1)$ and $S(r, f_2) = S(r, f_1)$.

(II) If $f_1 \neq f_2$, then

$$2T(r, f_1) = \sum_{j=1}^{5} \overline{N}(r, 1/(f_1 - a_j)) + S(r, f_1).$$

(I) is trivial when $f_1 = f_2$. We can prove (I) and (II) when $f_1 \neq f_2$ by using Lemma 2.

Lemma 7. Under the same circumstances as in Lemma 6, if $f_1 \neq f_2$,

(a) $T(r, f_1) \leq 2\{\overline{N}(r, 1/(f_1 - a_i)) + \overline{N}(r, f_1 - a_j))\} + S(r, f_1)$ $(1 \leq i \neq j \leq 5)$.

(b) $\overline{N}(r, 1/(f_1 - a_i)) + \overline{N}(r, 1/(f_1 - a_j)) \leq T(r, f_1) + S(r, f_1)$ $(1 \leq i \neq j \leq 5)$.

(c) $\overline{N}(r, 1/(f_1 - a_j)) \leq (2/3)T(r, f_1) + S(r, f_1)$ $(1 \leq j \leq 5)$.

Proof. (a) We apply Lemma 1(II) to f_1 and a_i, a_j, a_k, where i, j, k are distinct. We then have

$$T(r, f_1) \leq \overline{N}(r, 1/(f_1 - a_i)) + \overline{N}(r, 1/(f_1 - a_j)) + \overline{N}(r, 1/(f_1 - a_k)) + S(r, f_1).$$

Adding them for $k \neq i, j$, we have

$$3T(r, f_1) \leq \sum_{k=1}^{5} \overline{N}(r, 1/(f_1 - a_k) + 2\{\overline{N}(r, 1/(f_1 - a_i)) + \overline{N}(r, 1/(f_1 - a_j))\} + S(r, f_1).$$

Here, we use Lemma 6(II) to obtain (a) of this lemma.

(b) By Lemma 1(II) and Lemma 6(II) we easily obtain (b) of this lemma.

(c) For example we prove this inequality for $j = 5$. Applying Lemma 1(II) to any three elements of a_1, a_2, a_3, a_4 to obtain the inequality

$$(4/3)T(r, f_1) \leq \sum_{j=1}^{4} \overline{N}(r, 1/(f_1 - a_j)) + S(r, f_1)$$

Here we use Lemma 6(II) to obtain (c) of this lemma for $j = 5$.

Let $[z_1, z_2, z_3, z_4]$ be the cross-ratio of four distinct points z_1, z_2, z_3, z_4 in $\overline{\boldsymbol{C}}$. We note that when $[z_1, z_2, z_3, z_4] = \lambda$, for any permutation (p_1, p_2, p_3, p_4) of $(1, 2, 3, 4)$, the cross-ratio of $[z_{p_1}, z_{p_2}, z_{p_3}, z_{p_4}]$ is equal to one of the following six values:

$$\lambda, \quad 1/\lambda, \quad 1 - \lambda, \quad 1/(1 - \lambda), \quad \lambda/(\lambda - 1), \quad (\lambda - 1)/\lambda.$$

Let $f(z)$ be a transcendental meromorphic function in the complex plane and $a_1, \cdots, a_5$ distinct elements in $\boldsymbol{S}(f) \cup \{\infty\}$ such that $a_j \neq \infty$ $(j = 1, 2, 3, 4)$ and we put as follows:

$$\hat{f} = [f, a_2, a_1, a_5], \ a = [a_3, a_2, a_1, a_5], \ b = [a_4, a_2, a_1, a_5].$$

By the mapping $f \to \hat{f}$, a_1, a_2, a_3, a_4 and a_5 correspond to $0, 1, a, b$ and ∞ respectively.

Lemma 8. Under these circumstances, the following relations hold.

(a) $0, 1, a, b$ are distinct and $a, b \in \boldsymbol{S}(f)$;

(b) $T(r, \hat{f}) = T(r, f) + S(r, f)$ and $S(r, \hat{f}) = S(r, f)$.

(c) For each $1 \leq j \leq 5$

$$\overline{N}(r, 1/(\hat{f} - \alpha_j)) = \overline{N}(r, 1/(f - a_j)) + S(r, f),$$

where $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = a, \alpha_4 = b$ and $\alpha_5 = \infty$.

We can easily prove these relations.

3. Theorem

Let f_1 and f_2 be transcendental meromorphic functions in the complex plane with five distinct, small meromorphic functions $a_1, \cdots, a_5 \in \{\boldsymbol{S}(f_1) \cap \boldsymbol{S}(f_2)\} \cup \{\infty\}$ such that

(i) $a_j \neq \infty$ $(j = 1, 2, 3, 4)$;

(ii) $E(f_1 = a_j) = E(f_2 = a_j)$ $(j = 1, \cdots, 5)$.

We shall give several uniqueness theorems for meromorphic functions under these circumstances in this section. We put $A = \{a_1, a_2, a_3, a_4, a_5\}$.

Theorem 1. Suppose that one of the following condtions $(C_1), (C_2)$ and (C_3) holds:

(C_1) There is an element $a_{j_1} \in A$ satisfying

$$uT(r, f_1) \leq \overline{N}(r, 1/(f_1 - a_{j_1})) + S(r, f_1)$$

for a constant $u \in (2/3, 1]$.

(C_2) There are two elements $a_{j_1}, a_{j_2} \in A$ satisfying

$$uT(r, f_1) \leq \overline{N}(r, 1/(f_1 - a_{j_1})) + \overline{N}(r, 1/(f_1 - a_{j_2})) + S(r, f_1)$$

for a constant $u \in (1, 2]$.

(C_3) There are two elements $a_{j_1}, a_{j_2} \in A$ satisfying

$$\overline{N}(r, 1/(f_1 - a_{j_1})) + \overline{N}(r, 1/(f_1 - a_{j_2})) \leq uT(r, f_1) + S(r, f_1)$$

for a constant $u \in [0, 1/2)$.

Then, $f_1 = f_2$.

Proof. Suppose that $f_1 \neq f_2$.

(C_1) By Lemma 7(c) and the condition (C_1) we have the inequality

$$uT(r,f_1)\le \overline{N}(r,1/(f_1-a_{j_1}))+S(r,f_1)\le (2/3)T(r,f_1)+S(r,f_1),$$

which reduces to the inequality $(u-\frac{2}{3})T(r,f_1)\le S(r,f_1)$. This is a contradiction, since $u-\frac{2}{3}>0$. This means that $f_1=f_2$ must hold under the condition (C_1).

(C_2) By Lemma 7(b) and the condition (C_2) we have the inequality

$$uT(r,f_1)\le \overline{N}(r,1/(f_1-a_{j_1}))+\overline{N}(r,1/(f_1-a_{j_2}))+S(r,f_1)\le T(r,f_1)+S(r,f_1),$$

which reduces to the inequality $(u-1)T(r,f_1)\le S(r,f_1)$. This is a contradiction, since $u-1>0$. This means that $f_1=f_2$ must hold under the condition (C_2).

(C_3) By Lemma 7(a) and the condition (C_3) we have the inequality

$$(1/2)T(r,f_1)\le \overline{N}(r,1/(f_1-a_{j_1}))+\overline{N}(r,1/(f_1-a_{j_2}))+S(r,f_1)\le uT(r,f_1)+S(r,f_1),$$

which reduces to the inequality $(\frac{1}{2}-u)T(r,f_1)\le S(r,f_1)$. This is a contradiction, since $\frac{1}{2}-u>0$. This means that $f_1=f_2$ must hold under the condition (C_3).

Theorem 2. If there is a permutation (p_1,p_2,p_3,p_4,p_5) of $(1,2,3,4,5)$ such that

$$[a_{p_1},a_{p_2},a_{p_3},a_{p_4}]$$

is equal to a constant, then $f_1=f_2$.

Proof. Suppose that $f_1\ne f_2$. According to Theorem B,

$$\overline{N}(r,1/(f_1-a_j))\ne S(r,f_1)\quad (j=1,\cdots,5). \tag{3.1}$$

We put for brevity $a_{p_j}=b_j\ \ (j=1,2,3,4,5)$.

Put for $i=1,2$ as in the last part of Section 2

$$g_i=[f_i,b_2,b_3,b_4],\quad c=[b_1,b_2,b_3,b_4]\ \text{ and }\ d=[b_5,b_2,b_3,b_4].$$

and

$$\beta_1=0,\quad \beta_2=1,\quad \beta_3=c,\quad \beta_4=d,\quad \beta_5=\infty.$$

Then, $g_1\ne g_2,\ c,d\in \boldsymbol{S}(g_1)\cap \boldsymbol{S}(g_2)$ and $0,1,c,d$ are distinct.

Let $[b_1,b_2,b_3,b_4]=\alpha\in \boldsymbol{C}$. α is neither zero nor 1. By Lemma 1(I) and Lemma 8, we have the followings:

$$\begin{aligned}
2T(r,f_1)+S(r,f_1)=2T(r,g_1) &\le \sum_{j=1,j\ne 4}^{5}\overline{N}(r,1/(g_1-\beta_j))+S(r,g_1);\\
2T(r,f_1)+\overline{N}(r,1/(f_1-b_5))+S(r,f_1) &= 2T(r,g_1)+\overline{N}(r,1/(g_1-\beta_4))\\
&\le \sum_{j=1}^{5}\overline{N}(r,1/(g_1-\beta_j))+S(r,g_1)\\
&= \sum_{j=1}^{5}\overline{N}(r,1/(f_1-a_j))+S(r,f_1),
\end{aligned}$$

from which we have by Lemma 6 $\overline{N}(r,1/(f_1-b_5))=S(r,f_1)$, which contradicts with (6). This means that $f_1=f_2$ must hold.

Remark. For two functions a,b given in the last part of Section 2, the following relations hold:

1) (i) $[a_3,a_2,a_1,a_5]=a$, (ii) $[a_4,a_2,a_1,a_5]=b$, (iii) $[a_3,a_4,a_1,a_5]=a/b$,
(iv) $[a_2,a_3,a_4,a_5]=(1-b)/(a-b)$, (v) $[a_1,a_2,a_3,a_4]=a(1-b)/b(1-a)$.

2) If $[a_2,a_3,a_4,a_5]\equiv\alpha\ (\ne 0,1)$, then $\alpha a+(1-\alpha)b=1$.

3) If $[a_1,a_2,a_3,a_4]\equiv\alpha\ (\ne 0,1)$, then $\lambda\frac{1}{a}+(1-\lambda)\frac{1}{b}=1$, where $\lambda=\alpha/(1-\alpha)$.

Corolary 1. (I) If any four of $a_1,\cdots a_5$ are in $\overline{\boldsymbol{C}}$, then $f_1=f_2$.

(II) Suppose that $a_5=\infty$. If one of the following conditions (i),(ii), (iii) and (iv) holds for a constant $\alpha(\ne 0,1)$:

(i) $a_3=(1-\alpha)a_1+\alpha a_2$, (ii) $a_4=(1-\alpha)a_1+\alpha a_2$, (iii) $a_4=(1-\alpha)a_1+\alpha a_3$,

(iv) $a_2 = (1-\alpha)a_4 + \alpha a_3$,
then $f_1 = f_2$.

Proof. (I) It is easy to see that at least one of the cross ratios in Theorem 2 is equal to a constant which is not equal to 0 nor 1 in this case.
(II) For each k (=i, ii, iii, iv), from the condition (k) of this corollary we easily have that the cross ratio (k) of 1) in Remark 1 is equal to α, which is neither equal to 0 nor 1.

Theorem 3. If There is an element $a_j \in A$ satisfying

(C) $\overline{N}(r, 1/(f_1 - a_j)) \le uT(r, f_1) + S(r, f_1)$

for a constant $u \in [0, 2/17)$, then $f_1 = f_2$.

Proof. Suppose that $f_1 \ne f_2$. For convenience we suppose without loss of generality that $j = 5$ in (C). We have only to prove this theorem under the condition $\overline{N}(r, 1/(f_1 - a_5)) \ne S(r, f_1)$ according to Thoerem B. We can then apply Lemma 4(II) here.

For each j $(1 \le j \le 4)$, we have by Lemma 4(II) and by Lemma 6 that

$$\begin{aligned} 2T(r, f_1) + \overline{N}_{(2}(r, 1/(f_1 - a_j)) &\le \sum_{k=1}^{5} \overline{N}(r, 1/(f_1 - a_k)) + \overline{N}(r, 1/(f_1 - a_5)) + S(r, f_1) \\ &= 2T(r, f_1) + \overline{N}(r, 1/(f_1 - a_5)) + S(r, f_1), \end{aligned}$$

which reduces to

$$\overline{N}_{(2}(r, 1/(f_1 - a_j)) \le \overline{N}(r, 1/(f_1 - a_5)) + S(r, f_1) \ (j = 1, 2, 3, 4). \tag{3.2}$$

Put for i=1,2

$$g_i = [f_i, a_2, a_1, a_5], \quad a = [a_3, a_2, a_1, a_5], \quad b = [a_4, a_2, a_1, a_5]$$

as in the last part of Secton 2. We have only to prove this theorem when the condition of Theorem 2 is not satisfied. Namely, according to Remark 1 we consider the folowing two cases here:

(A) $1, a$ and b are linearly independent.

(B) $1, a$ and b are linearly dependent and

$$1 = \alpha a + \beta b \quad (\alpha + \beta \ne 1, \alpha\beta \ne 0).$$

When $1, a$ and b satisfy (A) or (B), it is easy to see that two functions in each class given below are linearly independent over $\boldsymbol{C}$:

$$\{1, a\}, \{1, b\}, \{a, b\}, \{a-1, b-1\}.$$

We apply Lemma 5 to the followings:

(A_1) g_1 and $0, 1, a$; (A_2) g_1 and $0, 1, b$; (A_3) g_1 and $0, a, b$;
(A_4) $g_1 - 1$ and $0, a-1, b-1$.

We apply Lemma 5 to these four cases. In these four cases, $q = 3$ and $m = 2$ in Lemma 5. Using Lemma 8 we have the following inequality for the case (A_{5-i})

$$2T(r, f_1) \le \sum_{j=1, j\ne k}^{4} N_2(r, 1/(f_1 - a_j)) + 2\overline{N}(r, 1/(f_1 - a_5)) + S(r, f_1) \tag{3.3}$$

for $k = i$ $(i = 1, 2, 3, 4)$.

From (7) and (8) we obtain the inequalities

$$2T(r, f_1) \le \sum_{j=1, j\ne k}^{4} \overline{N}(r, 1/(f_1 - a_j)) + 5\overline{N}(r, 1/(f_1 - a_5)) + S(r, f_1) \tag{3.4}$$

for $k = 1, 2, 3, 4$.

Adding these four inequalities in (9) side by side and using Lemma 6(II) we have the inequality

$$\begin{aligned} 8T(r, f_1) &\le 3\sum_{j=1}^{5} \overline{N}(r, 1/(f_1 - a_j)) + 17\overline{N}(r, 1/(f_1 - a_5)) + S(r, f_1) \\ &\le 6T(r, f_1) + 17\overline{N}(r, 1/(f_1 - a_5)) + S(r, f_1); \end{aligned}$$

that is to say,

$$2T(r, f_1) \leq 17\overline{N}(r, 1/(f_1 - a_5)) + S(r, f_1).$$

Then, by (C) for $j = 5$ we have the inequality

$$2T(r, f_1) \leq 17uT(r, f_1) + S(r, f_1),$$

which reduces to

$$(2 - 17u)T(r, f_1) \leq S(r, f_1).$$

This is a contradiction since $2 - 17u > 0$. This means that $f_1 = f_2$ must hold under the condition (C).

Corollary 2. If for some j $(1 \leq j \leq 5)$

$$\delta(a_j, f_1) > 15/17,$$

then $f_1 = f_2$.

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Nevanlinna theory and linear differential equations

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This work was supported in part by a Grant-in-Aid for General Scientific Research from the Ministry of Education, Science and Culture 09740095

1. Introduction

In this note, some relations between the classical Nevanlinna theory and linear differential equations,

$$w^{(n)} + a_{n-1}w^{(n-1)} + \cdots + a_1 w' + a_0 w = 0\,, \tag{1.1}$$

are discussed. For the sake of simplicity, the coefficients a_j are supposed to be polynomials or transcendental entire functions. Hence we are to observe the behavior of these *entire* solutions near the point at infinity. Our main purpose is to make a comparison of their value distribution between when all the coefficients a_j are polynomials and when there exist some transcendental coefficients. Concerning this topic, there are a lot of nice reference works, for example, the article [**?**] by Eremenko and the book [**?**], where we find almost all fundamental results and notations frequently utilized in our discussion. As concluding remarks we observe some open problems.

2. Polynomial coefficients

Let us consider the equation (**??**), whose coefficients a_j are polynomials. Then the following result, called the *Wiman-Valiron method*, is an indispensable device for our purpose:

Lemma 1. *Let g be a transcendental entire function and $\nu(r)$ its central index. Then there exists a set $\boldsymbol{F} \subset \mathbf{R}_+$ of finite logarithmic measure, $\int_{\boldsymbol{F}} dr/r < \infty$, such that*

$$\frac{g^{(j)}(z_0)}{g(z_0)} = \left(\frac{\nu(r)}{z_0}\right)^j \Big(1 + o(1)\Big), \tag{2.1}$$

for $j \in \mathbf{N}$ and for a point $z_0, |z_0| = r \notin \boldsymbol{F}$ being chosen as

$$|g(z_0)| = M(r, g) := \max_{|z|=r} |g(z)|.$$

Here the *central index* $\nu(r) = \nu(r, g)$ of an entire function $g(z) = \sum_{n=0}^{\infty} c_n z^n$ is defined as the greatest exponent m such that

$$|c_m| r^m = \max_{n \geq 0} |c_n| r^n.$$

Remark 1. The *order* $\rho(g)$ of an entire function g,

$$\rho(g) := \limsup_{r \to \infty} \frac{\log \log M(r, g)}{\log r},$$

is also given by

$$\rho(g) = \limsup_{r \to \infty} \frac{\log \nu(r)}{\log r}.$$

Applyig the Wiman-Valiron method, we have

Theorem A. (Wittich [?] *The coefficients a_j of Eq.* (**??**) *are polynomials if and only if all solutions of Eq.* (**??**) *are entire functions of finite (rational) order.*

Remark 2. Gundersen-Steinbart-Wang [?] proved the number, which had been obtained by Wittich,

$$\gamma := 1 + \max_{0 \leq j \leq n-1} \frac{\deg a_j}{n - j}$$

to be the sharp upper bound for the possible orders.

In order to detail our study, we shall consider some results on the distribution of values of entire solutions of Eq. (**??**). We now introduce a fundamental notation of the Nevanlinna theory of meromorphic functions on the complex plane **C**.

Definition. *For a meromorphic function f,*

- **Counting function:**

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ is the sum of multipilicites at poles of f in $|z| \leq t$.

- **Proximity function:**

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x := \log \max(x, 1)$.

- **Nevanlinna's order function:**

$$T(r, f) := N(r, f) + m(r, f).$$

Remark 3. The order $\rho(f)$ of a meromorphic function f is defined by

$$\rho(f) := \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

Nevanlinna's first main theorem,

$$T(r,f) = T\left(r, \frac{1}{f-c}\right) + O(1)\,, \quad c \in \mathbf{C}$$

tells us the "equivalence" of each value distribution for a given function f, even if it is holomorphic.

By definition, each of the three functionals satisfies asymptotic inequalities:

$$\begin{aligned} \phi(r, af+bg) &\leq \phi(r,f) + \phi(r,g) + O(1)\,, \quad a,b \in \mathbf{C}\setminus\{0\}\,, \\ \phi(r, f\cdot g) &\leq \phi(r,f) + \phi(r,g) \quad f,g \text{ meromorphic}\,. \end{aligned}$$

There is a significant estimate, called *Lemma on Logarithmic Derivative*, which plays the role corresponding to the Wiman-Valiron method for meromorphic functions.

Lemma 2. *For a nonconstant function $f(z)$ meromorphic on $\mathbf{C}$, put*

$$Q(r, f:j) := m\left(r, \frac{f^{(j)}}{f}\right)\,, \quad j \in \mathbf{N}.$$

If the order $\rho(f)$ is finite, $Q(r,f:j)/\log r$ is finite as $r \to \infty$. Even if $\rho(f) = \infty$, $Q(r,f:j)/\log\big(rT(r,f)\big)$ is bounded on a set Ω such that $\mathbf{R}_+ \setminus \Omega$ is of finite Lebesgue measure.

Remark 4. We distinguish functions f according to the growth of their order function. The most primitive criterion is whether the limit of $T(r,f)/\log r$ is finite or not, as $r \to \infty$. This is the case if and only if f is rational.

Theorem B (Wittich [?]). *Let $a_0 \not\equiv 0$ in Eq. (**??**). We can deduce the following estimates to any nontrivial solution w:*

$$m\left(r, \frac{1}{w-c}\right) = O(\log r)\,, \quad c \in \mathbf{C}\setminus\{0\}\,. \tag{2.2}$$

$$m\left(r, \frac{w}{w'}\right) = O(\log r)\,. \tag{2.3}$$

$$T(r,w') = T(r,w) + O(\log r)\,. \tag{2.4}$$

$$N\left(r, \frac{1}{w'}\right) + m\left(r, \frac{1}{w}\right) = T(r,w) + O(\log r)\,. \tag{2.5}$$

$$N\left(r, \frac{1}{w'}\right) = N\left(r, \frac{1}{w}\right) + O(\log r)\,, \tag{2.6}$$

and so on.

Proof. These are verified by the direct application of Lemma 2 and Theorem A. In fact:

- (**??**) follows by virtue of the identity

$$\frac{1}{w-c} = -\frac{1}{a_0}\left(\frac{w^{(n)}}{w-c} + \cdots + a_0\right)\,.$$

- (**??**) follows from the identity

$$\frac{w}{w'} = -\frac{1}{a_0}\left(\frac{w^{(n)}}{w'} + \cdots + a_2\frac{w''}{w'} + a_1\right)\,.$$

- (**??**) follows from (**??**), i.e.

$$\begin{aligned} m(r,w') &\leq m(r,w'/w) + m(r,w) = m(r,w) + O(\log r)\,, \\ m(r,w) &\leq m(r,w/w') + m(r,w') = m(r,w') + O(\log r)\,. \end{aligned}$$

- Nevanlinna's first main theorem shows

$$m\left(r, \frac{w'}{w}\right) + N\left(r, \frac{w'}{w}\right) = m\left(r, \frac{w}{w'}\right) + N\left(r, \frac{w}{w'}\right) + O(1)\,,$$

  from which (**??**) and (**??**) follow. □

We should note that these asymptotic *equalities* imply the further *equality* to the following Nevanlinna's second main theorem, which can never occur in general: *Let f be a nonconstant meromorphic function, let $q \geq 2$ and let $c_1, \cdots, c_q \in \mathbf{C}$ be distinct points. Then*

$$(q-1)T(r,f) \leq N(r,f) \quad + \quad \sum_{n=1}^{q} N\left(r, \frac{1}{f-c_n}\right) + \\ + \quad \sum_{n=1}^{q} Q(r, f-c_n : 1) + Q(r, f : 1) + O(1)\,.$$

This is one of the reasons why we study the value distribution of solutions to liner differential equations. (cf. Nevanlinna [?])

The distribution of *zeros* of solutions still remains to be studied. We now select two from among the known results.

Theorem C (Frank [?]**).** *If all coefficients in Eq.* (**??**) *are polynomials, and if $a_{n-1} \equiv 0$, then the existence of n linearly independent solutions for which 0 is a Picard exceptional value implies that the coefficients are constant.*

Theorem D (Petrenko [?]**).** *If $W = (w_1, \cdots, w_n)$ is an entire curve of finite order such that all functions $\langle W, \mathbf{c}\rangle := \sum_j c_j w_j$, $\mathbf{c} \in \mathbf{C}^n$, have no zeros of multiplicity larger than $n-1$, then $w_1, \cdots, w_n$ is a fundamental system of solutions of Eq.* (**??**) *with polynomial coefficients. (The converse is obvious.)*

3. Transcendental coefficients

We begin this section with

Theorem E Frei [?]**).** *Let a_j be the least transcendental function in the sequence $a_0, \cdots, a_{n-1}$ of coefficients of Eq.* (**??**)*. Then Eq.* (**??**) *possesses at most j linearly independent solutions of finite order.*

From now on, we deal with a specific case that all the coefficients a_j $(j = 0, \cdots, n-1)$ are spanped by a unique transcendental function over the ring of polynomials $\mathbf{C}[z]$. We are concerned with the maximal number of its fundamental solutions having 0 as it Picard exceptional value.

Theorem. *Let $A(z)$ be a transcendental entire function and $p_j(z)$, $q_k(z)$ $(j = 0, \cdots, n-1;\ k = 0, \cdots, s-1)$ be polynomials with $0 \leq s \leq n-1$. Then any non-trivial solution $f(z)$ to the differential equation*

$$w^{(n)} + \sum_{j=0}^{n-1} p_j(z) w^{(j)} = A(z)\left\{ w^{(s)} + \sum_{k=0}^{s-1} q_k(z) w^{(k)} \right\}, \tag{3.1}$$

satisfies the following inequality as $r \to \infty$ outside a set of finite Lebesgue measure [2],

$$m\left(r, \frac{f'}{f}\right) \leq 2\left\{ N\left(r, \frac{f'}{f}\right) + N\left(r, \frac{A'}{A}\right)\right\} + o\left\{ T\left(r, \frac{f'}{f}\right)\right\} + O(\log r)\,, \tag{3.2}$$

except for the "extremal" case when $A(z) = \exp\{(n-s)P(z)\}$ for a nonconstant polynomial $P(z)$ and $f(z)$ is one of $(n-s)$ linearly independent zero-free solutions

$$w_j(z) = \exp\left\{ \eta_j \int^z \exp\{P(t)\}dt - \ell_{n,s} P(z) + P_{n,s}(z) \right\}, \tag{3.3}$$

with a $(n-s)$th root of unity η_j, the integer $\ell_{n,s} = \{n(n-1) - s(s-1)\}/\{2(n-s)\}$ and the polynomial

$$P_{n,s}(z) := \frac{1}{n-s} \int_0^z \{p_{n-1}(t) - q_{s-1}(t)\} dt\,.$$

[2] from now on, we say this simply that *nearly ever where (n.e.)* as $r \to \infty$

Proof (Sketch for the case $n = 2$). We consider Eq.

$$w'' + p_1(z)w' + p_0(z)w = A(z)\Big(w' + q_0(z)w\Big)\,. \tag{3.4}$$

Take a solution f to this Eq. (**??**) such that $f' + q_0 f \not\equiv 0$. To prove our result, we just need to rewrite Eq. (**??**) by means of logarithmic derivatives $\phi := f'/f$ and $\alpha := A'/A$, into Eq.

$$\begin{aligned} & \left\{\alpha' - \alpha^2 - (p_1 - q_0)\alpha + (p_1 - q_0)' + \phi' - \alpha\phi\right\}\phi^2 \\ = \; & [\text{linear differential polynomial in } \phi \text{ over the field } \mathcal{S}(\phi)]\,, \end{aligned}$$

where the field $\mathcal{S}(\phi)$ is given by

$$\begin{aligned} \mathcal{S}(\phi) := \{h \text{ meromorphic} \quad : \quad & m(r,h) = o\{T(r,\phi)\} \\ & \text{as } r \to \infty, r \notin \Omega(h), \textstyle\int_{\Omega(h)} dr < \infty\}\,. \end{aligned}$$

The conclusion follows then by the application of a slight modification of Lemma of Logarithmic derivatives and the argument principle. □

We note that we can similarly deal with a *non-homogeneous* linear differential equation of the form

$$y^{(m)} + \sum_{j=0}^{m-1} \{p_j(z) - q_j(z)A(z)\}\, w^{(j)} = F(z)\,,$$

when a meromorphic function $F(z)$ satisfies, for example,

$$T\left(r, \frac{F'}{F}\right) = O(\log r)\,.$$

Remark 5. The $(n - s)$ fundamental solutions as in (**??**) are entire functions of finite (integral) *hyper-order*,

$$\rho_2(w_j) := \limsup_{r\to\infty} \frac{\log\log T(r, w_j)}{\log r} = \deg P = \rho(A)\,.$$

When A is a transcendental entire function which has a Picard exceptional value, each solution of Eq. (**??**), which has 0 as its unique finite Picard exceptional value, is either of finite order or one of the above $n-s$ fundamental solutions w_j's. Theorem C says that Eq. (**??**) has at most s fundamental solutions of finite order, which may be given by solutions to the simultaneous equations

$$w^{(n)} + \sum_{j=0}^{n-1} p_j(z)w^{(j)} = w^{(s)} + \sum_{k=0}^{s-1} q_k(z)w^{(k)} = 0\,.$$

In the exceptional case, for every fundamental system of solutions $\{f_1, \cdots, f_n\}$ of Eq. (**??**), there exist at least $n - s$ linerly independent vectors $\boldsymbol{c}_k \in \mathbf{C}^n$ such that each function $\langle W, \boldsymbol{c}_k\rangle$ has 0 as a Picard exceptional value for the entire curve W given by $W := (f_1, \cdots, f_n)$.

Example 1. *When $n = 2$, consider the equation*

$$w'' - w' = \left(e^{4z} + 2ie^{3z}\right) w$$

and the solution $f(z) = \exp\,(e^{2z}/2 + ie^z)$. Then the asymptotic equality holds in Inequation (**??**).

Remark 6. In this example, coefficients can be regarded as elements of the linear space spanned by two transcendental functions e^{2z} and e^z. This may partly tell us the complexity of the observation on more general transcendental coefficients.

4. Open problems

In the last section, concentrating the second order case, we present some open problems related to our result.

One of the most challenging problems is the following known as a Bank-Laine Conjecture.

Conjecture (Bank-Laine [?]). *Let f_1, f_2 denote two linearly independent solutions of the second order differential equation*

$$w'' + A(z)w = 0, \quad A(z) \text{ entire and transcendental}. \tag{4.1}$$

Let $\lambda(f)$ denote the exponent of convergence for the zeros of a meromorphic function $f(z)$,

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log N(r, 1/f)}{\log r}.$$

Then $\max(\lambda(f_1), \lambda(f_2)) = \infty$ should hold whenever $2\rho(A) \in (0, \infty)\backslash\mathbf{N}$.

Concerning this conjecture, there are many educational results. Here we present only one of them.

Theorem F (Bank-Laine-Langley) [?]). *Let P be a polynomial of degree $\lambda > 0$, and Q be an entire function of order $\sigma(Q) < \lambda$. Suppose that*

$$f'' + (e^P + Q)f = 0 \tag{4.2}$$

admits a non-trivial solution f such that the exponent of convergence for its zero-sequence $\lambda(f) < \lambda$. Then f has no zeros, Q is a polynomial and

$$Q = -\frac{1}{16}(P')^2 + \frac{1}{4}P''. \tag{4.3}$$

*Moreover, (**??**) admits in this case two linearly independent zero-free solutions.*

This result is sometimes called the "$\frac{1}{16}$-theorem", which is self-explanatory by the special case $P(z) \equiv z$. We may call this exceptional case (**??**) to be *"extremal"* to the $\frac{1}{16}$-theorem. These zero-free solutions are explicitly given by

$$f_{1,2}(z) = H(z)e^{g(z)} = \exp\left(\pm i\int^z e^{P(t)/2}\,dt - \frac{1}{4}P(z)\right).$$

The method used in the proof for our theorem yields a direct extention of this result (see [?]). As its immediate consequence, we also obtain

Corollary *Let A and Π be entire functions of order $\sigma(A)$ and $\sigma(\Pi)$, respectively. Suppose that $\sigma(A) > \sigma(\Pi)$, and for a number $K < 1/4$,*

$$\overline{N}\left(r,\ \frac{1}{A}\right) \le \Big(K + o(1)\Big)m(r,\ A), \qquad n.e..$$

Then there exists a subset Ω of $[1,\ \infty)$ having positive upper logarithmic density, such that

$$\liminf \frac{\overline{N}\left(r,\ \frac{1}{f}\right)}{m(r,\ A)} > 0, \tag{4.4}$$

as $r \to \infty$, $r \in \Omega$ n.e., for any non-trivial solution f of the equation

$$w'' - \{A(z) - \Pi(z)\}w = 0,$$

without the "extremal" case,

$$\begin{aligned} A(z) &= \exp\{2\alpha(z)\}, \\ \Pi(z) &= \big(\alpha'(z)/2\big)' - \big(\alpha'(z)/2\big)^2, \quad \textit{and,} \\ f(z) &= Ce^{-\alpha(z)/2}\exp\left\{\pm\int_0^z e^{\alpha(t)}dt\right\}. \end{aligned}$$

*Especially, if $\sigma(\Pi)$ is strictly less than the lower order of A, then we obtain $\lambda(f) \ge \sigma(A)$ instead of (**??**).*

Our result however gives just a partial answer to the Bank-Laine conjecture. On the other hand, the above corollary applies directly to the

Problem (Bank-Laine). *Suppose that the differential equation (??) has a solution $f(z)$ such that the exponent of convergence $\lambda(f)$ for the zeros of f is less than the order $\rho(A)$ of the entire function $A(z)$. What can be said about the zeros of the solutions of*

$$w'' + \Big(A(z) + \Pi(z)\Big)w = 0 \tag{4.5}$$

if the order of Π is less than the order of A. Especially, under what additional conditions for Π, $\lambda(w) = \infty$ for all solutions of Eq. (??)?

For the other partial result, see Bank, Laine and Langley [?] and Laine [?].

Remark 7. It is, of course, very interesting to consider the corresponding problems to the higher order case, but to formulate them in somewhat natural and solvable context looks quite complicated. Once it can be done, Cartan's theory for holomorphic curves may be very helpful to consider these problems, that is, to two curves, one of which consists of a fundamental system of solutions and the other consists of a sequence of coefficients. It seems to be a natural idea in view of Petrenko's theorem. In a rough statement, the existence of k zero-free fundamental solutions is equivarent to the existence of k omitted hyperplanes of the holomorphic curve given by any fundamental system.

It is still remaining open what relationships on the distribution of values are there between coefficients and fundamental solutions. Making a Polynomial-Transcendental dictionary will be the final interesting problem mentioned in this note. For example, to what number does Wittich's γ as in Remark 2 correspond in the transcendental coefficient case? We close this note by listing some of extremal cases especially when $p_{n-1}(z) \equiv 0$, $P(z) \equiv z$ and $s = 0$ in Theorem.

List: "Extremal" and their "1-perturbed" cases

	$f(z) := \exp\left(\eta_j^{(k)} e^z - \frac{k-1}{2} z\right)$	$F(z) := \left(e^{-z} - \eta_j^{(k)}\right) f(z)$
$k=2$	$w'' - \left(e^{2z} + \frac{1}{4}\right) w = 0$	$w'' - \left(e^{2z} + \frac{9}{4}\right) w = 0$
$k=3$	$w^{(3)} - w' - e^{3z} w = 0$	$w^{(3)} - 4w' - e^{3z} w = 0$
$k=4$	$w^{(4)} - \frac{5}{2} w'' - \left(e^{4z} - \frac{9}{16}\right) w = 0$	$w^{(4)} - \frac{13}{2} w'' - \left(e^{4z} - \frac{25}{16}\right) w = 0$
$k=5$	$w^{(5)} - 5w^{(3)} + 4w' - e^{5z} w = 0$	$w^{(5)} - 10w^{(3)} + 9w' - e^{5z} w = 0$

In this List, functions $f(z)$ are solutions of the "extremal" equation, and functions $F(z)$ are solutons of its perturbed equation so that

$$m(r, A) = \Big(k + o(1)\Big)\left\{\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{A}\right)\right\} \quad r \to \infty, \ \text{ n.e.,}$$

with $A(z) = e^{kz}$ for $k = 2, \cdots, 5$. Here $\{\eta_j^{(k)}\}_{j=1}^k$ is the set of kth roots of unity ($2 \leq k \leq 5$).

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Some Further Results on Factorization of Entire Functions

Hironobu Urabe

For considerable years, we have studied the state of affairs or the manners on factorization (under composition) of entire functions which belong to the class $\mathbf{J}\,(2\pi i)$ (consisting of entire functions of the form: $z + H(z)$, where $H(z + 2\pi i) \equiv H(z)$ is entire).

In this talk, we wish to show some further results on the unique factorizability and the primeness for certain kinds of such entire functions.

The main results are as follows.

Theorem 1. Let h be entire with the order $\rho(h(e^z)) < \infty$ and Q be a polynomial. Then the function F , defined by

$$F(z) \ = \ (z \, + \, \exp\,[h(e^z)]) \circ (z \, + \, Q(e^z)),$$

is uniquely factorizable (up to linear factors).

Theorem 2. Let h be an entire function of finite order $(\rho(h) < \infty)$ and Q be a polynomial ($\neq 0$, identically). Then the function F, defined by

$$F(z) \ = \ z \, + \, Q(e^z) \exp\,[h(e^z)],$$

is prime.

For the proof of these results, we shall make use of Theorems due to W. Bergweiler as well as N. Steinmetz, together with some, already known, fundamental facts.
Actually, we see from Bergweiler's Theorem that the following fact is valid: when $Q_j(z)$ $(j = 1, 2)$ are polynomials such that Q_1 is non-constant and h is a non-constant entire function, the function F, defined by

$$F(z) \ := \ Q_1(z) \, + \, Q_2(z)\, e^{h(z)},$$

is pseudo-prime (in entire sense).

Now, concerning the unique factorizabilty, for instance, we have been unable to prove whether or not the function

$$F(z) \ := \ (z \, + \, \exp\,[e^z]) \circ (z \, + \, \exp\,[e^z])$$

is uniquely factorizable ?

Moreover, it is an open question whether the function

$$F(z) \ := \ z \, + \, Q(e^z)\, e^{H(z)},$$

where Q is a polynomial and H is a periodic entire function with period $2\pi i$, is prime or not ?

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abc estimate, integral points, and geometry of P^n minus hyperplanes

Julie Tzu-Yueh Wang

1. Introduction

Let F be a number field and $\mathcal{H}$ be a set of hyperplanes in$P^n(F)$. Let S be a finite set of valuations of F including all the archimedean valuations. When $\mathcal{H}$ is in general position and the number of hyperplanes in $\mathcal{H}$ is at least $2n+1$, Ru and Wong [RW] proved that the number of the $(S,\mathcal{H})$-integral points is finite; later the author [Wa2] provided an explicit bound on the number. Ru then find a necessary and sufficient condition on $\mathcal{H}$ such that the number of the $(S,\mathcal{H})$-integral points of $P^n(K)-\mathcal{H}$ is finite; he also showed that this is a necessary and sufficient condition of Brody hyperbolicity. However, an explicit bound on the number of the $(S,\mathcal{H})$-integral points is not obtained in [Ru].

Let C be an irreducible nonsingular projective algebraic curve of genus g defined over an algebraically closed field k of characteristic $p \geq 0$. Let K be the function field of C and $\mathcal{H}$ be a set of hyperplanes in $P^n(K)$. Let S be a set consisting of finitely many points of C. When $p=0$, the author [Wa2] showed that if $\mathcal{H}$ is in general position and the number of hyperplanes in $\mathcal{H}$ is at least $2n+1$ then the height of the $(S,\mathcal{H})$-integral points is bounded and the bound is a linear function of $|S|$. When $p>0$, the author [Wa3] showed that if $\mathcal{H}$ is in general position and the number of hyperplanes in $\mathcal{H}$ is at least $2n+2$ then under certain condition the height of the $(S,\mathcal{H})$-integral points is bounded and the bound is a linear function of $|S|$.

Recently, motivated by the abc theorem for function fields (cf. [Ma], [BM], [Vol], [Wa1] and [No]), Buium defined abc varieties and proved that any affine open subset of an abelian variety over function fields (of characteristic 0) with trace zero is an abc variety.(cf. [Bu]) The definition of abc varieties is closely related to the integral points of an affine varieties. It turns out that the previous results on function fields done by the author are all abc varieties.

In the geometric case, as mentioned before that Ru gave a necessary and sufficient condition for $P^n(\mathfrak{C})-\mathcal{H}$ to be Brody hyperbolic. A more general question to consider is when the hyperplanes in $\mathcal{H}$ are moving, i.e. the coefficients of the linear forms corresponding to $\mathcal{H}$ are holomorphic functions. In [Wa4], the author apply the method in [Wa2] and obtained a generalization of the Picard's theorem with moving targets.

In this paper, we will apply an effective method to [Ru] in the number field case and obtain an explicit bound on the number of the $(S,\mathcal{H})$-integral points. In the function field case of zero characteristic, we will show that the condition on $\mathcal{H}$ given by Ru is also necessary and sufficient for the height of the $(S,\mathcal{H})$-integral points to be bounded; and is also a necessary and sufficient condition for $P^n(K)-\mathcal{H}$ to be an abc variety. Therefore, we will prove that $P^n(K)-\mathcal{H}$ is an abc variety if and only if the height of the $(S,\mathcal{H})$-integral points of $P^n(K)-\mathcal{H}$ is bounded. Finally, in the geometric case we will adapt this method to the situation when the coefficients of the linear forms corresponding to the hyperplanes in $\mathcal{H}$ are holomorphic functions.

2. ABC VARIETIES AND (S, D)-INTEGRAL POINTS

In this section we will restrict ourselves to function fields. However, the definition of abc varieties and (S, D)-integral points can be easily adapted to number fields.

Let C be an irreducible nonsingular projective algebraic curve of genus g defined over an algebraically closed field k of characteristic $p \geq 0$. Let K be the function field of C. Now for any affine variety $\mathcal{U}$ over K, we may consider an affine embedding $\mathcal{U} \subset \mathfrak{A}_K^m$. Then for any point $\tau \in \mathcal{U}(K)$ with affine coordinates $x = (x_1, ..., x_m)$, we may define height and conductor as following:

$$h(\tau) = -\sum_{P \in C} \min\{0, v_P(x_1), \ldots, v_P(x_m)\}$$
$$\mathrm{Cond}(\tau) = \{P \in C : \min\{v_P(x_1), \ldots, v_P(x_m)\} < 0\}$$
$$\mathrm{cond}(\tau) = |\mathrm{Cond}(\tau)|.$$

Definition . *We say that $\mathcal{U}$ satisfies the abc estimate if*

$$h(\tau) \ll cond(\tau) + O(1), \quad \text{for every } \tau \in \mathcal{U}(K),$$

where " $\ll$ " means the inequality holds up to multiplication with a positive constant.

Remark . *This definition does not depend on the choice of h and cond. (cf. [Bu])*

Definition . *An affine variety X defined over the algebraic closure K_a of K is an abc variety if it satisfies the abc estimate over any finite extension L of K over which it is defined.*

We now recalled the definition of $(S, D)-$ integral points. (cf. [Voj])

Definition . *Let D be a very ample effective divisor on a projective variety V and let $1 = x_0, x_1, ..., x_N$ be a basis of the vector space:*

$$\mathcal{L}(D) = \{f | f \text{ is a rational function on } V \text{ such that } f = 0 \text{ or } (f) \geq -D\}.$$

Then $\tau \to (x_1(\tau), ..., x_N(\tau))$ defines an embedding of $V(K) - D$ into K^N. A point τ of $V(K) - D$ is said to be an (S, D)-integral point if $v_P(x_i(\tau)) \geq 0$, $1 \leq i \leq N$, f or every $P \notin S$.

We recall some definitions and results from [Ru].

Note . Let $\mathcal{L}$ be a set of linear forms in $n + 1$ variables which are pairwise linearly independent. We denote by $(\mathcal{L})_F$ the vector space generated by the linear forms in $\mathcal{L}$ over F.

Definition . *Let F be a field and $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. We let $\mathcal{L}$ be the corresponding set of linear forms in $n+1$ variables. (We note here that all linear forms in $\mathcal{L}$ are pairwise linearly independent over F.) $\mathcal{H}$ is said to be nondegenerate over F if $\dim(\mathcal{L})_F = n + 1$ and for each proper nonempty subset $\mathcal{L}_1$ of $\mathcal{L}$*

$$(\mathcal{L}_1)_F \cap (\mathcal{L} - \mathcal{L}_1)_F \cap \mathcal{L} \neq \emptyset.$$

Remark . *If $\mathcal{H}$ is in general position and the number of hyperplanes of $\mathcal{H}$ is no less than $2n+1$, then $\mathcal{H}$ is nondegenerate over F.*

Definition . *Let F be a field and $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. Let V be a subspace of $P^n(F)$. V is called $\mathcal{H}$-admissible if V is not contained in any hyperplane in $\mathcal{H}$.*

Proposition (Ru)**.** *Let $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. Then $\mathcal{H}$ is nondegeneate over F if and only if for every $\mathcal{H}$-admissible subspace V of $P^n(F)$ of projective dimension greater than or equal to one, $\mathcal{H} \cap V$ contains at least three distinct hyperplanes which are linearly dependent over F.*

We also need the following version of the abc theorem for function fields.

Theorem (Brownawell-Masser)**.** *Let the characteristic of K is zero. If $f_0, ..., f_n$ are S-units and $f_0 + \cdots + f_n = 1$, then either some proper subsum of $f_0 + \cdots + f_n$ vanishes or*

$$h(f_0, ..., f_n) \leq \frac{n(n+1)}{2} \max\{0,\ 2g - 2 + |S|\}.$$

The main result in this section is the following.

Theorem 2.1. *Let K be the function field of a nonsingular projective algebraic curve C which is defined over an algebraically closed field k with zero characteristic. Let S be a set consisting of finitely many points of C such that there exist nonconstant S-units. Let $\mathcal{H}$ be a set of hyperplanes in $P^n(K)$. Then the following are equivalent*

(a) *$\mathcal{H}$ is nondegenerate over K.*
(b) *The height of the $(S,\mathcal{H})$-integral points of $P^n(K)-\mathcal{H}$ is bounded.*
(c) *The height of the $(S,\mathcal{H})$-integral points of $P^n(K)-\mathcal{H}$ is bounded linearly in $|S|$.*
(d) *$P^n(K)-\mathcal{H}$ is an abc variety.*

Theorem 2.2. *Suppose that the characteristic of K is $p>0$. Let L_i, $0\le i\le 2n+1$, be the linear forms corresponding to $\mathcal{H}$. Let $L_i=X_i$, $0\le i\le n$, and $L_{n+1+i}=\sum_{j=0}^{n}a_{ij}X_j$, $0\le i\le n$, where a_{ij} are elements of K. Let S_n be the permutation group of $\{0,1,2,..,n\}$. Assume that $\mathcal{H}$, are in general position and the set $\{\Pi_{i=0}^{n}a_{i\sigma(i)}|\ \sigma\in S_n\}$ is linearly independent over k, then $P^n(K)-\mathcal{H}$ is an abc variety.*

3. The explicit bound for number fields

The proof of Theorem 1 can be adapted to the number field case directly. However, the S-unit theorem for number fields only provides an explicit bound on the number of S-unit solutions. Therefore, our method can provide explicit bound on the number of $(S,\mathcal{H})$-integral points, but can not say anything about the abc estimate. Let F be a number field of degree d. Denote by M_F as the set of valuations of F and by M_∞ as the set of archimedean valuations of F. We first recall the S-unit theorem by Schlickewei [Sc]:

Theorem (Schlickewei)**.** *Let $a_1,...,a_n$ be nonzero elements of F. Suppose S is a finite subset of M_F of cardinality s, containing M_∞. Then the equation*

$$a_1x_1+\cdots+a_nx_n=3D1$$

has no more than

$$(4sd!)2^{36nd!s^6}$$

solutions in S-units $x_1,...,x_n$ such that no proper subsum $a_{i_1}x_{i_1}+\cdots+a_{i_m}x_{i_m}$ vanishes.

Together with Ru's result (cf. [Ru]) we have the following

Theorem 3.1. *Let F be a number field of degree d. Suppose that S is a finite subset of M_F of cardinality s, containing M_∞. Let $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. $\mathcal{H}$ is nondegenerate if and only if the number of $(S,\mathcal{H})$-integral points of $P^n(F)-\mathcal{H}$ is finite. Furthermore the number of $(S,\mathcal{H})$-integral points of $P^n(F)-\mathcal{H}$ is bounded by*

$$(4sd!)^n2^{36n(n+1)d!s^6}.$$

4. A generalization of the Picard's theorem

A complex space M is called Brody hyperbolic if every holomorphic curve$f:\ \mathfrak{C}\to M$ is constant. Ru proved the following (cf. [Ru]).

Theorem (Ru)**.** *Let $\mathcal{H}$ be a set of hyperplanes in $P^n(\mathfrak{C})$. Then $P^n(\mathfrak{C})-\mathcal{H}$ is Brody hyperbolic if and only if $\mathcal{H}$ is nondegenerate over $\mathfrak{C}$.*

In [Wa4] we extended the classical Picard's theorem to the case where the coefficients of the linear forms corresponding to $\mathcal{H}$ are holomorphic functions. In this section we will improve the results by adapting the proof of Theorem 1.

First, we explain our notation and terminology. Let $L_i(z)(X)=\sum_{j=0}^{n}g_{ij}(z)X_j$, $1\le i\le q$, for all $z\in\mathfrak{C}$, where g_{ij} are holomorphic functions and for each i, $g_{i0},...,g_{in}$ has no common zeroes . Denote by $H_i(z)=\{(x_0,...,x_n)\in P^n(\mathfrak{C})|\ L_i(z)(x_0\ ,...,x_n)=0,\ z\in\mathfrak{C}\}$ as the corresponding moving hyperplane of $L_i(z)$, $1\le i\le q$, and let $\mathcal{H}(z)=\{H_1(z),...,H_q(z)\}$. Let f_0,...,f_n be holomorphic functions on $\mathfrak{C}$ without common zeroes. We say that a holomorphic map f represented by $(f_0,...,f_n)$ is a holomorphic map omitting $\mathcal{H}(z)$ if $L_i(z)(f_0(z),...,f_n(z))\neq 0$ for each $z\in\mathfrak{C}$ and $i=1,...,q$.

Denote by Hol($\mathfrak{C}$) as the ring consisting of all holomorphic functions on $\mathfrak{C}$, and Mero($\mathfrak{C}$) as the field consisting of all meromorphic functions on $\mathfrak{C}$. One can also identify H_i as a hyperplane in $P^n(\mathrm{Hol}(\mathfrak{C}))$. Then $\mathcal{L} = \{L_1, ..., L_q\}$ can be identified as a set of linear forms with holomorphic functions as coefficients. Suppose that the set $\{L_{i_1}, ..., L_{i_m}\}$ is linearly dependent over Hol($\mathfrak{C}$) and any proper subset of $\{L_{i_1}, ..., L_{i_m}\}$ is linearly independent over Mero($\mathfrak{C}$). Then we have a minimal relation

$$a_{i_1} L_{i_1}(X) + \cdots + a_{i_m} L_{i_m}(X) \equiv 0, \tag{A}$$

where a_{i_j} is a nonzero holomorphic function and $a_{i_1}, \ldots, a_{i_m}$ has no common zeros.

Definition . *$\mathcal{H}$ is said to be unitary related if every holomorphic function a_{i_j} which appears in any of the minimal relations (A) has no zero.*

We also need the following Unit Theorem which is a consequence of the Borel's Lemma.

Theorem (Unit Theorem)**.** *Let $u_0, ..., u_m$ be holomorphic functions without zeros and $u_0 + \cdots + u_m = 1$. Suppose that no proper subsum $u_0 + \cdots + u_m - 1 = 0$ vanishes, then $u_0, ..., u_m$ are all constants.*

The main result in this section is the following.

Theorem 4.1. *Let $L_i(z)(X) = \sum_{j=0}^{n} g_{ij}(z) X_j$, $1 \le i \le q$, for all $z \in \mathfrak{C}$, where g_{ij} are holomorphic functions. Denote by $H_i(z)$ the corresponding moving hyperplane of $L_i(z)$, $1 \le i \le q$. Let $\mathcal{H} = \{H_1, ..., H_q\}$ be unitary related. Then $\mathcal{H}$ is nondegenerate over Mero($\mathfrak{C}$) if and only if there exist finitely many $(n+1) \times (n+1)$ invertible matrices with holomorphic functions as entries such that every holomorphic map omitting $\mathcal{H}(z)$ multiplied by one of the matrices is constant. In addition, this set of matrices depends only on the hyperplanes and can be determined effectively.*

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Value distribution theory and its applications.

C.C.Yang

Abstract

Value distribution quantifies,in a simplified sense, zeros of certain equations of transcendental functions. Modern value distribution theory was shaped by R. Nevanlinna around 1925,and since

then it has remained an active research field, among many contemporary complex analysts all over the world. Its applicatory to various problems in complex analysis is obvious, especially when it combines with classical function theory.

In this talk, the speaker ,besides the revealing some of his most recent research results, will give a survey on the applications of value distribution theory obtained by the speaker and his co-workers over the past decade ,in the following four areas :

1.Factorization theory of meromorphic functions . 2. Value sharing and uniqueness of meromorphic functions. 3. complex dynamics. 4. growth of solutions of algebraic differential equations.

It is reminded that many of the results reported here have been extended by the speaker and his co-workers to functions of several complex variable or holomorphic mappings.

Distributions on infinite dimensional vector spaces over p-adics

Kumi Yasuda

1. Extension of measures

If K is an extension field over $\mathfrak{Q}_p$ of finite degree, p-adic norm has a unique extension to K, which we denote by $\|\cdot\|$ again. Let e_K be the ramification degree, f_K the degree of residue class field over $\mathfrak{F}_p$, $N_K := e_K f_K$ the extension degree, and put $r_K := p^{1/e_K}$, $q_K := p^{f_K}$. If π_K is a prime element, then K is interpreted as the set of formal power series

$$\sum_{i=m}^{\infty} \alpha_i \pi_K^i, \qquad m \in \mathfrak{Z},\ \alpha_i \in \{0, 1, \dots, q_K - 1\},$$

and the norm $\|\cdot\|$ is given by

$$\left\| \sum_{i=m}^{\infty} \alpha_i \pi_K^i \right\| = r_K^{-m} \qquad \text{if } \alpha_m \neq 0.$$

Let $L \supset K$ be a field extension and the extension degree $[L : K]$ be finite.

Definition 1.1

We define a K-linear map T_K^L on L to K by

$$T_K^L(x) := \mathrm{Tr_{L,K}([L:K]^{-1}x) = [L:K]^{-1}Tr_{L,K}(x)} = \frac{1}{\mathrm{k}} \sum_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{x_i}, \qquad \mathrm{x \in L},$$

where $k = [K(x) : K]$, and $x = x_1, x_2, \cdots, x_k$ are all distinct conjugates of x over K.

Lemma 1.2.

(i) T_K^L *is a continuous map of* L *onto* K.
(ii) *If* $L \supset F \supset K$ *then* $T_K^L = T_K^F \circ T_F^L$.
(iii) $T_K^L(x) = x$ *for* $x \in K$.

Definition 1.3

Let $\mathfrak{Q}_p^{\mathrm{alg}}$ stand for the algebraic closure of $\mathfrak{Q}_p$. For each extension $K \supset \mathfrak{Q}_p$ of finite degree, define a map T_K on $\mathfrak{Q}_p^{\mathrm{alg}}$ to K by

$$T_K(x) = T_K^L(x) \qquad \text{if } x \in L,\ L \supset K.$$

Put $K_1 = \mathfrak{Q}_p$, and fix an increasing sequence $\mathcal{S} = \{K_n\}_{n=1}^{\infty}$ of extension fields over $\mathfrak{Q}_p$ of finite degrees. Put $B := \cup_{n=1}^{\infty} K_n \subset \mathfrak{Q}_p^{\mathrm{alg}}$.

Examples

E 1.1: $K_n :=$ the smallest field containing all extensions of degrees less than n. $B = \mathbb{Q}_p^{\mathrm{alg}}$.

E 1.2: $K_n :=$ the unramified extension of degree $n!$. B is the maximal unramified extension of $\mathfrak{Q}_p$.

We shall abbreviate subscripts and superscripts K_n to n, e.g. $R_n := R_{K_n}$, $T_n^m := T_{K_n}^{K_m}$. For each n, we denote by T_n the restriction of T_{K_n} to B. We put on B the topology induced by T_n, $n \geq 1$, i.e. the weakest topology relative to which T_n are continuous for all n. Let $\overline{B}$ be the completion of B, and we denote by T_n again the continuation of T_n to $\overline{B}$. Our aim is to extend measures to $\overline{B}$.

We say $\{\mu_n\}_{n=1}^{\infty}$ is a consistent sequence of probability measures (associated with $\mathcal{S} = \{K_n\}_{n=1}^{\infty}$), if μ_n is a probability measure on K_n such that

$$\mu_n(A_n) = \mu_{n+1}\left(\left(T_n^{n+1}\right)^{-1}(A_n)\right)$$

for all n and any Borel set A_n in K_n.

Theorem 1.4. *Assume that we are given a consistent sequence $\{\mu_n\}_{n=1}^{\infty}$ of Borel probability measures. Then there exists a unique Borel probability measure μ_∞ on $\overline{B}$ such that*

$$\mu_\infty\left(T_n^{-1}(A_n)\right) = \mu_n(A_n)$$

for any n and Borel set A_n in K_n.

2. Fourier tramsforms and Consistent measures

Let $K \supset \mathfrak{Q}_p$ be an extension of finite degree. We denote by K^* the group consisting of all characters of K. Let φ_0 be the element of $\mathfrak{Q}_p^*$ defined by

$$\varphi_0\left(\sum_{i=m}^{\infty}\alpha_i p^i\right) = \begin{cases} \exp\left(2\pi\sqrt{-1}\sum_{i=m}^{-1}\alpha_i p^i\right), & \text{if } m \leq -1, \\ 1, & \text{otherwise,} \end{cases}$$

then $\varphi_0(\mathfrak{Z}_p) = \{1\}$ and $\varphi_0(p^{-1}\mathfrak{Z}_p) \neq \{1\}$. For each extension K over $\mathfrak{Q}_p$ of finite degree, $\psi_K^1 := \varphi_0 \circ T_{\mathfrak{Q}_p}^K$ belongs to K^*. Put $l_K := \mathrm{ord}\left(\psi_{\mathrm{K}}^1\right)$. We can identify K^* with K by means of the correspondence

$$x \in K \leftrightarrow \psi_K^x(\cdot) := \psi_K^1(x\cdot) \in K^*,$$

(Theorem 3 and following Corollary in II of [9]).

For a probability measure μ_K on K, we interpret its Fourier transform $\widehat{\mu_K}$ as the function on K by

$$\widehat{\mu_K}(x) = \int_K \psi_K^x(y)\mu_K(dy).$$

A function g on K is the Fourier transform of a probability measure on K, if and only if it is positive definite, continuous, and $g(0) = 1$, (see Theorem 3.2 in IV of [7]).

We have seen in the previous section that a consistent sequence of probability measures can be extended to a probability measure on $\overline{B}$. In order to find extensible measures we shall give a correspondence between probability measures on $\overline{B}$ and functions on B. Let $\mathcal{G}$ be the set of positive definite functions g on B such that $g(0) = 1$ and the restriction to K_n is continuous for every n. We shall particularly observe the case that μ_n is symmetric, i.e. $\mu_n(u_n\cdot) = \mu_n(\cdot)$ for all $u_n \in K_n$ of norm 1. We say a function $g \in \mathcal{G}$ is symmetric if $g(u\cdot) = g(\cdot)$ for any $u \in B$ of norm 1.

Proposition 2.1.

(i) *Probability measures on $\overline{B}$ correspond in one-to-one way to consistent sequences $\{\mu_n\}_{n=1}^{\infty}$.*

(ii) *Consistent sequences $\{\mu_n\}_{n=1}^{\infty}$ correspond in one-to-one way to functions belonging to $\mathcal{G}$. μ_n is symmetric for every n if and only if the correponding function is symmetric.*

By the above proposition, every function g in $\mathcal{G}$ corresponds to a probability measure μ_∞ on $\overline{B}$. The correspondence is given by

$$g(x) = \int_{\overline{B}} \varphi_0 \circ T_1^n(xT_n(w))\mu_\infty(dw), \qquad \text{if } x \in K_n.$$

Here let us give some examples of symmetric functions g in $\mathcal{G}$ and the corresponding consistent sequence of symmetric probability measures.

Examples

[E 2.1] For $\lambda > 0$, put

$$g^{(1)}(x) = \begin{cases} 1, & \text{if } \|x\| \le \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding sequence $\{\mu_n^{(1)}\}_{n=1}^{\infty} = \{\mu_n^{(1)}(\lambda)\}_{n=1}^{\infty}$ is given by

$$\frac{d\mu_n^{(1)}}{dx}(x) = \begin{cases} q_n^{-l_n + \lfloor \frac{\log \lambda}{\log r_n} \rfloor}, & \text{if } \|x\| \le r_n^{l_n - \lfloor \frac{\log \lambda}{\log r_n} \rfloor}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor a \rfloor$ stands for the integer part of a. $\mu_n^{(1)}$ is a Gaussian measure on K_n.

[E 2.2] For $\alpha, \beta > 0$, put

$$g^{(2)}(x) = \exp\left(-\alpha \|x\|^{\beta}\right).$$

The corresponding sequence $\{\mu_n^{(2)}\}_{n=1}^{\infty} = \{\mu_n^{(2)}(\alpha, \beta)\}_{n=1}^{\infty}$ is given by

$$\frac{d\mu_n^{(2)}}{dx}(x) = \|x\|^{-N_n} \sum_{i=0}^{\infty} q_n^{-i} \left\{ \exp\left(-\alpha r_n^{\beta(l_n - i)} \|x\|^{-\beta}\right) - \exp\left(-\alpha r_n^{\beta(l_n - i + 1)} \|x\|^{-\beta}\right) \right\}.$$

$\mu_n^{(2)}$ is a stable law on K_n.

Now consider the case that for every n, $K_n \supset \mathfrak{Q}_p$ is an abel extension with Galois group G_n. Then $B \supset \mathfrak{Q}_p$ is an abel extension and its Galois group consists of sequences $\sigma = (\sigma_1, \sigma_2, \dots)$ of $\sigma_n \in G_n$ satisfying $\sigma_{n+1}|_{K_n} = \sigma_n$, whose action being defined by $\sigma(x) = \sigma_n(x)$ provided $x \in K_n$. The action of G on B is continuous, and hence it can be uniquely extended to a continuous action on $\overline{B}$.

We shall show results concerning with G-invariance of a probability measure on $\overline{B}$.

Proposition 2.2.

A probability measure μ_∞ on $\overline{B}$ is G-invariant if and only if the corresponding function $g \in \mathcal{G}$ is G-invariant.

Corollary 2.3.

- **(i):** *If $\{\mu_n\}_{n=1}^{\infty}$ is a consistent sequence of symmetric probability measures, then the extension μ_∞ is G-invariant.*
- **(ii):** *If ν is a probability measure on $\mathfrak{Q}_p$, then the function $g_\nu := \hat{\nu} \circ T_1$ belongs to $\mathcal{G}$, and the corresponding measure on $\overline{B}$ is G-invariant.*

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