ON BOUNDARY REGULARITY OF THE DIRICHLET PROBLEM FOR PLANE DOMAINS

TOSHIYUKI SUGAWA

ABSTRACT. In this paper, we propose a notion of the local harmonic measure decay (LHMD) property with exponent α at finite boundary points of open sets Ω in the Riemann sphere $\widehat{\mathbb{C}}$. Using this property, we show that Green's function of Ω is Hölder continuous with exponent α at such a point as well as the boundary regularity of the Dirichlet problem in Ω for the usual Laplacian at the point in the sense of Hölder continuity with exponent less than α. We further explain that the LHMD property can be regarded as a localization of the notion of uniform perfectness for the boundary. We also provide several applications to the theory of conformal mappings.

1. INTRODUCTION

1.1. **Hölder regularity of a boundary point.** We consider the boundary regularity of the *bounded* solution of the Dirichlet problem in a plane domain (or, more generally, an open set) Ω in the sense of Perron-Wiener-Brelot:

$$\Delta u = 0 \quad \text{in } \Omega \quad \text{and} \quad u = \varphi \quad \text{on } \partial\Omega, \tag{1.1}$$

where Δ denotes the usual Laplacian $(\partial/\partial x)^2+(\partial/\partial y)^2$ and φ is a given bounded function. Throughout this article, we consider only bounded harmonic functions for solutions of the Dirichlet problem.

For a positive constant α, a point $a \in \partial\Omega \setminus \{\infty\}$ will be called an *α-Hölder regular* boundary point of Ω if the solution u of the Dirichlet problem in Ω is Hölder continuous with exponent α at a whenever the boundary function φ is Hölder continuous with exponent α. More precisely, this means that if $\varphi(\zeta) = \varphi(a) + O(|\zeta - a|^\alpha)$ as $\zeta \to a$ in $\partial\Omega$ then $u(z) = \varphi(a) + O(|z-a|^\alpha)$ as $z \to a$ in Ω.

The Hölder regularity of boundary points has been investigated by many authors even for more general uniformly elliptic (possibly non-homogeneous) linear partial differential equations (see [1], [14] and their bibliographies).

When Ω is a disk or a half plane (a ball or a half space in the higher dimensional case), it is known that each finite boundary point is α-Hölder regular for any α with $0 < \alpha < 1$ (see [34, Proposition 3.4] for the unit disk in the plane, or [1, Theorem 3.1] for more general differential equations). On the one hand, any boundary point of the disk is never 1-Hölder regular (see §6.9 and Remark made there).

When the boundary is of class C^2 and of bounded curvature, O. D. Kellogg [24] showed the same result (in the case of 3-dimensional space).

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For domains with exterior cone condition, K. Miller [31] gave a satisfactory result on the Hölder regularity of the boundary for uniformly elliptic partial differential equations. We remark that C. Pucci [35] obtained partial results in this direction earlier.

In the case when the boundary is not necessarily smooth, however, such a property seems to be less known. As one exception, we should draw the attention of the reader to the paper [19] by A. Hinkkanen, in which he proved α-Hölder regularity of each point of a boundary continuum for $0 < \alpha < 1/2$. Actually, he obtained more detailed, almost sharp global results on the Dirichlet solutions for bounded plane domains each boundary component of which has at least a definite size of diameter.

We should note also that, in the case of higher dimensions ($n > 2$), there is pioneer work of V. G. Maz'ya (see [29], [30] and also [16]) in this direction even for uniformly elliptic differential operators.

1.2. **Control of boundary behaviour.** In order to control the boundary behaviour of a harmonic function, the notion of barrier has been effectively used. In this article, we propose "local harmonic measures" as good substitutions for barriers. The author learned this notion from the paper [3] of A. Ancona. The precise definition will be given in Section 2. The local harmonic measures have advantage in the sense that they enjoy

1. monotonicity with respect to domain extension and circular symmetrization, and
2. availability to use classical conformal mappings.

In fact, by virtue of these properties, we deduce many estimates for local harmonic measures in Section 5. These have applications to the theory of conformal mappings, see Section 7.

We investigate the boundary regularity using the local harmonic measures. As a result, we give a condition, called "LHMD property," for a boundary point to be Hölder regular in terms of the local harmonic measures in Section 2.

1.3. **LHMD property and uniform perfectness.** The global LHMD property, which means the LHMD property in a uniform sense, is shown to be same as the uniform Δ-regularity in the sense of Ancona [3] (see Theorem 3.1 below). Ancona [3] proved that a closed set E in $\mathbb{R}^n$ has uniformly Δ-regular complement if and only if E satisfies a uniform capacity density condition with respect to the Newtonian capacity in the case $n \geq 3$. The Newtonian capacity being replaced by the logarithmic capacity, the same result holds even if $n = 2$ as noted by several authors (cf. [12]). The proof for the case $n = 2$ requires slightly different (but standard) techniques from that of [3], so we include it in Section 3 for the convenience of the reader as well as for getting explicit estimates with the emphasis on the relation with the LHMD property.

Noting a result of Pommerenke (see §4.1), we see that the global LHMD property of a domain is nothing but the uniform perfectness of the boundary. Here, a closed set E in the Riemann sphere $\widehat{\mathbb{C}}$ containing at least two points is said to be *uniformly perfect* if there exists a positive constant c such that $E \cap \{z \in \mathbb{C}; cr < |z - a| < r\} \neq \emptyset$ for any point $a \in E$ and $0 < r < \operatorname{diam} E$, where $\operatorname{diam} E$ denotes the Euclidean diameter of E ($\operatorname{diam} E = +\infty$ if $\infty \in E$).

This notion was introduced by [6], and systematically investigated by Ch. Pommerenke [32], [33]. Nowadays, a number of equivalent conditions for the uniform perfectness are known. Uniformly perfect objects naturally arise in many branches of complex analysis.

Indeed, the limit set of a non-elementary Kleinian group of Lehner type (containing finitely generated case, see [40]), the Julia set of a rational function of degree at least two (see [8]), and some kind of self-similar fractals (see [39] and [38]) are uniformly perfect. For a survey on the uniform perfectness, see [41].

As a consequence of the above chain of ideas, we give another proof for the known fact that Green's function of a domain with uniformly perfect boundary is Hölder continuous up to the boundary (see Corollary 6.3).

1.4. **Structure of this article.** Now we briefly describe the organization of the present paper.

Section 2 gives the definition and fundamental properties of local harmonic measures as well as basic notions in potential theory. We prove there our key result (Theorem 2.2) saying that the LHMD property with exponent α at a finite boundary point a implies Hölder regularity with exponent less than α at the point a.

Section 3 is devoted to (analytic) characterizations of the LHMD property such as a (lower) capacity density condition with concrete estimates for related quantities.

In Section 4, we further show that the global LHMD property of Ω is actually equivalent to the uniform perfectness of the boundary $\partial\Omega$ (Theorem 4.1). In connection with the uniform perfectness, we also give local, geometric conditions which are sufficient or necessary for the LHMD property at a given point.

Section 5 treats several geometric conditions which ensure the LHMD property with a given exponent at a boundary point such as the exterior circle condition, the generalized exterior wedge condition. We also consider the case when the domain is bounded by (positively oriented) sufficiently smooth Jordan curves whose curvature is bounded below. Then, we require a global property (Theorem 5.4) of simple plane curves with curvature bounded below. For the convenience of the reader, we shall give a proof for that property in the appendix (Section 9). This might be of independent interest.

Section 6 explains how to globalize the local results obtained in preceding sections in the case when Ω satisfies the uniform LHMD property with exponent α. Specifically, we show α-Hölder continuity of Green's function of the domain and the boundedness of the harmonic extension operator H^Ω on the Lipschitz spaces of exponent less than α.

As an application, we have a lower estimate of Hausdorff dimension of a totally disconnected uniformly perfect set (Corollary 6.4). Compare with the similar result in [41].

In Section 7, we will give several applications of Hölder continuity of Green's function to the theory of conformal mappings of finitely connected bounded domains. As results, we present a natural generalization of a theorem due to M. Masumoto [28], and L^p-integrability conditions for the derivatives of conformal mappings from the domain onto a standard domain. The latter is closely related to the Brennan conjecture when the domain is simply connected.

In Section 8, we will give a simple example of a bounded domain carrying Green's function which is Hölder (actually Lipschitz) continuous up to the boundary, whereas the boundary is not uniformly perfect. This example answers a question raised by Siciak in [37] and produces another example of domain which preserves the global Markov inequality but not the local Markov inequality (see [26]).

Section 9 is additional and serves a proof of the above-mentioned result on simple plane curves with curvature bounded below.

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2. Local harmonic measures

Let Ω be a subdomain, or, more generally, an open subset, of the Riemann sphere. Throughout this article, to avoid the trivial case, we assume that Ω is of hyperbolic type, in other words, the boundary $\partial\Omega$ consists of at least three points unless otherwise indicated. Here and hereafter, the boundary $\partial\Omega$ and the complement Ω^{c} of Ω will be taken in the Riemann sphere $\widehat{\mathbb{C}}$. However, the point at infinity ∞ will play a special role in our arguments below. So, we will employ the special notation $\partial_{\mathrm{b}}\Omega = \partial\Omega \setminus \{\infty\}$.

2.1. **PWB-solutions for the Dirichlet problem.** First of all, we recall several notions from potential theory. The reader is referred to [17] and [11] for standard textbooks in potential theory. However, contrary to the tradition and to the above textbooks, we allow a superharmonic function to be constantly $+\infty$ in a component of Ω. More precisely, a function $s : \Omega \to (-\infty, +\infty]$ will be called *superharmonic* if s is lower semi-continuous and satisfies the condition that, for any closed Jordan domain Δ in Ω and for any harmonic function u on a neighbourhood of Δ, the condition $u \le s$ on $\partial\Delta$ implies the stronger one $u \le s$ on Δ. We also adopt the similar rule for subharmonic functions.

For a function φ on $\partial\Omega$, let $\overline{\mathcal{P}}(\varphi)$ be the set of upper functions for φ in Ω, where s is said to be an *upper function* for φ in Ω if s is superharmonic and bounded below in Ω and satisfies

$$\underline{s}(\zeta) := \liminf_{z\to\zeta \text{ in } \Omega} s(z) \ge \varphi(\zeta)$$

for each $\zeta \in \partial\Omega$. Similarly, the set $\underline{\mathcal{P}}(\varphi)$ of lower functions for φ in Ω can be defined, namely, $s \in \underline{\mathcal{P}}(\varphi)$ if and only if $-s \in \overline{\mathcal{P}}(-\varphi)$. We note that for any $u \in \overline{\mathcal{P}}(\varphi)$ and any $v \in \underline{\mathcal{P}}(\varphi)$ we have $v \le u$ in Ω by the usual minimum principle for superharmonic functions. Perron's principle says that the function

$$u(z) = \inf_{s\in\overline{\mathcal{P}}(\varphi)} s(z)$$

is harmonic unless $u \equiv +\infty$ in each component of Ω. We call the function u the *upper solution* of (1.1) for φ in Ω and denote it by $\overline{H}^{\Omega}\varphi$. Similarly, $\underline{H}^{\Omega}\varphi(z) = \sup_{s\in\underline{\mathcal{P}}(\varphi)} s(z) = -\overline{H}^{\Omega}(-\varphi)(z)$ is called the *lower solution* of (1.1). By the above notice, we see that $\underline{H}^{\Omega}\varphi \le \overline{H}^{\Omega}\varphi$ always holds in Ω. A boundary function φ on $\partial\Omega$ is called *resolutive* in Ω if $\overline{H}^{\Omega}\varphi = \underline{H}^{\Omega}\varphi$, and then this common function will be denoted by $H^{\Omega}\varphi$, which is called the solution of Dirichlet problem (1.1) in the sense of Perron-Wiener-Brelot, or *PWB solution* for φ in Ω. We make sure that the PWB solution is always a (finite-valued) harmonic function in Ω if it exists.

2.2. **Regularity and barriers.** A point a in $\partial\Omega$ is called a *regular* boundary point of Ω if $H^\Omega\varphi$ is continuously extended to $\varphi(a)$ at a for each bounded resolutive function φ on $\partial\Omega$ which is continuous at a. The above regularity is known to be characterized by the existence of a barrier of Ω at a. In particular, the regularity is a local property. Here, a function b is called *barrier* of Ω at a if there exists an open neighbourhood V of a satisfying the following three properties:

1. b is a positive superharmonic function in $\Omega \cap V$,
2. $\lim_{z\to a \text{ in } \Omega} b(z) = 0$, and
3. $\inf_{z\in\Omega\cap V\setminus W} b(z) > 0$ for all neighbourhood W of a.

We note here Kellogg's theorem stating that the set of irregular boundary points of a domain is polar, i.e., of capacity zero (cf. [17, Theorem 8.34]). Therefore, if a harmonic function u in Ω is the PWB solution for a bounded continuous function φ on $\partial_{\mathrm{b}}\Omega$, then $\lim_{z\to\zeta \text{ in } \Omega} u(z) = \varphi(\zeta)$ for each point ζ in $\partial_{\mathrm{b}}\Omega$ except for a polar set. In such a situation, we will conventionally say that $u = \varphi$ quasi-everywhere (q.e.) in $\partial_{\mathrm{b}}\Omega$. Moreover, for a bounded resolutive measurable function φ, we can say that $H^\Omega\varphi = \varphi$ q.e. in the set of continuity of φ.

2.3. **The (generalized) minimum principle.** Note also the following generalization of the minimum principle for superharmonic functions (see [43, Theorem III.28] or [17, Theorem 7.10]), which will be frequently used in the sequel: *Let s be a continuous superharmonic function in a domain Ω which is bounded below. If $\varliminf_{z\to\zeta \text{ in } \Omega} s(z) \geq 0$ for each point ζ in $\partial\Omega$ except for a polar set, then $s \geq 0$ in Ω.* Therefore, a polar set is negligible for bounded harmonic functions. Since the point at infinity is polar in the two-dimensional case, bounded solutions of the Dirichlet problem in Ω are determined by the data of the boundary function on finite boundary $\partial_{\mathrm{b}}\Omega$. Hence, we consider only bounded Borel (or continuous) functions on $\partial_{\mathrm{b}}\Omega$ as boundary functions in the following.

We record here another important fact about the resolutivity of standard class of boundary functions. The following result can be seen from [11, Part 1, Chapter VIII] (the case when Ω is a bounded domain follows also from [17, Theorem 8.13]).

Theorem A. *Suppose that the boundary of an open set Ω in $\widehat{\mathbb{C}}$ is of positive capacity. Then, all the bounded Borel functions on $\partial\Omega$ are resolutive in Ω.*

2.4. **Harmonic measures.** For a while, we assume that a domain (or an open set) Ω has boundary of positive capacity. For a Borel subset E of $\partial\Omega$, the harmonic measure of E relative to Ω, which will be denoted by $\omega(\cdot, E, \Omega)$, is defined as the PWB solution in Ω for the boundary value 1_E, where 1_E denotes the defining function of E. Here we note that 1_E is resolutive in Ω by the above theorem. As is well known, the set function $E \mapsto \omega(z, E, \Omega)$ for a fixed point $z \in \Omega$ is a Borel probability measure representing the point-evaluation of harmonic extensions at z, namely,

$$H^\Omega\varphi(z) = \int_{\partial\Omega} \varphi(\zeta)\omega(z, d\zeta, \Omega) \tag{2.1}$$

holds for every bounded Borel function φ on $\partial\Omega$.

Let a be a finite point of the boundary of Ω. To measure the size of the boundary of Ω near the point a, the harmonic measure is convenient. A standard quantity is

$$\hat{\omega}_{a,r,\Omega} = \omega(\cdot, \partial\Omega \setminus B(a,r), \Omega),$$

the harmonic measure of $\partial\Omega \setminus B(a,r)$ relative to Ω, where $B(a,r) = \{z \in \mathbb{C}; |z-a| \le r\}$. Intuitively, we can think that the size of $\partial\Omega \cap B(a,r)$ is large if $\hat{\omega}_{a,r,\Omega}(z)$ tends to 0 rapidly when z approaches a. For example, we have the following estimate for a bounded continuous function φ on $\partial_{\mathrm{b}}\Omega$ in terms of $\hat{\omega}_{a,r,\Omega}$:

$$\begin{aligned} |H^{\Omega}\varphi(z) - \varphi(a)| &= \left| \int_{\partial_{\mathrm{b}}\Omega} (\varphi(\zeta) - \varphi(a))\omega(z, d\zeta, \Omega) \right| \\ &\le 2\|\varphi\|_{\infty}\hat{\omega}_{a,r,\Omega}(z) + \sup_{|\zeta-a|\le r} |\varphi(\zeta) - \varphi(a)|(1 - \hat{\omega}_{a,r,\Omega}(z)). \end{aligned}$$

This simple estimate is, however, sometimes not sufficient to obtain a sharp order estimate of $H_{\varphi}(z) - \varphi(a)$ even if we know the precise asymptotic behaviour of $\hat{\omega}_{a,r,\Omega}(z)$ as $r \to +0$ or as $z \to a$. Furthermore, this quantity is global, and hence, not easy to calculate directly.

2.5. **Local harmonic measures.** On the other hand, we will have a great advantage if we consider the *local harmonic measure*

$$\omega_{a,r,\Omega} := \omega(\cdot, \Omega \cap \partial B(a,r), \Omega \cap B^{\circ}(a,r))$$

for $a \in \partial_{\mathrm{b}}\Omega$ and $r > 0$ (usually we take $r < \operatorname{diam} \partial\Omega$). The subscript Ω in the above will be omitted when we do not need to express Ω explicitly. Noting that each point in $\partial(\Omega \cap B^{\circ}(a,r)) \cap \partial B(a,r)$ is regular with respect to $\Omega \cap B^{\circ}(a,r)$ (cf. Theorem 5.2 below), we see that $\omega_{a,r,\Omega} = 1$ in $\Omega \cap \partial B(a,r)$. On the other hand, $\omega_{a,r,\Omega} = 0$ q.e. in $\partial\Omega \cap B^{\circ}(a,r)$.

The relation between $\omega_{a,r,\Omega}$ and $\hat{\omega}_{a,r,\Omega}$ is as follows:

$$\hat{\omega}_{a,r,\Omega}(z) \le \omega_{a,r,\Omega}(z) \tag{2.2}$$

for $z \in \Omega \cap B^{\circ}(a,r)$. Thus the local harmonic measure is stronger than the global one in this sense. To emphasize the contrast, we will call $\hat{\omega}_{a,r,\Omega}$ the global harmonic measure for a and r in this article. Inequality (2.2) is shown by applying the (generalized) minimum principle in $\Omega \cap B^{\circ}(a,r)$ (see also the proof of Lemma 5.1 below). This kind of inequality is called Carleman's principle of domain extension (see, for instance, [15, Chap. VIII §4]). We will mention the global harmonic measures in §4.4 again.

2.6. **Characterization of the boundary regularity in terms of local harmonic measures.** The local harmonic measures could be good substitutions for barriers in the following sense, although $\omega_{a,r,\Omega}$ does not satisfy the third condition of the barrier above.

Proposition 2.1. *Let Ω be an open subset of $\widehat{\mathbb{C}}$ and let $a \in \partial_{\mathrm{b}}\Omega$. Then the following are equivalent.*

(a) *a is a regular boundary point of Ω.*
(b) *There exists a barrier of Ω at a.*
(c) *There exists a positive constant r_0 such that $\omega_{a,r,\Omega}(z) \to 0$ as $z \to a$ in Ω for any r with $0 < r < r_0$.*

Proof. The equivalence of (a) and (b) is classical. In particular, from this it follows that the regularity is a local property. Since (a)$\Rightarrow$(c) is clear, we give the proof only for (c) $\Rightarrow$(a). The technique presented in this proof will be used in several times in the sequel. So, we explain it in detail here and we will omit the details after the second appearance.

Assume condition (c) and suppose that a bounded Borel function φ on $\partial_{\mathrm{b}}\Omega$ is continuous at a. Without loss of generality, we may assume $\varphi(a) = 0$. Set $u = H^{\Omega}\varphi$ and let $M = \sup_{\zeta\in\partial_{\mathrm{b}}\Omega} |\varphi(\zeta)|$. For a given number $\varepsilon > 0$, we can choose a number δ with $0 < \delta \le r_0$ such that $|\varphi(\zeta)| \le \varepsilon$ for all $\zeta \in \partial\Omega$ with $|\zeta - a| \le \delta$.

Now consider the function $\varphi_0 = \varphi \cdot 1_{B(a,\delta)}$ and $\varphi_1 = \varphi - \varphi_0$ and set $u_j = H^{\Omega}\varphi_j$ for $j = 0, 1$. Since $|\varphi_0| \le \varepsilon$, we see that $|u_0| \le \varepsilon$ in Ω. On the other hand, we have $-M\omega_{a,r,\Omega} \le u_1 \le M\omega_{a,r,\Omega}$ in $\Omega \cap B^{\circ}(a, r)$ for any $r < \delta$. Indeed, $u_1 = 0$ q.e. in $\partial\Omega \cap B^{\circ}(a, \delta)(\supset \partial\Omega \cap B(a, r))$ and $\omega_{a,r,\Omega} = 1$ in $\Omega \cap \partial B(a, r)$. Since $|u_1| \le M$ and $\omega_{a,r,\Omega} \ge 0$, we see $\limsup_{z\to\zeta}(M\omega_{a,r,\Omega}(z) - u_1(z)) \ge 0$ for each $\zeta \in \partial(\Omega \cap B^{\circ}(a, r))$ except for a polar set. Now the (generalized) minimum principle implies $M\omega_{a,r,\Omega} - u_1 \ge 0$ in $\Omega \cap B^{\circ}(a, r)$. The other inequality $-M\omega_{a,r,\Omega} \le u_1$ can be obtained similarly.

By condition (c), we now obtain $\lim_{z\to a} u_1(z) = 0$. Noting $u = u_0 + u_1$, we therefore have

$$\limsup_{z\to a} |u(z)| = \limsup_{z\to a} |u_0(z)| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude $\lim_{z\to a} u(z) = 0 = \varphi(a)$. Now condition (a) has been shown. $\square$

2.7. **LHMD property.** As the reader may guess from the above proof, if $\omega_{a,r,\Omega}(z)$ decay in some specific way when z tends to a the boundary behaviour of $H^{\Omega}\varphi$ may be controlled at a in a definite way. Indeed, we will show this in the following form.

For a point $a \in \partial_{\mathrm{b}}\Omega$ and for a constant $\alpha \in (0, +\infty)$, consider the condition that

$$\omega_{a,r,\Omega}(z) \le C\left(\frac{|z - a|}{r}\right)^{\alpha} \quad \text{for all } \ z \in \Omega \cap B^{\circ}(a, r) \text{ and for all } 0 < r < r_0, \tag{2.3}$$

where C and r_0 are positive constants. This will be called the *local harmonic measure decay property* or LHMD property, for short, at a with exponent α. By the above proposition, a point satisfying LHMD property is a regular boundary point. This kind of property was implicitly used for NTA(=Non-Tangentially Accessible) domains in [23, Lemma 4.1] by Jerison and Kenig, and for uniformly John domains with uniformly perfect boundary in [5, Lemma 1.1] by Balogh and Volberg.

2.8. **Main result.** The following is our fundamental result. In the course of preparation of this article, H. Aikawa suggested a simplification of the original proof to the author. We give here the simplified proof due to Aikawa.

Theorem 2.2. *Let Ω be an open subset of the Riemann sphere with boundary of positive capacity. Suppose that a finite boundary point a of Ω satisfies LHMD property* (2.3) *with exponent $\alpha \in (0, 1]$ for constants $C \ge 1$ and $r_0 > 0$. For a constant $\gamma \in (0, \alpha)$, if a bounded real-valued boundary function φ satisfies that $\varphi \le K$ on $\partial_{\mathrm{b}}\Omega$ and that $\varphi(\zeta) \le L|\zeta - a|^{\gamma}$ for* *each $\zeta \in \partial_{\mathrm{b}}\Omega \cap B(a, r_0)$ for positive constants K and L, then the upper solution $u = \overline{H}^{\Omega}\varphi$ of φ satisfies the inequality*

$$u(z) \le 2^{\alpha} C \left(\frac{K}{r_0^{\gamma}} + \frac{L}{\alpha - \gamma} \right) |z - a|^{\gamma}, \quad z \in \Omega. \tag{2.4}$$

Furthermore if $\varphi = 0$ on $\partial\Omega \cap B(a, r_1)$ for some $0 < r_1 \le r_0$, then we have

$$u(z) \le CK \left(\frac{|z - a|}{r_1} \right)^{\alpha}, \quad z \in \Omega. \tag{2.5}$$

Proof. Fix a point z_0 in Ω and set $\delta_j = 2^j |z_0 - a|$ for $j = 0, 1, \ldots$. First we suppose that $|z_0 - a| \le r_0$. Let N be the smallest integer such that $\delta_N > r_0$. Note the inequality

$$2^{-N} < |z_0 - a|/r_0. \tag{2.6}$$

Set $\varphi_j = \varphi \cdot 1_{B(a,\delta_j)\setminus B(a,\delta_{j-1})}$ for $j = 1, 2, \ldots, N - 1$, $\varphi_0 = \varphi \cdot 1_{B(a,\delta_0)}$, and $\varphi_N = \varphi \cdot 1_{\partial\Omega \setminus B(a,\delta_{N-1})}$.

Since $\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_N$, the subadditivity of the operator $\overline{H}^{\Omega}$ implies $u \le u_0 + u_1 + \cdots + u_N$, where $u_j = \overline{H}^{\Omega}\varphi_j$. The inequality $\varphi_j \le L\delta_j^{\gamma}$ implies $u_j \le L\delta_j^{\gamma}$ for $j = 0, 1, \ldots, N-1$. In addition, we have $u_N \le K$. Since $u_j = 0$ q.e. in $\partial\Omega \cap B^{\circ}(a, \delta_{j-1})$, we have $u_j \le L\delta_j^{\gamma}\omega_{a,r}$ on $\Omega \cap B^{\circ}(a, r)$ for any r with $|z_0 - a| < r < \delta_{j-1}$ and for $j = 1, \ldots, N-1$ by the maximum principle. Now (2.3) yields $u_j(z_0) \le CL\delta_j^{\gamma}(\delta_0/r)^{\alpha}$. Letting r tend to δ_{j-1}, we obtain

$$u_j(z_0) \le CL\delta_j^{\gamma} \left(\frac{\delta_0}{\delta_{j-1}} \right)^{\alpha} = 2^{\alpha} CL 2^{(\gamma-\alpha)j} |z_0 - a|^{\gamma}$$

for $j = 1, \ldots, N - 1$. We note that $u_0(z_0) \le L\delta_0^{\gamma} = L|z_0 - a|^{\gamma} \le 2^{\alpha}CL|z_0 - a|^{\gamma}$ and thus the above inequality is valid also for $j = 0$. For $j = N$, by (2.6), we have the similar inequality

$$u_N(z_0) \le CK \left(\frac{\delta_0}{\delta_{N-1}} \right)^{\alpha} = 2^{\alpha} CK 2^{-\alpha N} < 2^{\alpha} CK \left(\frac{|z_0 - a|}{r_0} \right)^{\alpha} \le 2^{\alpha} CK \left(\frac{|z_0 - a|}{r_0} \right)^{\gamma}.$$

By summing up these inequalities, we obtain

$$u(z_0) \le u_0(z_0) + \cdots + u_N(z_0) \le 2^{\alpha} C \left(\frac{K}{r_0^{\gamma}} + \frac{L}{1 - 2^{\gamma-\alpha}} \right) |z_0 - a|^{\gamma}.$$

Secondly, if $|z_0 - a| > r_0$, then we have $u(z_0) \le K \le K(|z_0 - a|/r_0)^{\gamma}$, in particular, the above inequality is still valid in this case. Thus we have shown the first part of the theorem.

Finally, consider the case that $\varphi = 0$ on $\partial\Omega \cap B(a, r_1)$ for some $0 < r_1 \le r_0$. We may assume $|z_0 - a| < r_1$. In the same way as above, we can see that $u \le K\omega_{a,r,\Omega}$ in $\Omega \cap B^{\circ}(a, r)$ for any r with $|z_0 - a| < r < r_1$. Using condition (2.3) and letting $r \to r_1$, we obtain (2.5) for $z = z_0$. $\square$

Under the same circumstances in the above theorem, suppose that a Borel function φ on $\partial_{\mathrm{b}}\Omega$ satisfies $|\varphi - \varphi(a)| \le K$ on $\partial_{\mathrm{b}}\Omega$ and $|\varphi(\zeta) - \varphi(a)| \le L|\zeta - a|^{\gamma}$ for constants

K, L and γ with $0 < \gamma < \alpha$. Then, by (2.4), the PWB solution u for φ in Ω satisfies the inequality

$$|u(z) - \varphi(a)| \le 2^\alpha C \left(\frac{K}{r_0^\gamma} + \frac{L}{\alpha - \gamma} \right) |z - a|^\gamma, \quad z \in \Omega. \tag{2.7}$$

In particular, we have the following.

Corollary 2.3. *If a finite boundary point of a plane domain Ω satisfies the LHMD property with exponent α, then the point is γ-Hölder regular with respect to Ω for all γ with $0 < \gamma < \alpha$.*

3. Characterizations of LHMD property

We state here characterizations of LHMD property in connection with the capacity density condition and with the Δ-regularity.

3.1. **Capacity density.** First, we recall the notion of (logarithmic) capacity. For a compact subset E of $\mathbb{C}$, the (logarithmic) capacity $\mathrm{Cap}(E)$ is defined to be the number e^{-c} such that

$$G_D(z, \infty) = \log|z| + c + o(1)$$

as $z \to \infty$, where $G_D(\cdot, \infty)$ denotes Green's function of the (unique) unbounded connected component D of $\widehat{\mathbb{C}} \setminus E$ with pole at infinity. The constant $c = -\log \mathrm{Cap}(E)$ is called the Robin constant of E.

For a closed set E in $\widehat{\mathbb{C}}$, define the (lower) *capacity density* at $a \in E$, denoted by $\mathrm{CD}(a, E)$, by

$$\begin{aligned} \mathrm{CD}(a, E) &= \liminf_{r \to 0} \frac{\mathrm{Cap}(E \cap B(a, r))}{r} \\ &= \lim_{r_0 \to 0} \mathrm{CD}_{r_0}(a, E), \end{aligned}$$

where $\mathrm{CD}_{r_0}(a, E) = \inf\{\mathrm{Cap}(E \cap B(a, r))/r; 0 < r < r_0\}$ for $0 < r_0 \le +\infty$. Since $\mathrm{Cap}(B(a, r)) = r$, we see $0 \le \mathrm{CD}_{r_0}(a, E) \le \mathrm{CD}(a, E) \le 1$. For an open set Ω in $\widehat{\mathbb{C}}$, a finite boundary point a of it is said to satisfy the *capacity density condition* with respect to Ω if $\mathrm{CD}(a, \Omega^c) > 0$. We remark that several kinds of capacity density conditions have been considered in the literature (see, for instance, [27], [21] and [37]).

3.2. **Δ-regularity.** A. Ancona introduced the notion of (uniform) Δ-regularity in [3]. In this article, a finite boundary point a of an open set Ω will be called Δ*-regular* if there exist positive constants ε and r_0 such that $\omega_{a,r,\Omega} \le 1 - \varepsilon$ in $\Omega \cap \partial B(a, r/2)$ for all $0 < r < r_0$.

When we try to interpret the Δ-regularity at the point a as a notion of how the local harmonic measure at a decreases as we approach a, we come to see that this is nothing but the LHMD property in the following theorem. As was stated, a prototype of the next theorem was stated by A. Ancona [3], however his statement handles only the case that the dimension is greater than two.

3.3. **Characterizations.**

Theorem 3.1. *For a fixed r_0 with $0 < r_0 \le +\infty$ and for a finite boundary point a of an open set Ω in $\widehat{\mathbb{C}}$, the following are equivalent.*

(1) *a satisfies the capacity density condition with respect to Ω, namely, there exists a positive constant c such that $\mathrm{Cap}(B(a,r)\cap\Omega^{\mathrm{c}}) \ge cr$ holds for any $0 < r < r_0$.*
(2) *For any $0 < t < 1$, there exists a constant $\varepsilon \in (0,1)$ such that $\omega_{a,r,\Omega} \le 1-\varepsilon$ on $\Omega \cap \partial B(a,tr)$ for any $0 < r < r_0$.*
(3) *There exist constants $t,\ \varepsilon \in (0,1)$ such that $\omega_{a,r,\Omega} \le 1-\varepsilon$ on $\Omega\cap\partial B(a,tr)$ for any $0 < r < r_0$.*
(4) *There exist positive constants C and α such that*
$$\omega_{a,r,\Omega}(z) \le C\left(\frac{|z-a|}{r}\right)^{\alpha}$$
for any $0 < r < r_0$ and $z \in B^{\circ}(a,r)\cap\Omega$.

Moreover, these constants depend only on each other, not on the particular point a.

Remark. In [43, p. 104], as a consequence of Wiener's criterion, it is stated that the condition $\limsup_{r\to 0}\mathrm{Cap}(B(a,r)\cap\Omega^{\mathrm{c}})/r > 0$ implies the regularity of a with respect to Ω.

3.4. **Some technical quantities.** To make our statement brief, we define several technical quantities. Let $\alpha_{r_0}(a,\Omega)$ denote the supremum of possible exponents of LHMD property at $a \in \partial_{\mathrm{b}}\Omega$ with respect to Ω for $0 < r < r_0$, in other words,
$$\alpha_{r_0}(a,\Omega) = \lim_{\delta\to 0}\inf_{0<r<r_0}\inf_{z\in\Omega\cap B(a,\delta r)}\frac{\log\omega_{a,r,\Omega}(z)}{\log(|z-a|/r)}.$$
By the monotonicity $\omega_{a,r,\Omega} \le \omega_{a,r',\Omega}$ in $\Omega\cap B^{\circ}(a,r')$ for $0 < r' < r$ (proved by the minimum principle), it is clear that this quantity does not depend on the particular choice of $r_0 > 0$ if r_0 is finite. Hence we can write simply $\alpha_{r_0}(a,\Omega) = \alpha(a,\Omega)$ for $0 < r_0 < +\infty$.

Furthermore, we use the more technical quantities $\varepsilon_{r_0}(a,t,\Omega)$ and $\varepsilon(a,t,\Omega)$ for $0 < t < 1$, $a \in \partial_{\mathrm{b}}\Omega$ and $0 < r_0 \le +\infty$ which are defined by
$$\varepsilon_{r_0}(a,t,\Omega) = 1 - \sup_{0<r<r_0}\sup_{z\in\Omega\cap B^{\circ}(a,tr)}\omega_{a,r,\Omega}(z),$$
and $\varepsilon(a,t,\Omega) = \lim_{r_0\to 0}\varepsilon_{r_0}(a,t,\Omega)$. The Δ-regularity at a means that $\varepsilon(a,1/2,\Omega) > 0$.

By using these quantities, apart from the dependence of the constants in the statement, we can restate the above theorem as the equivalence of the following four conditions:

(1') $\mathrm{CD}_{r_0}(a,\Omega^{\mathrm{c}}) > 0$.
(2') $\varepsilon_{r_0}(a,t,\Omega) > 0$ for all $0 < t < 1$.
(3') $\varepsilon_{r_0}(a,t,\Omega) > 0$ for some $0 < t < 1$.
(4') $\alpha_{r_0}(a,\Omega) > 0$.

In the case $r_0 < +\infty$, as we remarked, $\alpha_{r_0}(a,\Omega) = \alpha(a,\Omega)$. Therefore, these conditions do not depend on the particular choice of r_0 as far as $r_0 < +\infty$. In particular, we can eliminate r_0 from the above four conditions.

From now on, we concentrate on the proof of the equivalence of the above four conditions. The dependence of the constants can be observed in the proof below, so we will omit the detailed discussion for it. The implication (2')⇒(3') is trivial.

3.5. **Proof of (3')⇔(4').** The equivalence (3')⇔(4') can be seen from the following lemma.

Lemma 3.2. *Let $0 < r_0 \le +\infty$. For any $t \in (0,1)$, we have the inequality*

$$\frac{\log(1-\varepsilon_{r_0}(a,t,\Omega))}{\log t} \le \alpha_{r_0}(a,\Omega) \le \liminf_{\delta\to 0} \frac{\log(1-\varepsilon_{r_0}(a,\delta,\Omega))}{\log\delta}. \tag{3.1}$$

As an immediate consequence of the above inequality, we see that the limit exists and satisfies the identity

$$\alpha_{r_0}(a,\Omega) = \lim_{t\to 0} \frac{\log(1-\varepsilon_{r_0}(a,t,\Omega))}{\log t}.$$

Proof of Lemma 3.2. Set $\varepsilon = \varepsilon_{r_0}(a,t,\Omega)$. Since $\omega_{a,r} \le 1-\varepsilon$ on $\Omega\cap\partial B(a,tr)$ for $0<r<r_0$, we can show

$$\omega_{a,r} \le (1-\varepsilon)\omega_{a,tr} \quad \text{on } \Omega\cap B^\circ(a,tr)$$

by the generalized minimum principle. The repetition of this gives us the estimate

$$\omega_{a,r} \le (1-\varepsilon)^n \omega_{a,t^n r} \quad \text{on } \Omega\cap B^\circ(a,t^n r)$$

for any $0<r<r_0$. Therefore, for $z\in\Omega\cap B^\circ(a,r)$ if we choose an integer n such that $t^{n+1}r \le |z-a| < t^n r$, then we have

$$\omega_{a,r}(z) \le (1-\varepsilon)^n \omega_{a,t^n r}(z) < (1-\varepsilon)^n \le \frac{1}{1-\varepsilon}\left(\frac{|z-a|}{r}\right)^\alpha,$$

where $\alpha = \log(1-\varepsilon)/\log t$. It is worthwhile recording this inequality separately:

$$\omega_{a,r,\Omega}(z) \le \frac{1}{1-\varepsilon_{r_0}(a,t,\Omega)}\left(\frac{|z-a|}{r}\right)^\alpha, \quad \text{for } z\in\Omega\cap B(a,tr),\ 0<r<r_0, \tag{3.2}$$

where

$$\alpha = \frac{\log(1-\varepsilon_{r_0}(a,t,\Omega))}{\log t}. \tag{3.3}$$

Hence, we obtain $\alpha_{r_0}(a,\Omega) \ge \log(1-\varepsilon_{r_0}(a,t,\Omega))/\log t$. Thus the left-hand side in (3.1) is proved.

In order to show the right-hand side, we take any α with $0<\alpha<\alpha_{r_0}(a,\Omega)$. Then, there exists a small number $\delta_0>0$ such that $\log\omega_{a,r}(z)/\log(|z-a|/r) > \alpha$, namely, $\omega_{a,r}(z) < (|z-a|/r)^\alpha$ for $z\in\Omega\cap B(a,\delta r)$, $0<r<r_0$ and for $0<\delta<\delta_0$. In particular, we have $\varepsilon_{r_0}(a,\delta,\Omega) \ge 1-\delta^\alpha$, in other words, $\alpha \le \log(1-\varepsilon_{r_0}(a,\delta,\Omega))/\log\delta$ for $0<\delta<\delta_0$. The last inequality implies $\alpha \le \liminf_{\delta\to0}\log(1-\varepsilon_{r_0}(a,\delta,\Omega))/\log\delta$. Thus we get the conclusion. □

3.6. **Proof of (4')⇒(1')⇒(2').** The implications (4')⇒(1')⇒(2') (and thus the completion of the proof of Theorem 3.1) follows from the arguments below. In particular, we will obtain the next result.

Proposition 3.3. *Let $a \in \partial_{\mathrm{b}}\Omega$ and let $0 < r_0 \le +\infty$. Suppose that LHMD property* (2.3) *with exponent $\alpha > 0$ holds for a constant $C \ge 1$. Then, we have the inequality*

$$-\log \mathrm{CD}_{r_0}(a, \Omega^{\mathrm{c}}) \le \frac{\varphi(C)+1}{\alpha}, \tag{3.4}$$

where $\varphi : [1, +\infty) \to [0, +\infty)$ is the inverse function of $x \mapsto e^x/(1+x)$. In particular, we have

$$\alpha_{r_0}(a, \Omega) \le \frac{\varphi(C)+1}{-\log \mathrm{CD}_{r_0}(a, \Omega^{\mathrm{c}})}.$$

Conversely, for any given number m with $0 < m < 1$, there exists a positive number K_0 depending only on m such that

$$\frac{m}{\min\{-\log \mathrm{CD}_{r_0}(a, \Omega^{\mathrm{c}}), K_0\}} \le \alpha_{r_0}(a, \Omega). \tag{3.5}$$

Remark. Note that the asymptotic behaviour of φ :

$$\varphi(C) = \log C + \log\log C + (1 + o(1))\frac{\log\log C}{\log C}, \quad C \to \infty.$$

On the other hand, it is quite elementary to show the inequality $\cosh(3x/4) < e^x/(1 + x) < \cosh(x)$ for all positive x. Therefore, we obtain the useful estimate $\cosh^{-1}(C) = \log(C + \sqrt{C^2 - 1}) < \varphi(C) < 4\cosh^{-1}(C)/3$ for $C > 1$.

For simplicity, we assume that $a = 0$ and write $B_r = B(0, r)$ and $\omega_r = \omega_{0,r,\Omega}$. Denote by g_r Green's function of Ω_r with pole at infinity, where Ω_r is the connected component of $\widehat{\mathbb{C}} \setminus (B_r \cap \Omega^{\mathrm{c}})$ containing ∞. Then $g_r(z) = \log|z| - \log C_r + o(1)$ as $z \to \infty$, where $C_r = \mathrm{Cap}(B_r \cap \Omega^{\mathrm{c}})$.

First we show inequality (3.4). Set $\varepsilon = \varepsilon_{r_0}(a, t, \Omega)$ and fix $0 < r < r_0$. Also set $M = \sup_{z\in\partial B_r\cap\Omega} g_r(z)$. Then, noting that $g_r = 0$ q.e. on $\partial\Omega \cap B_r$, we can see $M\omega_r - g_r \ge 0$ on $\Omega \cap B_r^{\circ}$ by the minimum principle. Therefore, $g_r \le M\omega_r \le (1 - \varepsilon)M$ on $\Omega \cap \partial B_{tr}$. In particular,

$$g_r(z) - \log(|z|/tr) \le (1 - \varepsilon)M \tag{3.6}$$

on $\Omega_r \cap \partial B_{tr}$. Since $g_r(z) - \log(|z|/tr)$ is harmonic near the point at infinity, we see inequality (3.6) remains valid also in $\Omega_r \setminus B_{tr}$. Using this, we have $M \le (1-\varepsilon)M + \log(1/t)$, which implies $\varepsilon M \le \log(1/t)$. By (3.6) again, $g_r(z) - \log(|z|/tr) \le (1/\varepsilon - 1)\log(1/t)$ for $z \in \Omega_r$ with $|z| > tr$. Applying this for $z = \infty$, we then have $-\log C_r \le (1/\varepsilon - 1)\log(1/t) + \log(1/tr) = \log(1/t)/\varepsilon - \log r$, thus $C_r \ge t^{1/\varepsilon} r$. The last inequality means $\mathrm{CD}_{r_0}(a, \Omega^{\mathrm{c}}) \ge t^{1/\varepsilon_{r_0}(a,t,\Omega)}$, in other words,

$$-\log \mathrm{CD}_{r_0}(a, \Omega^{\mathrm{c}}) \le \frac{\log(1/t)}{\varepsilon_{r_0}(a, t, \Omega)}. \tag{3.7}$$

Assume now condition (2.3) for α, C, r_0. Taking $0 < t < 1$ so that $Ct^\alpha < 1$, from inequality (3.7) we obtain

$$-\log \mathrm{CD}_{r_0}(a, \Omega^{\mathrm{c}}) \le \frac{\log(1/t)}{1 - Ct^\alpha} = \frac{x}{1 - Ce^{-\alpha x}} =: f(x),$$

where we set $t = e^{-x}$ for $x > 0$. Now we minimize $f(x)$. Let x_0 be the unique positive solution of the equation $f'(x) = 0$, equivalently, $e^{\alpha x} = C(1 + \alpha x)$. In terms of φ defined in the statement of Proposition 3.3, we can write $x_0 = \varphi(C)/\alpha$. Since $f(x_0) = x_0 + 1/\alpha$, we get (3.4).

Secondly, we show (3.5) and implication (1')⇒(2'). To prove these, we will use a variant of the Harnack inequality:

Lemma 3.4. *For a positive constant $r > 0$, let u be a positive harmonic function in the domain $\mathbb{D}_r^* := \{z \in \mathbb{C}; |z| > r\}$ which satisfies $u(z) = \log|z| + c + o(1)$ as $z \to \infty$. Then $u(z) > \log(|z|/r)$ for $z \in \mathbb{D}_r^*$, in particular, $c \ge \log 1/r$. Moreover, the inequality*

$$\frac{|z| - r}{|z| + r} \le \frac{u(z) - \log(|z|/r)}{c + \log r} \le \frac{|z| + r}{|z| - r}$$

holds for $z \in \mathbb{D}_r^*$.

Noting $v(z) = u(z) - \log(|z|/r) > 0$ by the minimum principle, we apply the ordinary Harnack inequality to the positive harmonic function $v(1/w)$ in $|w| < 1/r$ to show the above inequality.

Now fix an arbitrary number δ with $0 < \delta < t$ and set $K = -\log \mathrm{CD}_{r_0}(a, \Omega^{\mathrm{c}}) \ge 0$ for some $r_0 > 0$. By Lemma 3.4,

$$\frac{|z| - \delta r}{|z| + \delta r} \le \frac{g_{\delta r}(z) - \log|z|/\delta r}{\log \delta r / C_{\delta r}} \le \frac{|z| + \delta r}{|z| - \delta r},$$

for $|z| > \delta r$. In particular, we then have

$$L := \min_{|z|=r} g_{\delta r}(z) \ge \log\frac{1}{\delta} + \frac{1-\delta}{1+\delta}\log\frac{\delta r}{C_{\delta r}},$$

$$L' := \max_{|z|=tr} g_{\delta r}(z) \le \log\frac{t}{\delta} + \frac{t+\delta}{t-\delta}\log\frac{\delta r}{C_{\delta r}}.$$

Applying Lemma 3.4 to $g_{\delta r} - L$, we get the inequality $-\log C_{\delta r} - L + \log r \ge 0$. On the other hand, noting that $g_{\delta r} \ge L\omega_r$ on $\Omega \cap B_r$, we obtain $M := \sup_{z \in \Omega \cap \partial B_{tr}} \omega_r(z) \le \sup_{|z|=tr} g_{\delta r}(z)/L = L'/L$. Hence, using $C_{\delta r} \ge e^{-K}\delta r$, we have

$$M \le \frac{\log\frac{t}{\delta} + \frac{t+\delta}{t-\delta}\log\frac{\delta r}{C_{\delta r}}}{\log\frac{1}{\delta} + \frac{1-\delta}{1+\delta}\log\frac{\delta r}{C_{\delta r}}} \le 1 - \frac{\log\frac{1}{t} - K(\frac{t+\delta}{t-\delta} - \frac{1-\delta}{1+\delta})}{\log\frac{1}{\delta} + K\frac{1-\delta}{1+\delta}}.$$

Since r can be taken arbitrarily as long as $0 < r < r_0$, we conclude

$$\frac{\log(1 - \varepsilon_{r_0}(a, t, \Omega))}{\log t} \ge \frac{\varepsilon_{r_0}(a, t, \Omega)}{\log\frac{1}{t}} \ge \frac{1 - K(\frac{t+\delta}{t-\delta} - \frac{1-\delta}{1+\delta})/\log\frac{1}{t}}{\log\frac{1}{\delta} + K\frac{1-\delta}{1+\delta}} \tag{3.8}$$

for any pair (t, δ) with $0 < \delta < t < 1$. We can easily see that, for a fixed t, the right-most term in (3.8) is positive for sufficiently small δ, and thus, we have proved the implication (1')⇒(2').

In order to get inequality (3.5) we choose δ and t so that $t = \min\{(\mathrm{CD}_{r_0}(a, \Omega^c))^\beta, x\}$ and $\delta = t^2$, where the positive numbers x and β will be specified later. Set $K' = \log(1/t)/\beta = \max\{K, \log(1/x)/\beta\}$. By (3.8), we now have

$$\frac{\log(1-\varepsilon_{r_0}(a,t,\Omega))}{\log\frac{1}{t}} \geq \frac{1-K(\frac{1+t}{1-t}-\frac{1-t^2}{1+t^2})/\beta K'}{2\beta K' + K\frac{1-t^2}{1+t^2}} \geq \frac{1-(\frac{1+t}{1-t}-\frac{1-t^2}{1+t^2})/\beta}{(2\beta+\frac{1-t^2}{1+t^2})K'}$$
$$= \frac{h_\beta(t)}{K'} \geq \frac{h_\beta(x)}{K'},$$

where

$$h_\beta(x) = \frac{1+x^2-2x(1+x)/\beta(1-x)}{2\beta(1+x^2)+1-x^2}.$$

The last inequality follows from the fact that the function $h_\beta(x)$ is monotonically decreasing in $0 < x < 1$. Actually, a direct calculation yields $h'_\beta(x) = -2(1+x)^2(2\beta(1-x+x^2)+1-x^2)/(1-x)^2(2\beta(1+x^2)+1-x^2)^2 < 0$. Since $\lim_{\beta\to 0}\lim_{x\to 0} h_\beta(x) = 1$, we can choose sufficiently small $x > 0$ and $\beta > 0$ so that $h_\beta(x) > m$. By Lemma 3.2, if we set $K_0 = \log(1/x)/\beta$, we have (3.5).

Remark. It is not difficult to obtain practically an explicit value K_0 for a given m between 0 and 1. For example, suppose we are given $m = 1/4$. Then, if we choose $\beta = 1$ and $x = 0.1$, we have $h_1(0.1) = 0.254\cdots > 1/4$. Therefore, we can take $K_0 = -\log 0.1 \approx 2.3$.

4. LHMD property and uniform perfectness

4.1. **Global and uniform capacity density conditions.** Ch. Pommerenke [32] proved that a compact set E in the Riemann sphere $\widehat{\mathbb{C}}$ is uniformly perfect if and only if E satisfies the *global capacity density condition*: $\mathrm{CD}_d(E) > 0$, where $d = \operatorname{diam} E$ and we write $\mathrm{CD}_{r_0}(E) = \inf_{a\in E\setminus\{\infty\}} \mathrm{CD}_{r_0}(a, E)$ for $0 < r_0 \leq +\infty$. Moreover, he gave the explicit estimate

$$k(E)^2/32 \leq \mathrm{CD}_d(E) \leq k(E), \tag{4.1}$$

where $k(E)$ is the supremum of numbers $c \geq 0$ so that $E \cap \{z; cr \leq |z-a| \leq r\} \neq \emptyset$ holds for any $a \in E \setminus \{\infty\}$ and $0 < r < \operatorname{diam} E$.

We will say that E satisfies the uniform capacity density condition if $\mathrm{CD}(E) > 0$, where $\mathrm{CD}(E) = \lim_{r_0\to 0} \mathrm{CD}_{r_0}(E)$. When $\operatorname{diam} E < +\infty$, the global capacity density condition is same as uniform capacity density condition. Note that this is not true in general if $\operatorname{diam} E = +\infty$.

4.2. **Global and uniform LHMD properties.** In an analogous way, we introduce two kinds of LHMD properties for open sets. An open set Ω will be said to satisfy the *uniform LHMD property* with exponent $\alpha > 0$ if there exist constants $1 \leq C < +\infty$ and $0 < r_0 \leq +\infty$ such that $\omega_{a,r,\Omega}(z) \leq C(|z-a|/r)^\alpha$ for all r with $0 < r < r_0$, $a \in \partial_{\mathrm{b}}\Omega$ and $z \in \Omega \cap B(a, r)$. Furthermore if we can take $r_0 = d = \operatorname{diam}\partial\Omega$, we will say that Ω satisfies the *global LHMD property.* Note that the global LHMD property coincides with the uniform LHMD property if $d < +\infty$ (cf. §4.3).

By this terminology, we can say that the complement of an open subset Ω of $\widehat{\mathbb{C}}$ with $\#\partial\Omega \geq 2$ is uniformly perfect if and only if Ω satisfies the global LHMD property. In particular, the global LHMD property is preserved by quasiconformal mappings on Ω although the exponent may change, because the uniform perfectness is invariant under quasiconformal mappings (see, for instance, [41]).

4.3. **Example.** In order to illustrate the need to distinguish the term "global LHMD" from "uniform LHMD", we give a simple example. Let $\{a_n\}$ be a sequence of real numbers satisfying the condition $a_{n+1} - a_n \to +\infty$ as $n \to \infty$. Then, the domain $\Omega = \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B(a_n, 1)$ satisfies the uniform LHMD property with exponent 1 for any positive *finite* number $r_0 > 0$ (see Theorem 5.2 below), but not for $r_0 = +\infty$ because Ω^{c} is not uniformly perfect.

4.4. **Relation between local and global harmonic measures.** We mention here an important relation between local and global harmonic measures. In view of (2.2), if a boundary point a satisfies LHMD property (2.3) with exponent α, then the global harmonic measure $\hat{\omega}_{a,r,\Omega}$ satisfies the same inequality. It is a remarkable fact that the converse is also true in general. For a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, H. Aikawa [2] proved the following: Suppose that the global harmonic measures satisfy $\hat{\omega}_{a,r,\Omega}(z) \leq C(|z-a|/r)^{\alpha}$ for all $a \in \partial_{\mathrm{b}}\Omega$, r with $0 < r < r_0$ and $z \in \Omega \cap B^{\circ}(a,r)$, where α, C and r_0 are positive constants. Then $\omega_{a,r,\Omega}(z) \leq C'(|z-a|)^{\alpha}$ holds for all $a \in \partial_{\mathrm{b}}\Omega$, r with $0 < r < r_0'$ and $z \in \Omega \cap B^{\circ}(a,r)$, where C' and r_0' are constants.

4.5. **Characterization of uniform perfectness in terms of hyperbolic metric.** The uniform perfectness can be characterized also in terms of the hyperbolic metric. Let Ω be an open set in $\widehat{\mathbb{C}}$ of hyperbolic type, namely, $\#\partial\Omega \geq 3$. Then the hyperbolic metric $\lambda_\Omega(z)|dz|$ of curvature -4 can be defined on each connected component of Ω. Let δ_Ω be the distance function of Ω, namely $\delta_\Omega(z) = \operatorname{dist}(z, \partial_{\mathrm{b}}\Omega) = \min\{|z-a|; a \in \partial_{\mathrm{b}}\Omega\}$. For $0 < r \leq +\infty$, we consider the open subset $\Omega(r) = \{z \in \Omega; \delta_\Omega(z) < r\}$. The following result is essentially due to Beardon and Pommerenke [6]. (In the case when $\infty \in \Omega$, we need something more but we omit it.)

Theorem B. *Let Ω be an open subset of $\widehat{\mathbb{C}}$ of hyperbolic type. Then the complement Ω^{c} is uniformly perfect if and only if*

$$\inf_{z \in \Omega(d)} \delta_\Omega(z)\lambda_\Omega(z) > 0,$$

where $d = \operatorname{diam} \partial\Omega$.

4.6. **Localized quantities measuring boundary.** In view of the relation

$$\inf_{z \in \Omega(d)} \delta_\Omega(z)\lambda_\Omega(z) = \inf_{z \in \Omega(d)} \inf_{a \in \partial_{\mathrm{b}}\Omega} |z-a|\lambda_\Omega(z) = \inf_{a \in \partial_{\mathrm{b}}\Omega} \inf_{z \in \Omega(d)} |z-a|\lambda_\Omega(z),$$

it may be natural to introduce the quantities

$$\mu_r(a,\Omega) = \inf_{z \in \Omega \cap B(a,r)} |z-a|\lambda_\Omega(z) \quad \text{and} \quad \nu_r(a,\Omega) = \inf_{z \in \Omega \cap B(a,r)} \delta_\Omega(z)\lambda_\Omega(z)$$

for $0 < r \leq +\infty$ and $a \in \mathbb{C} \setminus \Omega$. We conventionally define these values to be $+\infty$ if $\Omega \cap B(a,r)$ is empty. Note that Zheng [44] introduced a similar quantity (which is, in fact, same as $\mu_\infty(a,\Omega)$) and proved several related results. Since $\delta_\Omega(z) \le |z-a|$, we always have $\nu_r(a,\Omega) \le \mu_r(a,\Omega)$. Then, actually, one can easily check

$$\inf_{z\in\Omega(r)} \delta_\Omega(z)\lambda_\Omega(z) = \inf_{a\in\partial_{\mathrm{b}}\Omega} \mu_r(a,\Omega) = \inf_{a\in\partial_{\mathrm{b}}\Omega} \nu_r(a,\Omega)$$

for any $0 < r \le +\infty$. Letting $r = \operatorname{diam}\partial\Omega$, we can summarize these observations as follows.

Theorem 4.1. *Let Ω be an open set in $\widehat{\mathbb{C}}$ with $\#\partial\Omega \ge 3$ and set $d = \operatorname{diam}\partial\Omega$. Then, the following conditions are equivalent.*

(a) *$\partial\Omega$ is uniformly perfect.*
(b) *Ω satisfies global LHMD property. In other words, there exist constants $C \ge 1$ and $\alpha > 0$ such that $\omega_{a,r,\Omega}(z) \le C(|z-a|/r)^\alpha$ for all $a \in \partial_{\mathrm{b}}\Omega$, $0 < r < d$ and $z \in \Omega \cap B^\circ(a,r)$.*
(c) $\inf\{\mathrm{CD}_d(a,\Omega^{\mathrm{c}}); a \in \partial_{\mathrm{b}}\Omega\} > 0$.
(d) $\inf\{\mu_d(a,\Omega); a \in \partial_{\mathrm{b}}\Omega\} > 0$.
(e) $\inf\{\nu_d(a,\Omega); a \in \partial_{\mathrm{b}}\Omega\} > 0$.

Moreover, if $d < +\infty$, we can replace d by any positive number r_0 in the above conditions.

4.7. **A geometric localized quantity.** Now we further introduce a geometric quantity. We say that A is a round annulus centered at a if A has the form $\{z \in \mathbb{C}; r_1 < |z-a| < r_2\}$ for constants $0 \le r_1 < r_2 \le +\infty$ and set $M(A) = \sqrt{r_1 r_2}$ and $\operatorname{mod}(A) = \log(r_2/r_1)$. Let $a \in \mathbb{C} \setminus \Omega$. For $0 < r \le +\infty$, we denote by $\mathcal{A}_r(a,\Omega)$ the set of round annuli A in Ω centered at a with $M(A) \le r$. Note that, if $2r \le d = \operatorname{diam}\partial\Omega$, each annulus in $\mathcal{A}_r(a,\Omega)$ separates $\partial\Omega$. Finally, set

$$m_r(a,\Omega) = \sup_{A\in\mathcal{A}_r(a,\Omega)} \operatorname{mod}(A).$$

Here, we define $m_r(a,\Omega) = 0$ if $\mathcal{A}_r(a,\Omega)$ is empty. We note the obvious relation

$$-\log k(\Omega^{\mathrm{c}}) = \sup_{a\in\mathbb{C}\setminus\Omega} m_d(a,\Omega), \tag{4.2}$$

where $d = \operatorname{diam}\partial\Omega$.

Using the inequality due to J. A. Hempel [18] and J. A. Jenkins [22]:

$$\lambda_{D_{a,b}}(z) \ge \frac{1}{|z-a|(2|\log|\frac{z-a}{b-a}|| + K)}, \tag{4.3}$$

where $D_{a,b} = \mathbb{C} \setminus \{a,b\}$ and $K = \Gamma(1/4)^4/2\pi^2 \approx 8.7537$, we can show the following result.

Proposition 4.2. *Let $a \in \mathbb{C} \setminus \Omega$ and $0 < r \le +\infty$. Then*

$$\frac{1}{m_r(a,\Omega) + K} \le \mu_r(a,\Omega) \le \frac{\pi}{2m_r(a,\Omega)}.$$

Proof. The proof can proceed as in [6]. Let $A \in \mathcal{A}_r(a,\Omega)$ and take a point $z_0 \in A$ so that $|z_0 - a| = M(A)(\le r)$. Since $\lambda_\Omega(z_0) \le \lambda_A(z_0) = \pi/2\operatorname{mod}(A)M(A) = \pi/2|z_0-a|\operatorname{mod}(A)$ (see [6]), we see $\mu_r(a,\Omega) \le |z_0-a|\lambda_\Omega(z_0) \le \pi/2\operatorname{mod}(A)$. Thus, the right-hand side has been obtained.

Next, we prove the left-hand side. Take any number m with $m > m_r(a,\Omega)$. Fix an arbitrary point z_0 in $\Omega \cap B(a,r)$ and set $r_0 = |z_0 - a|$. Then, the annulus $A = \{z; e^{-m/2}r_0 <$

$|z-a| < e^{m/2}r_0\}$ must not belong to $\mathcal{A}_r(a,\Omega)$, so we can find a point $b \in \partial_{\mathrm{b}}\Omega \cap A$. Since $e^{-m/2} < |(z_0-a)/(b-a)| < e^{m/2}$, by (4.3), we have $\lambda_\Omega(z_0) \geq \lambda_{D_{a,b}}(z_0) \geq 1/|z_0-a|(m+K)$. Now the conclusion easily follows. $\square$

Corollary 4.3. *Let $a \in \partial_{\mathrm{b}}\Omega$. Then*

$$\liminf_{z\to a \ in \ \Omega} |z-a|\lambda_\Omega(z) > 0 \quad \Leftrightarrow \quad \limsup_{r\to 0} m_r(a,\Omega) < +\infty.$$

4.8. **Comparisons between localized quantities.** As was seen before, the LHMD property of Ω at a can be characterized by the condition $\mathrm{CD}(a,\Omega) > 0$. Occasionally, a geometric characterization of it would be more preferable for us. In view of Theorem 4.1, one may expect that $\mathrm{CD}_r(a,\Omega^{\mathrm{c}}) > 0$ is equivalent to $\mu_r(a,\Omega) > 0$ or to $\nu_r(a,\Omega) > 0$. Unfortunately, neither is true in general as we shall see later. However, these conditions are good geometric criteria to test the validity of the inequality $\mathrm{CD}_r(a,\Omega^{\mathrm{c}}) > 0$.

Theorem 4.4. *Let $a \in \partial_{\mathrm{b}}\Omega$ and $0 < r \leq \operatorname{diam}\partial\Omega$. Then we have*

$$\frac{1}{\mu_r(a,\Omega)} - \frac{\Gamma(1/4)^4}{2\pi^2} \leq -\log \mathrm{CD}_r(a,\Omega^{\mathrm{c}}) \leq \frac{\pi}{\nu_r(a,\Omega)} + 5\log 2$$

Proof. Obviously, $-\log \mathrm{CD}_r(a,\Omega^{\mathrm{c}}) \leq m_r(a,\Omega)/2$, so we get the left-hand side inequality from Proposition 4.2. Next, we prove the right-hand side. Set $\nu = \nu_r(a,\Omega)$. First, note the inequality $\nu \leq 1/2$. Actually, if we choose z_0 in $\Omega \cap B(a,r/2)$, then there exists a point $b \in \partial\Omega \cap B(a,r)$ such that $r_0 := \delta_\Omega(z_0) = |z_0-b|$. Note that the disk $\Delta = \{z \in \mathbb{C}; |z-z_0| < r_0\}$ is contained in Ω. Set $z_t = z_0 + t(b-z_0)$ for each $t \in [0,1)$. Since, $\delta_\Omega(z_t) = |z_t - b| = (1-t)r_0$, we obtain $\delta_\Omega(z_t)\lambda_\Omega(z_t) \leq \delta_\Omega(z_t)\lambda_\Delta(z_t) = (1-t)r_0 \cdot 1/r_0(1-t^2) = 1/(1+t)$. Letting $t \to 1$, we have $\nu \leq 1/2$.

Consider the open subset $\Omega_0 = \Omega \cap B^\circ(a,r)$ of Ω. We show the inequality

$$\inf_{z\in\Omega_0} \delta_{\Omega_0}(z)\lambda_{\Omega_0}(z) \geq \nu.$$

Clearly, $\delta_{\Omega_0}(z) = \min\{\delta_\Omega(z), r-|z-a|\}$ holds for $z \in \Omega_0$. Since $\delta_\Omega(z)\lambda_{\Omega_0}(z) \geq \delta_\Omega(z)\lambda_\Omega(z) \geq \nu$, and since $(r-|z-a|)\lambda_{\Omega_0}(z) \geq (r-|z-a|)\lambda_{B^\circ(a,r)}(z) = r/(r+|z-a|) > 1/2 \geq \nu$, we obtain the desired inequality.

Set $d_0 = \operatorname{diam}\Omega_0 \leq 2r$. From Proposition 4.2, we deduce $\nu \leq \nu_{d_0}(b,\Omega_0) \leq \mu_{d_0}(b,\Omega_0) \leq \pi/2m_{d_0}(b,\Omega_0)$ for each $b \in \mathbb{C}\setminus\Omega_0$. Noting the relation (4.2), we get $-\log k(\Omega_0^{\mathrm{c}}) \leq \pi/2\nu$. Now we can use (4.1) to prove $-\log \mathrm{CD}_\infty(\Omega_0^{\mathrm{c}}) \leq \log(32/k(\Omega_0^{\mathrm{c}})^2) \leq \pi/\nu + 5\log 2$. Since $\mathrm{CD}_r(a,\Omega^{\mathrm{c}}) \geq \mathrm{CD}_\infty(a,\Omega_0^{\mathrm{c}})$, we conclude the final inequality. $\square$

As a direct consequence of the above theorem, we have the following implications:

$$\begin{aligned}
&\lim_{r\to 0}\nu_r(a,\Omega) = \liminf_{z\to a \text{ in } \Omega} \delta_\Omega(z)\lambda_\Omega(z) > 0 \\
\Rightarrow \quad &\mathrm{CD}(a,\Omega^{\mathrm{c}}) = \liminf_{r\to 0} \frac{\mathrm{Cap}(\Omega^{\mathrm{c}} \cap B(a,r))}{r} > 0 \\
\Rightarrow \quad &\lim_{r\to 0}\mu_r(a,\Omega) = \liminf_{z\to a \text{ in } \Omega} |z-a|\lambda_\Omega(z) > 0.
\end{aligned}$$

However, as we announced, these implications are both strict. We end this section with showing it by simple examples.

Example 4.1. Let $\mathbb{H}$ denote the right half plane $\{z \in \mathbb{C}; \operatorname{Re} z > 0\}$ and set $\Omega = \mathbb{H} \setminus \{2^n; n \in \mathbb{Z}\}$, where $\mathbb{Z}$ stands for the set of integers. Since each 2^n is an isolated boundary point of Ω, we can see that $\delta_\Omega(z)\lambda_\Omega(z) \to 0$ as $z \to 2^n$. Therefore, $\liminf_{z\to 0} \delta_\Omega(z)\lambda_\Omega(z) = 0$.

On the other hand, since a countable set is polar, we obtain $\operatorname{Cap}(\Omega^c \cap B(0,r)) = \operatorname{Cap}(\mathbb{H}^c \cap B(0,r)) = C_0 r$ for $r > 0$, where $C_0 = \operatorname{Cap}(\mathbb{H}^c \cap B(0,1)) > 0$. Hence, we have $\operatorname{CD}(0,\Omega^c) = C_0 > 0$. This example shows that the first implication is strict.

Example 4.2. Next consider the domain $\Omega = \mathbb{C}^* \setminus \{2^n; n \in \mathbb{Z}\}$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then, $m_r(0,\Omega) = \log 2$ for all $0 < r < +\infty$. Therefore, by Corollary 4.3, we see $\liminf_{z\to 0} |z|\lambda_\Omega(z) > 0$.

On the other hand, clearly, we have $\operatorname{CD}(0,\Omega^c) = 0$. Hence, this example shows that the second implication is strict.

We remark that, in the above examples, we can replace each singleton $\{2^n\}$ by a sufficiently small closed disk $B(2^n, r_n)$ to obtain a domain without isolated boundary point.

5. Estimates of local harmonic measures

In the last section, we investigated geometric and analytic characterizations of the LHMD property. In particular, we saw that a domain has the global LHMD property if and only if the boundary of the domain is uniformly perfect. The exponent of LHMD is, however, not very explicit with respect to the uniform perfectness constant, or at least, it is not easy to calculate it accurately.

So, in this section, for a given exponent $\alpha > 0$, we present several geometric conditions which guarantee the LHMD property with the exponent α at a boundary point.

In our argument, it is crucial to use some monotonicity principles for harmonic measures, one of which is the following.

Lemma 5.1. *Suppose that domains Ω and $\tilde{\Omega}$ satisfy $\Omega \subset \tilde{\Omega}$ and $a \in \mathbb{C} \setminus \tilde{\Omega}$. Then, we have $\omega_{a,r,\Omega} \le \omega_{a,r,\tilde{\Omega}}$ on $\Omega \cap B^\circ(a,r)$.*

This can be easily verified by the generalized minimum principle.

5.1. Exterior circle condition.

We start with a sufficient condition for local harmonic measures to have linear growth.

In this article, for a number $\rho > 0$, a domain Ω will be said to satisfy the *exterior circle condition* with radius ρ at a finite boundary point a of Ω, if there exists a closed disk Δ with radius ρ satisfying $a \in \partial\Delta$ and $\Delta \cap \Omega = \emptyset$. This kind of condition was first considered by H. Poincaré.

If Ω satisfies the exterior circle condition with fixed radius ρ at every finite boundary point, then Ω is said to satisfy the *uniform exterior circle condition* with radius ρ. A domain Ω is convex if and only if Ω satisfies the exterior circle condition with arbitrarily large radius $\rho > 0$.

Theorem 5.2. *Let Ω be a domain in the Riemann sphere satisfying the exterior circle condition with radius ρ at $a \in \partial_{\mathrm{b}}\Omega$. Then the inequality*

$$\omega_{a,r,\Omega}(z) \le \frac{\arcsin\left(\sqrt{\frac{1-(r/2\rho)^2}{1+(|z-a|/r)^2+|z-a|/\rho}} \cdot \frac{|z-a|}{r}\right)}{\arcsin\sqrt{1-(r/2\rho)^2}} \le 2\frac{|z-a|}{r} \tag{5.1}$$

holds for any $0 < r < 2\rho$ and any point $z \in \Omega$ with $|z-a| < r$.

Proof. Without loss of generality, we may assume that $a = 0, r = 1$ and $\Delta = \{|z-\rho| \le \rho\}$, and hence $1 < 2\rho$. Let $\zeta_0 = e^{i\varphi}$ be the intersection point of the unit circle and $\partial\Delta$ with positive imaginary part (so $0 < \varphi < \pi/2$). The angle φ is determined by the relation $\cos\varphi = 1/2\rho$. We set $\tilde{\Omega} = \widehat{\mathbb{C}} \setminus \Delta$ and write $\omega = \omega_{0,1,\Omega}$ and $\tilde{\omega} = \omega_{0,1,\tilde{\Omega}}$. Since the function $[(1-\zeta_0 z)/(1-\bar{\zeta}_0 z)]^{\pi/\varphi}$ maps $\tilde{\Omega} \cap \mathcal{B}^\circ(0,1)$ conformally onto the upper half plane, we can compute $\tilde{\omega}$ directly as

$$\tilde{\omega}(z) = \frac{1}{\varphi}\arg\left(\frac{1-\zeta_0 z}{1-\bar{\zeta}_0 z}\right) = \frac{1}{\varphi}\left\{2\varphi - \arg\left(\frac{\zeta_0 - z}{\bar{\zeta}_0 - z}\right)\right\}.$$

If $|z| = t < 1$, then we have $\tilde{\omega}(z) \le \tilde{\omega}(-t) = 2(\varphi - \arg(\zeta_0 + t))/\varphi$, because the equipotential curves of $\tilde{\omega}$ consists of those open circular arcs in $\tilde{\Omega} \cap B^\circ(0,1)$ whose endpoints are ζ_0 and $\bar{\zeta}_0$. Setting $\theta = \varphi - \arg(\zeta_0 + t)$, by the sine formula, we obtain

$$\frac{t}{\sin\theta} = \frac{|\zeta_0 + t|}{\sin\varphi} = \frac{\sqrt{1+t^2+t/\rho}}{\sin\varphi} \ge \frac{1}{\sin\varphi}.$$

Hence,

$$\tilde{\omega}(z) = \frac{2\theta}{\varphi} = \frac{2}{\varphi}\arcsin\left(\sqrt{\frac{1-1/4\rho^2}{1+|z|^2+|z|/\rho}}|z|\right) \le \frac{2}{\varphi}\arcsin(|z|\sin\varphi) \le 2|z|.$$

Finally, applying Lemma 5.1, we get the conclusion. □

As we see from the proof, the first inequality in (5.1) is sharp.

5.2. **Domains with smooth boundary of curvature bounded below.** A bounded domain whose boundary is of class C^2, of course, satisfies the uniform exterior circle condition with sufficiently small constant ρ. However, in order to get an explicit estimate, we should impose on the curvature bound for the boundary curves. Now we formulate one of the results in this direction.

For an oriented regular C^2 simple closed plane curve $\xi : I \to \mathbb{C}$ parametrized by the arclength, we can define the curvature $\kappa : I \to \mathbb{R}$ by

$$\kappa(t) = \operatorname{Im}\frac{\xi''(t)}{\xi'(t)} = \frac{d}{dt}\arg\xi'(t).$$

As elementary differential geometry tells us, the curvature $\kappa(t)$ represents the (signed) reciprocal of curvature radius of the curve ξ at the point $\xi(t)$. If the curvature is nonnegative, then the domain bounded by the curve is convex. In particular, if the curvature is bounded by a positive constant from below, the domain is strictly convex.

Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain whose boundary in $\mathbb{C}$ consists of finitely many disjoint smooth regular simple curves of class C^2. We will always assume that the inclusion of each curve into $\mathbb{C}$ is a proper map. Therefore, if a boundary curve is not closed, it must terminate at the point at infinity in both directions. Note that we do not assume any differentiability of boundary curves at the point at infinity and admit the situation that several boundary curves meet at the point at infinity.

Here and hereafter, the boundary curves will be positively oriented with respect to the domain Ω, in other words, if we proceed along the boundary curve according to the orientation, we can see the domain Ω in the left. The curvature will be defined in accordance with this orientation.

We denote by p_Ω the *inner distance* on Ω, precisely, $p_\Omega(z,w) = \inf_\xi \operatorname{diam}\xi$, where the infimum is taken over all paths ξ connecting z and w in Ω and $\operatorname{diam}\xi$ denotes the Euclidean diameter of the curve ξ. Because Ω has smooth boundary, p_Ω can be extended continuously on $\overline{\Omega}\setminus\{\infty\}$. Note that $|z-w| \le p_\Omega(z,w)$ by definition.

Now we are ready to state our theorem.

Theorem 5.3. *Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain whose boundary in $\mathbb{C}$ consists of smooth curves of class C^2 with curvature not less than $-1/\rho$, where ρ is a positive constant or $+\infty$. Then, for each finite boundary point a of Ω, estimate* (5.1) *holds for any $0 < r < 2\rho$ and any point $z \in \Omega$ with $p_\Omega(a,z) \le r$.*

The assertion above can be shown by the same method as in the proof of Theorem 5.2 if we know the following global property of plane curves.

Proposition 5.4. *Let ξ be an oriented regular simple closed plane curve of class C^2 whose curvature κ satisfies $\kappa \ge -1/\rho$ at every point on ξ, where ρ is a positive constant. For each point a on ξ, let Δ denote the open disk of radius ρ which is tangent to and sitting on the right of the curve ξ at the point a. Suppose that a subarc ξ_1 of ξ starting from the point a intersects Δ. Then the Euclidean diameter of ξ_1 is at least 2ρ.*

This proposition is intuitively trivial, however it seems not to be easy to give a rigorous proof of it. We could not find a reference for this result, so we will include a proof of this proposition in Appendix for the convenience of the reader.

Remark. As is seen from the proof, the assumption on ξ can be weakened in the above proposition as follows: *ξ is an oriented regular simple closed plane curve of class C^1 for which ξ' is absolutely continuous and $\kappa = \operatorname{Im}(\xi''/\xi') \ge -1/\rho$ a.e., where ξ is parametrized by arclength.* In particular, if ξ is of class $C^{1,1}$, i.e., ξ is of class C^1 and ξ' is Lipschitz continuous, then the above proposition is applicable.

5.3. **Generalized exterior wedge condition.** Our next theorem treats domains with a generalized exterior wedge condition. For the precise statement, we prepare some notation. Let Ω be an open set in the Riemann sphere. For $a \in \mathbb{C}$ and $0 < r < \infty$, we set

$$L_r(a,\Omega) = \begin{cases} -\infty & \text{if} \quad \partial B(a,r) \subset \Omega, \\ |\partial B(a,r) \cap \Omega^{\mathrm{c}}|/r & \text{otherwise,} \end{cases}$$

here $|\cdot|$ denotes the one-dimensional Lebesgue measure. In particular, $0 \le L_r(a,\Omega) \le 2\pi$ or $L_r(a,\Omega) = -\infty$.

For $\beta \in [0, 2\pi)$, an open set Ω such that $L_r(a, \Omega) \geq \beta$ for $a \in \partial_{\mathrm{b}}\Omega$ and $0 < r < \rho$, where ρ is a positive constant, will be said to satisfy the *generalized exterior wedge condition* at a with opening β (and with height ρ). This notion is, of course, a generalization of the usual exterior wedge condition, which means that there is a (closed) wedge of opening β and height ρ with vertex at a which lies in the complement of the domain.

For $\beta \in [0, 2\pi)$, if there exists a positive constant ρ such that $L_r(a, \Omega) \geq \beta$ for any $a \in \partial_{\mathrm{b}}\Omega$ and for any $0 < r < \rho$, the domain Ω is said to satisfy the *uniform generalized exterior wedge condition* of opening β (with height ρ). Actually, it is clear that there does not exist a domain satisfying the uniform generalized exterior wedge condition of opening β with $\beta > \pi$.

Theorem 5.5. *Let Ω be an open subset in $\widehat{\mathbb{C}}$ and let $\beta \in [0, 2\pi)$ and $\rho > 0$ be constants. Suppose that $L_r(a, \Omega) \geq \beta$ for a finite boundary point a of Ω and for all r with $0 < r < \rho$. Then the estimate*

$$\omega_{a,r,\Omega}(z) \leq \frac{4}{\pi} \arctan\left(\left(\frac{|z-a|}{r}\right)^{\frac{\pi}{2\pi-\beta}}\right) < \frac{4}{\pi}\left(\frac{|z-a|}{r}\right)^{\frac{\pi}{2\pi-\beta}} \tag{5.2}$$

is valid for any $0 < r < \rho$ and $z \in \Omega \cap B^\circ(a, r)$. In particular, Ω satisfies the LHMD property with exponent $\pi/(2\pi - \beta)$ at the point a.

When the domain satisfies the (usual) exterior wedge condition, the above theorem agrees with a special case of the result of K. Miller [31] concerning the boundary Hölder regularity.

When the boundary component of Ω containing a is non-degenerate, namely, it is a continuum, then Ω satisfies the generalized exterior wedge condition at a with opening 0. Hence, by the above theorem, Ω enjoys the LHMD property with exponent $1/2$ at a. Combining with Theorem 2.2, we can see that a is an α-Hölder regular boundary point for any α with $0 < \alpha < 1/2$. This agrees with a result of A. Hinkkanen [19]. More specifically, if $\beta = 0,\ a = 0$ and $r = 1$, the first inequality in (5.2) takes the form

$$\omega_{0,1,\Omega}(z) \leq \frac{4}{\pi} \arctan\left(\sqrt{|z|}\right) = \frac{2}{\pi} \arccos\left(\frac{1-|z|}{1+|z|}\right).$$

This inequality is known in this case, namely, when $\partial\Omega$ contains a continuum connecting 0 with the unit circle ([36], see also [15, Chap. VIII §4]).

Proof. We may assume that $a = 0$ and $r = 1$. Let Ω^* denote the circular symmetrization of Ω, precisely, $\Omega^* = \{te^{i\theta}; 0 < t < \infty, 2|\theta| < 2\pi - L_t(a, \Omega)\}$. The following is a deep result due to A. Baernstein II.

Lemma 5.6 (Baernstein II [4, Theorem 7]). *$\omega_{0,1,\Omega}(z) \leq \omega_{0,1,\Omega^*}(|z|)$ for $z \in \Omega \cap B^\circ(0, 1)$.*

By assumption, we know $\Omega^* \cap B^\circ(0, 1) \subset \tilde{\Omega} \cap B^\circ(0, 1)$, where $\tilde{\Omega} = \{te^{i\theta}; 0 < t < 1, 2|\theta| < 2\pi - \beta\}$. The function $w = z^{\pi/(2\pi-\beta)}$ maps $\tilde{\Omega} \cap B^\circ(0, 1)$ conformally onto the right half unit disk. Therefore, we can see

$$\omega_{0,1,\tilde{\Omega}}(z) = \frac{2}{\pi}\left\{\pi - \arg\left(\frac{-i-w}{i-w}\right)\right\}.$$

In particular, $\omega_{0,1,\tilde{\Omega}}(|z|) = \frac{4}{\pi}(\frac{\pi}{2} - \arctan\frac{1}{|w|}) = \frac{4}{\pi}\arctan|w| = \frac{4}{\pi}\arctan(|z|^{\pi/(2\pi-\beta)})$. Now, combining Lemma 5.6 with Lemma 5.1, we obtain $\omega_{0,1,\Omega}(z) \le \omega_{0,1,\Omega^*}(|z|) \le \omega_{0,1,\tilde{\Omega}}(|z|)$, and thus we have shown the desired estimate. □

5.4. **Hölder regularity at outward pointing cusps.** From the above result, we know the existence of domains which admit α-Hölder regular boundary points for $\alpha \ge 1$. For example, consider the sector $\Omega = \{z \in \mathbb{C}^*; 0 < \arg z < \theta\}$ for $\theta \in (0,\pi)$. Then the origin is α-Hölder regular with respect to Ω for any $\alpha < \pi/\theta$ by Theorem 2.2. Of course, such points are not generic in the boundary.

Moreover, we can observe that an outward pointing cusp is α-Hölder regular for any positive α.

Theorem 5.7. *Let a be a finite boundary point of an open set Ω in $\widehat{\mathbb{C}}$. Suppose that $L_r(a,\Omega) \ge 2\pi - r/\rho$ for all $0 < r < r_0$, where ρ and r_0 are positive constants with $r_0 \le 2\rho$. Then, we have*

$$\omega_{a,r,\Omega}(z) < \frac{4}{\pi}\exp\left(\frac{\pi\rho(|z-a|-r)}{r|z-a|}\right) < \frac{4}{\pi}\exp\left(\frac{\pi\rho}{r_0}\left(1-\frac{r}{|z-a|}\right)\right) \tag{5.3}$$

for $0 < r < r_0$ and for $z \in \Omega \cap B^\circ(a,r)$. In particular, the point a is α-Hölder regular for any positive number α.

Proof. The proof is similar to that of Theorem 5.5. We may assume that $a = 0$. Let Ω^* be the circular symmetrization of Ω described in the proof of Theorem 5.5. Since $L_r(a,\Omega) \ge 2\pi - 2\arcsin(r/2\rho)$ for $0 < r < r_0$, we obtain $\Omega^* \subset \tilde{\Omega} = \{re^{i\theta}; 0 < r < r_0,\ |\theta| < \arcsin(r/2\rho)\}$. Geometrically, $\tilde{\Omega}$ is the right component of the open set $B^\circ(0,r_0)\setminus(B(i\rho,\rho)\cup B(-i\rho,\rho))$.

Now fix r with $0 < r < r_0$. Consider the domain $\tilde{\Omega}_r = \tilde{\Omega} \cap B^\circ(r/2,r/2)$ and set $u = \omega(\cdot,\partial\tilde{\Omega}_r \cap \partial B(r/2,r/2),\tilde{\Omega}_r)$. Then, by the minimum principle, we can see that $\omega_{0,r,\tilde{\Omega}} < u$ in $\tilde{\Omega}_r$. Therefore, using Lemmas 5.1 and 5.6, we obtain $\omega_{0,r,\Omega}(z) \le \omega_{0,r,\Omega^*}(|z|) \le \omega_{0,r,\tilde{\Omega}}(|z|) < u(|z|)$ for any $z \in \Omega \cap B^\circ(0,r)$.

We next compute the value $u(x)$ for positive x. Note that the Möbius transformation $z \mapsto \pi\rho(r-z)/rz$ maps $\tilde{\Omega}_r$ onto the domain $\{w \in \mathbb{C}; \operatorname{Re} w > 0, |\operatorname{Im} w| < \pi/2\}$. Hence, the function $f(z) = \exp(\pi\rho(r-z)/rz)$ maps $\tilde{\Omega}_r$ conformally onto the domain $W = \{w \in \mathbb{C}; \operatorname{Re} w > 0, |w| > 1\}$ in such a way that $f(\partial\tilde{\Omega}_r \cap \partial B(r/2,r/2)) = \{\zeta; |\zeta| = 1, \operatorname{Re}\zeta > 0\} =: I$. Since $\omega(y,I,W) = 2(\arg(y+i) - \arg(y-i))/\pi = 4\arctan(1/y)/\pi < 4/\pi y$ for $y > 1$, we obtain $u(x) < 4/\pi f(x) = 4\exp(-\pi\rho(r-x)/rx)/\pi$. Taking $x = |z|$, we get the desired inequality. □

As we can read from the above proof, the following geometric form also holds. *Suppose that, for $a \in \partial_b\Omega$, there exist two closed disks B_1, B_2 of the same radius ρ in the outside of Ω satisfying $B_1 \cap B_2 = \{a\}$. Then, the same inequality as* (5.3) *holds for $r_0 = 2\rho$.*

6. Globalization

In the previous sections, we mainly investigated the local behaviour of bounded harmonic functions at a boundary point. In this section, we use results proved there to obtain global Hölder continuity of Green's function and of PWB solutions of Dirichlet problems under the assumption of a uniform property of the boundary.

6.1. **Reduction of global Hölder continuity of Green's function.** Let Ω be a domain with boundary of positive capacity such that $\infty \in \Omega \subset \widehat{\mathbb{C}}$. Then, Ω admits Green's function with pole at infinity, which will be denoted by $G = G_\Omega$. Since $G(z) - \log|z|$ is harmonic and $\log|z|$ is Lipschitz continuous near the point at infinity, we only have to consider the Hölder continuity of G near the boundary of Ω. So, we may restrict our attention to the subset $\Omega(r_0) = \{z \in \Omega; \delta_\Omega(z) < r_0\}$ of Ω, where r_0 is a fixed positive number. Of course, the following results are still valid for Green's function with pole at an arbitrary point under a suitable modification of the statement near the pole.

The first assertion enables us to reduce the problem involving two variables to that of one variable. A similar, more general result can be found, for example, in [19, Lemma A].

Lemma 6.1. *Let Ω be a domain containing ∞ in $\widehat{\mathbb{C}}$ with regular boundary in the sense of Dirichlet. For constants $\alpha \in (0, 1]$ and $r_0 > 0$, the following conditions are equivalent:*

(i) *G_Ω is Hölder continuous with exponent α in $\Omega(r_0)$, namely, there exists a constant C_1 such that*
$$|G_\Omega(z) - G_\Omega(w)| \le C_1|z - w|^\alpha$$
for any pair of points $z, w \in \Omega(r_0)$,

(ii) *there exists a constant C_2 such that*
$$G_\Omega(z) \le C_2 \delta_\Omega(z)^\alpha$$
for any point $z \in \Omega(r_0)$.

In fact, we have $C_2 \le C_1 \le 2^{1+\alpha} C_2$ for the possible smallest constants C_1 and C_2 in the above.

Proof. For simplicity, we write $G = G_\Omega$. By regularity of the boundary, $G(z) \to 0$ as $z \to \zeta \in \partial\Omega$. Hence, condition (ii) immediately follows from (i) with the same constant. Now we show the opposite direction. Assume condition (ii) and let $z, w \in \Omega(r_0)$.

Case 1: $2|z - w| \ge \max\{\delta_\Omega(z), \delta_\Omega(w)\}$.

In this case, we have
$$|G(z) - G(w)| \le G(z) + G(w) \le C_2(\delta_\Omega(z)^\alpha + \delta_\Omega(w)^\alpha) \le 2C_2(2|z - w|)^\alpha.$$

Case 2: $2|z - w| < \max\{\delta_\Omega(z), \delta_\Omega(w)\}$.

We assume that $2|z - w| < \delta_\Omega(z)$ and take an arbitrary $\delta \in (0, \delta_\Omega(z))$.

For any $\zeta \in \mathbb{D}$, we have the Poisson's formula
$$G(z + \delta\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |\zeta|^2}{|e^{i\theta} - \zeta|^2} G(z + \delta e^{i\theta}) d\theta.$$

Here note the elementary estimate $|1-(1-|\zeta|^2)/|e^{i\theta}-\zeta|^2| \le 2|\zeta|/(1-|\zeta|) \le 4|\zeta|$ for $|\zeta| \le 1/2$. Letting δ sufficiently near to $\delta(z)$, we can make the modulus of $\zeta := (w-z)/\delta$ less than $1/2$. Thus

$$(6.1)\qquad |G(w)-G(z)| \le 4|\zeta| \cdot \frac{1}{2\pi}\int_0^{2\pi} G(z+\delta e^{i\theta})d\theta = 4|\zeta|G(z) = 4\frac{|w-z|}{\delta}G(z).$$

Letting $\delta \to \delta_\Omega(z)$, we have

$$\begin{aligned}
|G(w)-G(z)| &\le 4\frac{|w-z|}{\delta_\Omega(z)}G(z) \le 4C_2|w-z|\delta_\Omega(z)^{\alpha-1} \\
&= 4C_2|w-z|^\alpha \left(\frac{|w-z|}{\delta_\Omega(z)}\right)^{1-\alpha} \\
&< 4C_2|w-z|^\alpha \left(\frac{1}{2}\right)^{1-\alpha} = 2^{1+\alpha}C_2|w-z|^\alpha.
\end{aligned}$$

In any case, we have the first condition with $C_1 = 2^{1+\alpha}C_2$. □

6.2. **Hölder continuity of Green's function.** By the aid of the previous lemma, we can prove the following. Note that, in the case when $\infty \in \Omega$, the domain Ω satisfies the uniform LHMD property if and only if $\partial\Omega$ is uniformly perfect (see Theorem 4.1).

Theorem 6.2. *Let Ω be a plane domain satisfying the uniform LHMD property with exponent $\alpha > 0$. Then Green's function of Ω is Hölder continuous with exponent α near the boundary.*

Proof. As before, G denotes Green's function of Ω with pole at ∞. By assumption, the local harmonic measure $\omega_{a,r,\Omega}$ satisfies $\omega_{a,r,\Omega}(z) \le C(|z-a|/r)^\alpha$ for $a \in \partial\Omega,\ 0 < r \le r_0$, where C and r_0 are positive constants. We denote by K the supremum of $G(z)$ in the set $\Omega(2r_0)$. Note that G can be regarded as the PWB solution $H^{\Omega(2r_0)}\varphi$ for φ, where $\varphi = 0$ on $\partial\Omega$ and $\varphi = G$ on $\partial\Omega(2r_0) \cap \Omega$.

Let $z \in \Omega(r_0)$. Then we can take a point $a \in \partial\Omega$ with $|z-a| = \delta_\Omega(z) = \delta_{\Omega(2r_0)}(z)$. We now make use of the latter part of Theorem 2.2 to claim

$$G(z) \le CK\left(\frac{|z-a|}{r_0}\right)^\alpha = CKr_0^{-\alpha}\delta_\Omega(z)^\alpha.$$

Now Lemma 6.1 immediately yields the desired result. □

6.3. **Applications to uniformly perfect sets.** As we have seen, an open set Ω has uniformly perfect boundary if and only if Ω satisfies global LHMD property (Theorem 4.1). Therefore, the next result, which was first stated in [8], immediately follows as a corollary.

Corollary 6.3. *Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain whose boundary is uniformly perfect. Then Green's function of Ω is Hölder continuous up to the boundary.*

Using the quantitative results obtained before, we could have explicit estimates for the exponent of Hölder continuity of Green's function in terms of constants characterizing uniform perfectness, however those would be, more or less, indirect.

We also note that the following result is obtained by combining with a theorem of Carleson [7], which asserts the removability of a compact set of Hausdorff dimension less than α for Hölder continuous harmonic functions with exponent α off the set.

Corollary 6.4. *Let α be a number with $0 < \alpha \leq 1$. If an open set $\Omega \subset \mathbb{C}$ satisfies the uniform LHMD property with exponent α, then Hausdorff dimension of its boundary is at least α.*

In particular, a uniformly perfect set has positive Hausdorff dimension (cf. [8]). A similar result can be found in [41, Theorem 7.2]. If a compact set E contains a continuum, then, of course, E has Hausdorff dimension at least 1. Therefore, the above corollary actually makes sense only if $\partial\Omega$ is totally disconnected. We know abundant examples of totally disconnected, uniformly perfect sets such as limit sets in the complex dynamics as we noted in Introduction.

6.4. **Historical remarks.** The present research was originally motivated by the above-mentioned fact that Green's function on a domain with uniformly perfect boundary has a Hölder continuous extension to the boundary. We now explain a little bit more about the history.

In the book [8] of Carleson and Gamelin published in 1993, it is stated without proof that Green's function of a domain with uniformly perfect boundary can be extended to the boundary in a Hölder continuous way. (In [9], an outline of the proof is presented in the case that the domain is a Fatou set, i.e., the complement of a Julia set.) They remarked that, from this fact, it follows that the Julia set of a rational function of degree at least two is of positive Hausdorff dimension as was stated above.

The first complete proof of this fact appeared in J. Lithner's paper [26] in the literature. He showed that a compact set is uniformly perfect if and only if the set preserves the local Markov inequality. To prove that the local Markov inequality implies the global one, he used and proved the above fact (see also §8.1). His method consists of two steps; the first is to show that a uniformly perfect set necessarily contains a Cantor set with some uniform property, and the second is to show the complement of such a Cantor set carries Green's function with Hölder continuity property.

Later, J. Siciak gave in [37] an explicit exponent of Hölder continuity of Green's function in terms of condenser capacity density. He also noted in [37] the history of this result around 1994.

6.5. **Applications to domains with uniform exterior conditions.** We also have the following results from Theorems 5.2 and 5.5.

Corollary 6.5. *If a plane domain Ω satisfies the uniform exterior circle condition, Green's function of Ω is Lipschitz continuous near the boundary.*

Corollary 6.6. *If a plane domain Ω satisfies the uniform generalized exterior wedge condition with opening β, then Green's function of Ω is Hölder continuous with exponent $\pi/(2\pi - \beta)$ near the boundary.*

In particular, any hyperbolic simply connected domain satisfies the generalized exterior wedge condition with opening 0, and thus its Green's function is Hölder continuous with exponent 1/2, which is a classical result (see also Proposition 7.1 below).

6.6. **Lipschitz spaces.** Next, we consider the (global) Hölder continuity of PWB solutions for bounded, Hölder continuous boundary functions.

To make our results more comprehensive, it is desirable to employ the Banach spaces of Hölder continuous functions on subsets of $\mathbb{C}$. For a constant $\alpha \in (0,1]$ and for a subset E of $\mathbb{C}$, we denote by $\Lambda_\alpha(E)$ the (real) Banach space consisting of all (real-valued) functions φ on E such that

$$\|\varphi\|_{\Lambda_\alpha(E)} := \sup_{x\in E} |\varphi(x)| + \sup_{x,y\in E, x\neq y} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha} < \infty.$$

Sometimes, $\Lambda_\alpha(E)$ is called the Lipschitz space (or the Hölder space) on E with exponent α. Note that $\Lambda_\beta(E)$ is continuously embedded in $\Lambda_\alpha(E)$ if $\beta > \alpha$. For convenience, though somewhat artificially, we define $\Lambda_\alpha(E)$ as $\Lambda_\alpha(E\setminus\{\infty\})$ even for a subset E of $\widehat{\mathbb{C}}$ containing ∞.

6.7. **Hölder continuity of PWB solutions.** Using this notation, we can state one of our main results as in the following.

Theorem 6.7. *Let Ω be a plane domain satisfying uniform LHMD property with exponent α. Then, for any γ with $0<\gamma<\alpha$, there exists a constant B such that $\|H^\Omega\varphi\|_{\Lambda_\gamma(\Omega)} \le B\|\varphi\|_{\Lambda_\gamma(\partial_{\mathrm{b}}\Omega)}$ for all $\varphi \in \Lambda_\gamma(\partial_{\mathrm{b}}\Omega)$.*

Assume the uniform LHMD property with exponent α for Ω, namely, there are some constants $1 \le C < +\infty$ and $0 < r_0 \le +\infty$ such that $\omega_{a,r,\Omega}(z) \le C(|z-a|/r)^\alpha$ holds for all $0<r<r_0$ and $z \in \Omega \cap B^\circ(a,r)$. Let $z\in\Omega$ and take $a \in \partial_{\mathrm{b}}\Omega$ such that $|z-a| = \delta_\Omega(z)$. Let $\varphi \in \Lambda_\gamma(\partial_{\mathrm{b}}\Omega)$ for some $\gamma<\alpha$. Then, by Theorem 2.2, we have

$$\begin{aligned}|H^\Omega\varphi(z)-\varphi(a)| \le& 2^\alpha C\left(r_0^{-\gamma}\sup_{\zeta\in\partial_{\mathrm{b}}\Omega}|\varphi(\zeta)-\varphi(a)| + \frac{1}{\alpha-\gamma}\sup_{\xi,\eta\in\partial_{\mathrm{b}}\Omega}\frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|^\gamma}\right)|z-a|^\gamma \\ \le& 2^\alpha C\left(2r_0^{-\gamma} + 1/(\alpha-\gamma)\right)\|\varphi\|_{\Lambda_\gamma(\partial_{\mathrm{b}}\Omega)}\delta_\Omega(z)^\gamma.\end{aligned}$$

Therefore, we can get the expected conclusion by the following lemma.

Lemma 6.8. *Let $\alpha\in(0,1]$ and $\varphi\in\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)$. Suppose that a constant $C\ge 1$ satisfies the inequality*

$$|H^\Omega\varphi(z)-\varphi(a)| \le C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}\delta_\Omega(z)^\alpha$$

for all $\varphi\in\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)$, $z\in\Omega$ and $a\in\partial_{\mathrm{b}}\Omega$ such that $\delta_\Omega(z)=|z-a|$. Then $\|H^\Omega\varphi\|_{\Lambda_\alpha(\Omega)} \le 16C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}$ holds for all $\varphi\in\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)$.

Proof. We write $u = H^\Omega\varphi$ for a given $\varphi\in\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)$. Let $z,w\in\Omega$.

Case 1: $2|z-w| \ge \max\{\delta_\Omega(z),\delta_\Omega(w)\}$.

Take a and b in $\partial_{\mathrm{b}}\Omega$ such that $|z-a| = \delta_\Omega(z)$ and $|w-b| = \delta_\Omega(w)$. Then we have

$$\begin{aligned}|u(z)-u(w)| &\le |u(z)-\varphi(a)| + |\varphi(a)-\varphi(b)| + |u(w)-\varphi(b)| \\ &\le C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-a|^\alpha + \|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|a-b|^\alpha + C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|w-b|^\alpha \\ &\le (2^{1+\alpha}C + 5^\alpha)\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-w|^\alpha \le 9C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-w|^\alpha\end{aligned}$$

since $|a-b| \le |a-z| + |z-w| + |w-b| \le 5|z-w|$.

Case 2: $2|z-w| < \max\{\delta_\Omega(z), \delta_\Omega(w)\}$.

We assume $2|z-w| < \delta_\Omega(z)$ and take a point a in $\partial_{\mathrm{b}}\Omega$ such that $\delta_\Omega(z) = |z-a|$. Under the additional assumption that $\varphi(a) = 0$, we first show the following estimate:

$$|u(z)-u(w)| \le 8C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-w|^\alpha. \tag{6.2}$$

We can write $\varphi = \varphi^+ - \varphi^-$, where $\varphi^+ = (|\varphi| + \varphi)/2$ and $\varphi^- = (|\varphi| - \varphi)/2$. Here note that $\|\varphi^\pm\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)} \le \|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}$. Set $u^+ = H^\Omega\varphi^+$ and $u^- = H^\Omega\varphi^-$. Then, in the same way as in the proof of (6.1), we can show

$$|u^+(z) - u^+(w)| \le \frac{4|z-w|}{\delta_\Omega(z)}u^+(z).$$

By assumption $u^+(z) = u^+(z) - \varphi^+(a) \le C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-a|^\alpha$, we have

$$|u^+(z) - u^+(w)| \le 4C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}\left(\frac{|z-w|}{\delta_\Omega(z)}\right)^{1-\alpha}|z-w|^\alpha \le 4C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-w|^\alpha.$$

Similarly, we have $|u^-(z) - u^-(w)| \le 4C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-w|^\alpha$. Now we have estimate (6.2).

In the case $\varphi(a) \ne 0$, we replace φ by $\varphi - \varphi(a)$, then have

$$|u(z)-u(w)| \le 8C\|\varphi - \varphi(a)\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-w|^\alpha \le 16C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-w|^\alpha.$$

Summing up the above, we have $|u(z)-u(w)| \le 16C\|\varphi\|_{\Lambda_\alpha(\partial_{\mathrm{b}}\Omega)}|z-w|^\alpha$ for any z and w in Ω, and thus the proof is complete. □

Remark. By the above proof, we get information on the operator norm of $H^\Omega : \Lambda(\partial_{\mathrm{b}}\Omega) \to \Lambda(\Omega)$, namely,

$$\|H^\Omega\varphi\|_{\Lambda_\gamma(\Omega)} \le 2^{5+\alpha}C\left(2r_0^{-\gamma} + 1/(\alpha-\gamma)\right)\|\varphi\|_{\Lambda_\gamma(\partial_{\mathrm{b}}\Omega)}.$$

6.8. **Operator norm of harmonic extension.** Combining Theorems 5.2, 5.5 and the remark just above, we obtain the following results.

Corollary 6.9. *If a plane domain Ω satisfies the uniform exterior circle condition with radius ρ, for any $\alpha \in (0,1)$, the harmonic extension operator $H^\Omega : \Lambda_\alpha(\partial_{\mathrm{b}}\Omega) \to \Lambda_\alpha(\Omega)$ is bounded and its operator norm satisfies*

$$\|H^\Omega\|_{\Lambda_\alpha} \le C\left(\rho^{-\alpha} + \frac{1}{1-\alpha}\right), \tag{6.3}$$

where C is an absolute constant.

Corollary 6.10. *If a plane domain Ω satisfies the uniform generalized exterior wedge condition with opening β, for any $\alpha < \pi/(2\pi - \beta)$, the harmonic extension operator $H^\Omega : \Lambda_\alpha(\partial_{\mathrm{b}}\Omega) \to \Lambda_\alpha(\Omega)$ is bounded.*

In the proof of Theorem 6.7, for a given $z \in \Omega$, we needed only to consider a boundary point a with $\delta(z) = |z - a|$. In this case, we have $p_\Omega(z, a) = |z - a| = \delta(z)$. Hence, from Theorem 5.3, we also have the following corollary.

Corollary 6.11. *Let Ω be a plane domain whose boundary is of class C^2 and has curvature not less than $-1/\rho$, where ρ is a positive constant. Then the harmonic extension operator $H^\Omega : \Lambda_\alpha(\partial_{\mathrm{b}}\Omega) \to \Lambda_\alpha(\Omega)$ satisfies the same norm estimate as* (6.3) *for an absolute constant C.*

6.9. **Examples.** We cannot expect that the above results would be still valid in general when $\gamma = \alpha$, as the following simple examples show.

Example 6.1. We take the unit disk $\mathbb{D}$ as Ω. Consider the function $\varphi(\zeta) = |\operatorname{Im}\zeta|$ on $\partial\mathbb{D}$, which is, of course, Lipschitz continuous on $\partial\mathbb{D}$. But the harmonic extension u of φ is *not* Lipschitz continuous in $\mathbb{D}$ because $u(r)$, where $-1 < r < 1$, can be expressed by

$$u(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos t + r^2}|\sin t|dt = \frac{1 - r^2}{\pi r}\log\frac{1 + r}{1 - r}.$$

Therefore, the image of $\Lambda_1(\partial\mathbb{D})$ under the harmonic extension operator $H^{\mathbb{D}}$ to $\mathbb{D}$ is not contained in $\Lambda_1(\mathbb{D})$. Note that a similar example was constructed by Hinkkanen (Example (5.11) of [19]).

By transformation under the map $h(z) = (1 - z)/(1 + z)$, we obtain the right half plane $\Omega = \mathbb{H}_+ = \{z; \operatorname{Re} z > 0\}$ and the boundary function $\psi(iy) = 2|y|/(1 + y^2)$, whose harmonic extension v has the expression

$$v(x) = \frac{2x}{\pi(x^2 - 1)}\log x$$

on the positive real axis $\mathbb{R}_+$.

In order to show that Corollary 6.10 is sharp, we bend the above example.

Example 6.2. For $1/2 \le \alpha \le 1$, we consider the domain $\mathbb{H}_+^\alpha = \{z \in \mathbb{C} \setminus \{0\}; |\arg z| < \pi/2\alpha\}$. This satisfies the uniform (generalized) exterior wedge condition with opening $\beta = 2\pi - \pi/\alpha$. By Theorem 5.5, $\mathbb{H}_+^\alpha$ has the global LHMD property with exponent $\alpha = \pi/(2\pi - \beta)$. Hence Corollary 6.10 ensures boundedness of the harmonic extension operator H from $\Lambda_\gamma(\partial_{\mathrm{b}}\mathbb{H}_+^\alpha)$ into $\Lambda_\gamma(\mathbb{H}_+^\alpha)$ for $\gamma < \alpha$. However, the corresponding statement for $\gamma = \alpha$ no longer holds. Indeed, the function $\theta(re^{\pm\pi i/2\alpha}) = \psi(\pm ir^\alpha) = 2r^\alpha/(1 + r^{2\alpha})$ belongs to $\Lambda_\alpha(\partial_{\mathrm{b}}\mathbb{H}_+^\alpha)$ while its harmonic extension $w(z) = v(z^\alpha)$ has the expression

$$w(x) = \frac{2\alpha x^\alpha}{\pi(x^{2\alpha} - 1)}\log x$$

on the positive real axis $\mathbb{R}_+$, where ψ and v are the functions defined in the previous example, in particular, $w \notin \Lambda_\alpha(\mathbb{H}_+^\alpha)$.

To obtain a bounded domain Ω with the same property when $1/2 < \alpha < 1$, we have only to transform $\mathbb{H}_+^\alpha$ by $h(z) = (1-z)/(1+z)$. The resulting domain is

$$h^{-1}(\mathbb{H}_+^\alpha) = \left\{ z \in \mathbb{C}; \left| z + i \cot \frac{\pi}{2\alpha} \right| < \frac{1}{\sin \frac{\pi}{2\alpha}} \quad \text{or} \quad \left| z - i \cot \frac{\pi}{2\alpha} \right| < \frac{1}{\sin \frac{\pi}{2\alpha}} \right\}.$$

We note that we can modify this example to obtain a bounded domain Ω with global LHMD property with exponent $1/2$ for which H^Ω does not preserve Hölder spaces with exponent $1/2$. However, when $0 < \alpha < 1/2$, the author does not know if we could get a (bounded) domain satisfying global LHMD property with exponent α for which the harmonic extension operator does not preserve Hölder spaces with exponent α.

Remark. Quite recently, Aikawa [2] proved the remarkable result that there is no bounded domain Ω in $\mathbb{R}^n$ $(n \geq 2)$ for which $H^\Omega(\Lambda_1(\partial\Omega)) \subset \Lambda_1(\Omega)$ holds. Moreover, he established the converse of Theorem 6.7, in other words, for a bounded domain Ω with regular boundary, if the harmonic extension operator H^Ω preserves Λ_α for some $\alpha > 0$, then Ω enjoys the global LHMD property with exponent α. In particular, a bounded domain Ω with regular boundary has uniformly perfect boundary if and only if $H^\Omega(\Lambda_\alpha(\partial\Omega)) \subset \Lambda_\alpha(\Omega)$ for some $\alpha > 0$.

7. Application to conformal mappings

7.1. **Boundary behaviour of a conformal mapping and Green's function.** In this section, we present applications of Theorem 6.2 to the theory of conformal mappings. We consider a finitely connected *bounded* domain Ω in $\mathbb{C}$ without degenerate boundary components even if the boundary is bounded. As is well known, the domain Ω is can be mapped by a conformal mapping g onto a plane domain, say Ω_0, bounded by finitely many closed analytic curves. Let $f = g^{-1} : \Omega_0 \to \Omega$. Here and hereafter an analytic curve will mean a simple closed curve of class C^ω in the complex plane $\mathbb{C}$. (Furthermore, by Koebe's theorem, we can choose Ω_0 so that each component of its boundary is a circle.) Take a point $a \in \Omega$ and set $a_0 = g(a)$. We denote by G and G_0 the Green's functions of Ω and Ω_0 with pole at a and a_0, respectively. Hence, $G(f(z)) = G_0(z)$ for each $z \in \Omega_0$.

By conformal invariance of the hyperbolic metric, we note the identity $\lambda_\Omega(f(z))|f'(z)| = \lambda_{\Omega_0}(z)$. Since $\partial\Omega$ is uniformly perfect, by Theorem B,

$$c \leq \delta_\Omega(w)\lambda_\Omega(w) \leq 1 \tag{7.1}$$

holds for some positive constant c.

Let α be a constant with $0 < \alpha \leq 1$. By Lemma 6.1, G is Hölder continuous of exponent α near the boundary if and only if $G(w) \leq C\delta_\Omega(w)^\alpha$ holds near the boundary for some constant C. By (7.1), this property is equivalent to the condition

$$G_0(z) = G(f(z)) \leq \text{const.}\lambda_\Omega(f(z))^{-\alpha} = \text{const.}(|f'(z)|/\lambda_{\Omega_0(z)})^\alpha$$

when z is sufficiently close to $\partial\Omega_0$. By using Schwarz' reflection principle, we can see $G_0(z) \asymp \delta_{\Omega_0}(z) \asymp 1/\lambda_{\Omega_0}(z)$ near the boundary of Ω_0. Now we can easily verify the following proposition.

Proposition 7.1. *Let Ω be a finitely connected bounded domain in $\mathbb{C}$ without punctures. Let $\alpha \in (0,1]$ be a constant and $f : \Omega_0 \to \Omega$ be a conformal mapping, where Ω_0 is a plane domain bounded by finitely many analytic curves. Then, Green's function of Ω is Hölder continuous of exponent α near the boundary if and only if there exists a positive constant m such that*

$$\frac{1}{|f'(z)|} \le m\lambda_{\Omega_0}(z)^{\frac{1}{\alpha}-1} \tag{7.2}$$

for $z \in \Omega_0$.

If Ω is simply connected and if Ω_0 is the unit disk, the inequality (7.2) with $\alpha = 1/2$ immediately follows from the Koebe distortion theorem for univalent functions. This, of course, agrees with the statement of Corollary 6.6.

Now Corollary 6.6 yields the next result, which is a natural generalization of Theorem I in [28] shown by M. Masumoto.

Corollary 7.2. *Let $f : \Omega_0 \to \Omega$ be as above. Suppose that Ω satisfies the uniform generalized exterior wedge condition with opening $\beta \in [0, \pi]$. Then there exists a positive constant m such that*

$$\frac{1}{|f'(z)|} \le m\lambda_{\Omega_0}(z)^{1-\beta/\pi}$$

for $z \in \Omega_0$.

Note that an arbitrary, finitely connected, bounded domain without degenerate boundary components satisfies the generalized wedge condition with opening 0.

7.2. **Non-integrability of subharmonic functions.** By virtue of the above corollary, the same argument as in [28] yields the following result.

Theorem 7.3. *Let Ω be a finitely connected bounded domain in $\mathbb{C}$ without punctures. Suppose that Green's function of Ω is Hölder continuous of exponent $\alpha \in (0,1]$ near the boundary. We set $\gamma(p,\alpha) = 2 - \alpha \min\{1,p\}$ for $0 < p < \infty$. If a nonnegative subharmonic function s on Ω satisfies*

$$\iint_\Omega \delta_\Omega(z)^{-\gamma(p,\alpha)} s(z)^p dxdy < +\infty, \quad z = x + iy$$

for some $0 < p < \infty$, then s must vanish identically.

In the case when the boundary of Ω is of class $C^{1,1}$, N. Suzuki [42] proved the above result with $\alpha = 1$. Actually, our general statement can be deduced from Suzuki's special one as is described in [28]. For convenience, we now reproduce it.

Proof. Let $\gamma = \gamma(p,\alpha)$ and $\gamma_0 = \gamma(p,1)$. Note $\gamma \le 2$ and that $s_0 := s \circ f$ is subharmonic in the domain Ω_0. By (7.1), we may suppose that s satisfies $\iint_\Omega \lambda_\Omega(w)^\gamma s(w)^p dudv < +\infty$. Noting estimate (7.2), by a change of variables, we have

$$\begin{aligned}\iint_\Omega \lambda_\Omega(w)^\gamma s(w)^p dudv &= \iint_{\Omega_0} \lambda_{\Omega_0}(z)^\gamma |f'(z)|^{2-\gamma} s_0(z)^p dxdy \\ &\ge \frac{1}{m} \iint_{\Omega_0} \lambda_{\Omega_0}(z)^{\gamma_0} s_0(z)^p dxdy.\end{aligned}$$

Hence, by Suzuki's result, we have $s_0 = 0$. □

Corollary 7.4. *If a finitely connected bounded domain Ω satisfies the uniform generalized exterior wedge condition with opening $\beta \in [0, \pi]$, then there does not exist a non-negative subharmonic function $s \neq 0$ on Ω with*

$$\iint_\Omega \delta_\Omega(z)^{-\gamma_p} s(z)^p dxdy < +\infty, \quad z = x + iy$$

for some $0 < p < \infty$, where $\gamma_p = 2 - \min\{p, 1\}/(2 - \beta/\pi)$.

7.3. **Integrability of the derivatives of conformal mappings.** We now give another application of estimate (7.2). Consider the inverse mapping $g : \Omega \to \Omega_0$ of f. It is trivial that $\iint_\Omega |g'(w)|^2 dudv = \text{Area}\,(\Omega_0) < +\infty$. In our case, we can say more.

Theorem 7.5. *Let Ω be a finitely connected bounded domain without punctures of which Green's function is Hölder continuous with exponent α. Suppose that g is a conformal mapping from Ω onto a domain Ω_0 bounded by finitely many analytic curves. Then we have*

$$\iint_\Omega |g'(z)|^p dxdy < +\infty$$

for any $0 < p < 2 + \alpha/(1 - \alpha)$. In particular, for a domain Ω satisfying the generalized wedge condition with opening β, we have the above result for any $0 < p < 2 + \pi/(\pi - \beta)$.

Proof. We will give a proof for the case that Ω is simply connected and that Ω_0 is the unit disk $\mathbb{D}$. The general case can be treated in the quite same way as follows.

We may assume $2 < p < 2 + \alpha/(1 - \alpha)$. By (7.2), we have

$$\begin{aligned}\iint_\Omega |g'(w)|^p dudv &= \iint_{\mathbb{D}} |f'(z)|^{2-p} dxdy \leq M \iint_{\mathbb{D}} (1 - |z|)^{-(p-2)(1-\alpha)/\alpha} dxdy \\ &\leq 2\pi M \int_0^1 (1 - r)^{-(p-2)(1-\alpha)/\alpha} dr,\end{aligned}$$

where M is a constant. The last term is finite if $(p - 2)(1 - \alpha)/\alpha > 1$. □

In particular, if Ω is bounded and simply connected and if Ω_0 is the unit disk, the above yields $\iint_\Omega |g'(w)|^p dudv < +\infty$ for any $0 < p < 2 + \alpha/(1 - \alpha)$. Note again that $\alpha \geq 1/2$ always holds, and hence $2 + \alpha/(1 - \alpha) \geq 3$.

We should remark that the Brennan conjecture asserts that $\iint_\Omega |g'(w)|^p dudv < +\infty$ for any simply connected domain Ω and for any $0 < p < 4$, but this is unsolved yet so far. The best result up to now is $p < 3.39$, which is due to Ch. Pommerenke (see [34, Chapter 8]). Therefore, our result above must not be sharp.

Finally, we mention a couple of results involving the exterior wedge condition. F. D. Lesley [25] proved that for a bounded Jordan domain Ω with exterior wedge condition of opening $\beta > 0$ the inverse of the Riemann mapping function $g = f^{-1} : \Omega \to \mathbb{D}$ is Hölder continuous with exponent $1/(2 - \beta/\pi)$. Under the same condition, C. H. Fitzgerald and Lesley [13] proved that $1/f' \in H^p$ for $0 < p < 1/(1 - \beta/\pi)$, where H^p denotes the Hardy space on the unit disk with exponent p. Our results above are closely related to their results, however, there seem to be no immediate implications between them.

8. Counterexample

8.1. **Application to Markov inequalities.** We construct here a simple counterexample showing that Hölder continuity property of Green's function does not necessarily imply uniform perfectness of the boundary. Actually, our domain here carries Lipschitz continuous Green's functions, whereas its boundary is not uniformly perfect. This answers negatively one of the questions raised in [37]. J. Lithner [26] has constructed a certain kind of Cantor set preserving the local Markov inequality, but not preserving the global Markov inequality. Lithner [26] also proved that the global Hölder continuity of Green's function implies the global Markov inequality, as well as that uniform perfectness is equivalent to the preservation of local Markov inequality. Therefore, our example gives another example similar to Lithner's one (also note that our example has no degenerate boundary components).

8.2. **Construction.** Let a sequence r_n of positive numbers satisfying $r_n < 1/8$ be given. Let ε_n and M_n $(n = 1, 2, \dots)$ be sequences of positive numbers monotonically decreasing to 0, which we will specify later. We take $\varepsilon_1 < 1/2$. Set

$$\begin{aligned}\widehat{\Omega}_0 = &\{z \in \mathbb{C}; |\operatorname{Im} z| < 1/2, \operatorname{Re} z > -1\} \\ &\setminus \bigcup_{n=1}^{\infty} \big(B(2n - 1 + (1 + \varepsilon_n)i/2, 1/2) \cup B(2n - 1 - (1 + \varepsilon_n)i/2, 1/2)\big).\end{aligned}$$

Note that $\widehat{\Omega}_0$ is simply connected and satisfies the exterior circle condition with constant $\rho = 1/2$. We now consider the domain $\widehat{\Omega} := \widehat{\Omega}_0 \setminus \bigcup_{n=1}^{\infty} B(2n, r_n)$ and its inversion $\Omega := \{z \in \widehat{\mathbb{C}}; 1/z \in \widehat{\Omega}\}$. Let G be the Green's function of Ω with pole at infinity and $\hat{G}$ be the Green's function of $\widehat{\Omega}$ with pole at the origin, and hence $G(1/z) = \hat{G}(z)$. If we denote by $\hat{G}_0$ the Green's function of $\widehat{\Omega}_0$ with pole at the origin, then we have $\hat{G} < \hat{G}_0$ on $\widehat{\Omega}$ by the minimum principle. Note that the majorant $\hat{G}_0$ does not depend on the sequence r_n.

We begin with the comparison between $\delta := \delta_\Omega$ and $\hat{\delta} := \delta_{\widehat{\Omega}}$. Set $D_n := \{z \in \widehat{\Omega}; 2n-1 < \operatorname{Re} z < 2n + 1\}$ and $\tilde{D}_n := \{z \in \widehat{\Omega}_0; 2n - 1 < \operatorname{Re} z < 2n + 1\}$.

Lemma 8.1. *For $z \in D_n$ $(n \ge 0)$, we have $\hat{\delta}(z) \le 4(n + 1)^2 \delta(1/z)$.*

Proof. Let $a \in \partial\widehat{\Omega}$ such that $\delta(1/z) = |1/z - 1/a| = |z - a|/|az|$. Then the point a is contained in the boundary of a disk in $\widehat{\Omega}$ containing z. Thus $|a| \le 2n + 1 + \varepsilon(1 + \varepsilon/2) \le 2n + 2$. Now we have $\delta(1/z) \ge |z - a|/4(n + 1)^2 \ge \hat{\delta}(z)/4(n + 1)^2$. □

It is easy to see that, for a given sequence M_n of positive numbers, there exists a sequence ε_n such that $\hat{G}_0 \le M_n$ on $\tilde{D}_n$ $(n = 1, 2, \dots)$. The maximum principle implies that $\hat{G} \le M_n$ on D_n for each $n = 1, 2, \dots$. By the proof of Theorem 6.2, we can show $\hat{G}(z) \le C M_n \hat{\delta}_0(z)$ on D_n, where $\hat{\delta}_0 = \delta_{\widehat{\Omega}_0}$ and C is an absolute constant not less than one. If $z \in D_n$ satisfies $\hat{\delta}_0(z) = \hat{\delta}(z)$, then we have $\hat{G}(z) \le C M_n \hat{\delta}(z)$. If not, it is clear that the point z satisfies $|z - 2n| < 1/2$. We set $A_n := \{z \in \mathbb{C}; r_n < |z - 2n| < 1/2\}$ and denote by ω_n the harmonic measure of $\{|z - 2n| = 1/2\}$ relative to A_n for each $n = 1, 2, \dots$.

Explicitly, we can write $\omega_n(z) = (\log|z-2n|/r_n)/\mu_n$, where μ_n denotes the modulus of A_n, i.e., $\mu_n = \log 1/2r_n$. Then we see that $\hat{G} \le M_n\omega_n$ on A_n. In other words,

$$\hat{G}(z) \le \frac{M_n}{\mu_n}\log\frac{|z-2n|}{r_n} \le \frac{M_n}{\mu_n}\frac{|z-2n|-r_n}{r_n} = \frac{M_n}{\mu_n r_n}\hat{\delta}(z).$$

Now we take

$$M_n = \frac{\mu_n r_n}{4(n+1)^2} = \frac{r_n \log 1/2r_n}{4(n+1)^2}.$$

Then we have $\hat{G}(z) \le C\hat{\delta}(z)/4(n+1)^2$ for $z \in D_n$. This estimate can be made still valid for $n=0$ and $z \in D_0 \setminus B(0,1/4)$ by replacing the constant C by another one if necessary. Now, by Lemma 8.1, we have $G(z) \le C\delta(z)$ for $z \in \Omega$ with $|z| \le 4$, which implies the Lipschitz continuity of G near the boundary (cf. Lemma 6.1).

Choosing the sequence r_n so that $r_n \to 0$ as $n \to \infty$, the moduli μ_n of essential annuli A_n of $\widehat{\Omega}$ tend to infinity as $n \to \infty$, therefore $\widehat{\Omega}$, and hence Ω does not have uniformly perfect boundary (cf. [41]).

9. Appendix

In this appendix, we will present a proof of Proposition 5.4. To this end, first we give preparatory results which are true also for regular plane curves with the same curvature bound.

Let κ be a continuous function from an interval I with $0 \in I$ into $\mathbb{R}$. Denote by $\Delta(\rho)$ the open disk of radius ρ and centered at $-i\rho$. We set $\theta(t) = \int_0^t \kappa(s)ds$ and consider the regular curve $\xi : I \to \mathbb{C}$ given by $\xi(t) = \int_0^t e^{i\theta(s)}ds$. We write $\xi(t) = X(t) + iY(t)$, in other words,

$$X(t) = \int_0^t \cos\theta(s)ds \quad \text{and} \quad Y(t) = \int_0^t \sin\theta(s)ds. \tag{9.1}$$

Of course, the curve ξ is not necessarily simple, but the following lemmas can be shown. (In the case when $|\kappa| \le 1/\rho$, we can deduce stronger global properties for the curve. See, for example, [20] and the references cited therein.)

Lemma 9.1. *Suppose that $\theta(t_0) = 0, \theta(t_1) = -\pi/2, \kappa(t) = \theta'(t) \ge -1/\rho$ and $0 \ge \theta(t) \ge -\pi/2$ for $t_0 \le t \le t_1$, where ρ is a positive constant. Then*

$$X(t) - X(t_0) \ge \rho\sin\frac{t-t_0}{\rho} \quad \text{and} \quad Y(t) - Y(t_0) \ge \rho\left(\cos\frac{t-t_0}{\rho} - 1\right) \tag{9.2}$$

for $t_0 \le t \le t_1$, in particular, the curve $\xi(t) - \xi(t_0)$ is injective and does not intersect the disk $\Delta(\rho)$ on the interval $[t_0, t_1]$. Moreover, the following hold:

$$X(t_1) - X(t_0) \ge \rho \quad \text{and} \quad Y(t_1) - Y(t_0) \le -\rho. \tag{9.3}$$

Proof. Since $-\pi/2 \le \theta(t_0+s) \le -s/\rho$, we see $\sin\theta(t_0+s) \ge -\sin s/\rho$ and $\cos\theta(t_0+s) \ge 1 - \cos s/\rho$. Now inequalities (9.2) follow from the expression (9.1). In particular, letting $t = t_1$, we have the first inequality in (9.3). To prove the second inequality in (9.3), we need more analysis.

We define the monotone decreasing continuous function $\hat{\theta}$ by

$$\hat{\theta}(t) = \min_{t_0 \le s \le t} \theta(s).$$

Then $\hat{\theta}(t) \le \theta(t)$. Noting that $-\pi/2 - \theta(t) = \int_t^{t_1} \theta'(s)ds \ge -(t_1 - t)/\rho$, we have $t_1 - t \ge \rho(\theta(t) + \pi/2) \ge \rho(\hat{\theta}(t) + \pi/2)$ for $t_0 \le t \le t_1$.

Since $\theta = \hat{\theta}$ on the support E of the probability measure $-d\sin\hat{\theta}(t)$ on $[t_0, t_1]$ and since $\sin\theta(t) \le 0$ on $[t_0, t_1]$, by integrations by parts, we have

$$\begin{aligned}
Y(t_1) - Y(t_0) &= \int_{t_0}^{t_1} \sin\theta(t)dt \le \int_E \sin\hat{\theta}(t)dt \\
&= -\int_E (t - t_1)d\sin\hat{\theta}(t) = -\int_{t_0}^{t_1} (t - t_1)d\sin\hat{\theta}(t) \\
&\le \int_{t_0}^{t_1} \rho(\hat{\theta}(t) + \pi/2)d\sin\hat{\theta}(t) = \rho\int_0^{-\pi/2} (\theta + \pi/2)d\sin\theta = -\rho.
\end{aligned}$$

□

Lemma 9.2. *Suppose that $\theta(t_0) = 0, \theta'(t) \ge -1/\rho$ and $-\pi/2 \le \theta(t) \le \pi/2$ for $t_0 \le t \le t_1$. Then the curve $\xi(t) - \xi(t_0)$ is injective and does not intersect the disk $\Delta(\rho)$ on the interval $[t_0, t_1]$.*

Proof. First we observe, by assumption, $X(t)$ is non-decreasing in $[t_0, t_1]$. By the approximation argument, we may assume that the function θ has at most finitely many zeros in $[t_0, t_1]$. We divide the interval $[t_0, t_1]$ into subintervals $I_1, \ldots, I_N$ with $I_j = [a_j, b_j]$ such that $a_1 = t_0, b_j = a_{j-1}$ and $b_N = t_1$, $|\theta| > 0$ on (a_j, b_j) and that $\theta(a_j) = 0$ for $j = 1, \ldots, N$. We assume that the assertion in the lemma is true for $I_1 \cup \cdots \cup I_{n-1}$. In the case when $\theta > 0$ on I_n, $Y(t)$ is monotone increasing in this interval, therefore the assertion is true also for $I_1 \cup \cdots \cup I_n$. In the case when $\theta < 0$ on I_n, we can use Lemma 9.1 to conclude truth of the assertion for this case. Hence, by induction, we have shown the assertion on the whole interval $[t_0, t_1]$. □

Lemma 9.3. *Suppose that $\theta(t_0) = 0, \theta(t_1) = -\pi, \theta'(t) \ge -1/\rho$ and $0 \ge \theta(t) \ge -\pi$ for $t_0 \le t \le t_1$. Then $Y(t)$ is non-increasing, in particular, ξ is injective on this interval $[t_0, t_1]$. Furthermore the curve $\xi(t) - \xi(t_0)$ does not intersect the disk $\Delta(\rho)$ and the Euclidean diameter of $\xi([t_0, t_1])$ is at least 2ρ.*

Proof. The first assertion easily follows from $Y'(t) = \sin\theta(t) \le 0$. Next let $s_0 = \min\{s > 0; \theta(s) = -\pi/2\}$. By Lemma 9.1, $\xi([t_0, s_0])$ does not intersect $\Delta(\rho)$ and the endpoint $X(s_0) + iY(s_0)$ satisfies $X(s_0) \ge \rho$ and $Y(s_0) \le -\rho$. Now Lemma 9.2 is applicable for the curve $i\xi$ on the interval $[s_0, t_1]$ to conclude that $\xi(t) - \xi(t_0)$ does not intersect the disk $\Delta(\rho)$. Let $s_1 = \max\{s > 0; \theta(s) = -\pi/2\}$. Applying Lemma 9.1 to the curve $\xi([s_1, t_1])$, we obtain $Y(t_1) - Y(s_0) \le Y(t_1) - Y(s_1) \le -\rho$. Therefore, the last assertion now follows from this and $Y(s_0) \le -\rho$. □

Of course, to obtain the statement of Proposition 5.4, we must make essential use of simpleness of the curve ξ. Before the proof, observe that the mirror image $\tilde{\xi}$, which will be defined here by $\tilde{\xi}(t) = -\overline{\xi(-t)}$, has the same curvature bound because $\tilde{\kappa}(t) = \kappa(-t)$.

We consider the situation stated in the proposition. Let L be the length of the curve ξ. Since ξ is simple, the rotation index $\int_0^L \kappa(t)dt/2\pi$ is ± 1 by the theorem of turning tangents (cf. [10]). The signature depends only on the orientation of the curve. Note that this notion can easily be defined and the above is still true in the case that the curve has finitely many corners. See [34] for more detailed account.

Without loss of generality, we may assume that $a = \xi(0) = 0$ and that $\xi'(0) = 1$, and hence $\Delta = \Delta(\rho) = \{z; |z + i\rho| < \rho\}$. Then, by the function $\theta(t) = \int_0^t \kappa(s)ds$, the unit velocity vector $\xi'(t)$ can be represented as $\exp(i\theta(t))$.

Now suppose that the curve ξ intersects Δ. Set $t_* = \inf\{t > 0; \xi(t) \in \Delta\}$ and $\theta_* = \theta(t_*)$. We can define $\varphi(t) = \arg(\xi(t)+i\rho)-\pi/2$ as a continuous function on $[0, t_*]$ with $\varphi(0) = 0$. We set $\varphi_* = \varphi(t_*)$. From Lemma 9.2, $|\varphi_*| > 0$ follows. Since ξ is simple, we note that $|\varphi_*| < 2\pi$. If $|\varphi_*| \geq \pi$, we can quite easily show that $\operatorname{diam} \xi([t_0, t_*]) \geq 2\rho$. Hence, we assume $|\varphi_*| < \pi$ in the sequel.

Case 1: $-\pi < \varphi_* < 0$.

We now consider the simple closed curve ξ_* which consists of the subarc $\xi|_{[0,t_*]}$ and the two segments $[i\rho(e^{i\varphi_*} - 1), -i\rho]$ and $[-i\rho, 0]$. Let ω be the turning tangent of ξ_* at the point $\xi(t_*)$, then $\omega = \varphi_* - \pi/2 - \theta_* + 2n\pi$ for some integer n and $|\omega| \leq \pi/2$.

Then it is clear that the rotation index $(\theta_* + \omega + (-\pi - \varphi_*) - \pi/2)/2\pi$ equals -1. This forces $n = 0$, and hence $|\varphi_* - \theta_* - \pi/2| \leq \pi/2$, i.e.,

$$\varphi_* - \pi \leq \theta_* \leq \varphi_* < 0. \tag{9.4}$$

Next we consider the case when

$$\sup_{0 \leq t < t' \leq t_*} |\theta(t) - \theta(t')| \geq \pi. \tag{9.5}$$

In this case, there exist t_0 and t_1 with $t_0 < t_1$ in $[0, t_*]$ such that $\theta(t_1) - \theta(t_0) = \pm\pi$ and $\theta(t)$ lies between $\theta(t_0)$ and $\theta(t_1)$ for all $t_0 \leq t \leq t_1$. If $\theta(t_1) - \theta(t_0) = -\pi$, then, essentially, Lemma 9.3 can be applied to obtain $\operatorname{diam} \xi([t_0, t_1]) \geq 2\rho$, and thus the expected assertion follows. Otherwise, we have only to consider the mirror image $\tilde{\xi}$ to deduce the same assertion.

Finally, suppose that (9.5) does not hold. We set $\theta_{\max} = \max_{0 \leq t \leq t_*} \theta(t)$, $t_0 := \min\{t \in [0, t_*]; \theta(t) = \theta_{\max}\}$ and $\Delta' := \{\xi'(t_0)z + \xi(t_0); z \in \Delta\}$.

By Lemma 9.3, the curve $\xi([t_0, t_*])$ does not intersect the disk Δ'. By applying Lemma 9.3 to the mirror image $\tilde{\xi}$, we can show that the curve $\xi([0, t_0])$ does not intersect the disk Δ', too.

Now the curve $\xi([0, t_*]) \cup \{e^{i\theta}; 0 \leq t \leq t_*\}$ should enclose the domain $D := (\Delta')^\circ \setminus \Delta$, and thus $\operatorname{diam} \xi([0, t_*]) \geq \operatorname{diam} D \geq 2\rho$.

Case 2: $0 < \varphi_* < \pi$.

Suppose that $\operatorname{diam} \xi([0, t_*]) < 2\rho$. Let D be the domain bounded by this subarc $\xi([0, t_*])$ and the segment $[0, \xi(t_*)]$. Then we have $\operatorname{diam} D < 2\rho$.

We consider the mirror image $\tilde{\xi}(-t) = -\overline{\xi(t)}$ of ξ. Then the subarc $\tilde{\xi}([0, \tilde{t}_*])$, where $\tilde{t}_*$ is the parameter value determined for $\tilde{\xi}$ in the similar way to t_* above, would be contained in the mirror image of D, and hence $\operatorname{diam} \tilde{\xi}([0, \tilde{t}_*]) \le \operatorname{diam} D < 2\rho$, which is impossible by Case 1.

Now the proof of Proposition 5.4 is complete.

References

1. S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* I, Comm. Pure Appl. Math. **12** (1959), 623–727.
2. H. Aikawa, *Hölder continuity of Dirichlet solution for a general domain*, preprint, 2001.
3. A. Ancona, *On strong barriers and inequality of Hardy for domains in* $\mathbb{R}^n$, J. London Math. Soc. (2) **34** (1986), 274–290.
4. A. Baernstein II, *Integral means, univalent functions and circular symmetrization*, Acta Math. **133** (1974), 139–169.
5. Z. Balogh and A. Volberg, *Boundary Harnack Principle for separated semihyperbolic repellers, harmonic measure applications*, Rev. Math. Iberoamericana **12** (1996), 299–336.
6. A. F. Beardon and Ch. Pommerenke, *The Poincaré metric of plane domains*, J. London Math. Soc. (2) **18** (1978), 475–483.
7. L. Carleson, *Removable singularities of continuous harmonic functions on* $\mathbb{R}^m$, Math. Scand. **12** (1963), 15–18.
8. L. Carleson and T. W. Gamelin, *Complex Dyamics*, Springer-Verlag, 1993.
9. L. Carleson, P. W. Jones, and J.-C. Yoccoz, *Julia and John*, Bol. Soc. Brasil Mat. (N. S.) **25** (1994), 1–30.
10. S. S. Chern, *Curves and surfaces in euclidean space*, Global Differential Geometry (S. S. Chern, ed.), Mathematical Association of America, 1989, pp. 99–139.
11. J. L. Doob, *Classical potential theory and its probabilistic counterpart*, Springer-Verlag, 1984.
12. J. L. Fernández, *Domains with strong barrier*, Rev. Mat. Iberoamericana **5** (1989), 47–65.
13. C. H. Fitzgerald and F. D. Lesley, *Integrability of the derivative of the Riemann mapping function for wedge domains*, J. Analyse Math. **49** (1987), 271–292.
14. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Diffrential Equations of Second Order*, Springer-Verlag, 1977.
15. G. M. Goluzin, *Geometric theory of functions of a complex variable*, American Mathematical Society, Providence, R.I., 1969, Translations of Mathematical Monographs, Vol. 26.
16. L. I. Hedberg, *On Maz'ya's work in potential theory and the theory of function spaces*, The Maz'ya Anniversary Collection, Vol. 1 (J. Rossmann *et al*, ed.), Birkhäuser Verlag, 1999, pp. 7–16.
17. L. L. Helms, *Introduction to Potential Theory*, Wiley-interscience, New York, 1969.
18. J. A. Hempel, *The Poincaré metric on the twice punctured plane and the theorems of Landau and Schottky*, J. London Math. Soc. (2) **20** (1979), 435–445.
19. A. Hinkkanen, *Modulus of continuity of harmonic functions*, J. Analyse Math. **51** (1988), 1–29.
20. R. Howard and A. Treibergs, *A reverse isoperimetric inequality, stability and extremal theorems for plane curves with bounded curvature*, Rocky Mountain J. Math. **25** (1995), 635–684.
21. P. Järvi and M. Vuorinen, *Uniformly perfect sets and quasiregular mappings*, J. London Math. Soc. **54** (1996), 515–529.
22. J. A. Jenkins, *On explicit bounds in Landau's theorem.* II, Canad. J. Math. **33** (1981), 559–562.
23. D. S. Jerison and C. E. Kenig, *Boundary behavior of harmonic funcitons in non-tangentially accessible domains*, Adv. in Math. **46** (1982), 80–147.
24. O. D. Kellogg, *On the derivatives of harmonic functions on the boundary*, Trans. Amer. Math. Soc. **33** (1931), 486–510.
25. F. D. Lesley, *Conformal mappings of domains satisfying a wedge condition*, Proc. Amer. Math. Soc. **93** (1985), 483–488.
26. J. Lithner, *Comparing two versions of Markov's inequality on compact sets*, J. Approx. Theory **77** (1994), 202–211.

27. O. Martio and J. Sarvas, *Density conditions in the n-capacity*, Indiana Univ. Math. J. **26** (1977), 761–776.
28. M. Masumoto, *A distorsion theorem for conformal mappings with an application to subharmonic functions*, Hiroshima Math. J. **20** (1990), 341–350.
29. V. G. Maz'ya, *On the boundary regularity of solutions of elliptic equations and of a conformal mapping.* (Russian), Dokl. Akad. Nauk SSSR **152** (1963), 1297–1300, English translation in Soviet Math. **4** (1963) 1547–1551.
30. ______, *The modulus of continuity of a harmonic function at a boundary point* (Russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **135** (1984), 87–95.
31. K. Miller, *Extremal barriers on cones with Phragmén-Lindelöf theorems and other applications*, Ann. Mat. Pura Appl. **90** (1971), 297–329.
32. Ch. Pommerenke, *Uniformly perfect sets and the Poincaré metric*, Ark. Math. **32** (1979), 192–199.
33. ______, *On uniformly perfect sets and Fuchsian groups*, Analysis **4** (1984), 299–321.
34. Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, 1992.
35. C. Pucci, *Regolaritá alla frontiera di soluzioni di equazioni ellittiche*, Ann. Mat. Pura Appl. **65** (1964), 311–328.
36. E. Schmidt, *Über den Millouxschen Satz*, Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. **25** (1932), 394–401.
37. J. Siciak, *Wiener's type regularity criteria on the complex plane*, Ann. Polon. Math. **66** (1997), 203–221.
38. R. Stankewitz, *Uniformly perfect analytic and conformal attractor sets*, to appear in Bull. London Math. Soc.
39. ______, *Uniformly perfect sets, rational semigroups, Kleinian groups and IFS's*, Proc. Amer. Math. Soc. **128** (2000), 2569–2575.
40. T. Sugawa, *Uniform perfectness of the limit sets of Kleinian groups*, to appear in Trans. Amer. Math. Soc.
41. ______, *Various domain constants related to uniform perfectness*, Complex Variables **36** (1998), 311–345.
42. N. Suzuki, *Nonintegrability of harmonic functions in a domain*, Japan. J. Math. **16** (1990), 269–278.
43. M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.
44. J.-H. Zheng, *Uniformly perfect sets and distortion of holomorphic functions*, to appear in Nagoya Math J.

Department of Mathematics, Kyoto University, 606-8502 Kyoto, Japan

Current address: Department of Mathematics, University of Helsinki, P.O.Box 4 (Yliopistonkatu 5), FIN-00014, Helsinki, Finland

E-mail address: sugawa@kusm.kyoto-u.ac.jp