ON THE HOROCYCLIC COORDINATE FOR THE TEICHMÜLLER SPACE OF ONCE PUNCTURED TORI

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ABSTRACT. This paper gives geometric and analytic properties of the horocyclic coordinate (named by Irwin Kra) for the Teichmüller space of once punctured tori.

Introduction The horocyclic coordinate for the Teichmüller space $T_{1,1}$ of once punctured tori is one of holomorphic realizations of $T_{1,1}$, more precisely, the Maskit slice of once punctured torus groups, in the complex plane $\mathbb{C}$. Classically, it is known that the Teichmüller space (of Riemann surfaces) is contractible. This was first proved by O.Teichmüller in the case of compact Riemann surfaces (cf. [1], [6] and [10] etc.). Especially, the image $\mathcal{M}$ of $T_{1,1}$ for the horocyclic coordinate is simply connected. Thus, the topological property of $\mathcal{M}$ is well-known. Recently, Y.Minsky gives one of geometric properties of $\mathcal{M}$. He showed in his paper [12] that $\mathcal{M}$ is a Jordan domain in the Riemann sphere $\hat{\mathbb{C}}$. One of our aims of this paper is to prove a geometric property of $\mathcal{M}$ which is different from that of Minsky's as follows.

Theorem 1. *The image of the horocyclic coordinate for the Teichmüller space of once punctured tori is not a quasi-disk.*

This is proved in Section 2.5. To prove Theorem 1, we use a Minsky's theorem, called the "Pivot Theorem" (cf. Section 2.3 below). This result is essentially suggested by D.Wright (cf. [16, Section 5]). In this paper, Theorem 1 is showed by the study of the geometry near cusps in $\partial\mathcal{M}$ other than D.Wright's observations (cf. Theorem 2).

This paper is organized as follows: In Section 1, we recall the definition of the horocyclic coordinate for the Teichmüller space of once punctured tori after L.Keen and C.Series [2], and define some notations used in this paper. In Section 2, we treat the main theorem of this paper. Section 3 deals with results related to the cusp opening for the maximally parabolic groups in the boundary of Maskit slice. In Section 4 we give an analytic property of $\mathcal{M}$. Namely, we study the behavior at boundary of $\mathcal{M}$ of elements of the group of automorphism of $\mathcal{M}$.

The author would like to express his hearty gratitude to Professor Yoichi Imayoshi and Professor Hiroshi Yamaguchi for their constant encouragement. He has been greatly helped by conversations with Dr. Yohei Komori. He would like to thank him for his help. He also would like to thank Professor Toshiyuki Sugawa for his various comment for the first version of this paper.

Date: December 9, 1998.
1991 *Mathematics Subject Classification.* 30F40, 30F60, 32G15.

1. Preliminaries

1.1. In this subsection, we recall the definition of the horocyclic coordinate for the Teichmüller space of once punctured tori in accordance with L.Keen and C.Series [2, Section 2] (see also [5, Section 6.3] and [16]).

For $\mu \in \mathbb{C}$, let $S, T_\mu \in \mathrm{SL}_2(\mathbb{C})$ be

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad T_\mu = -\imath \begin{pmatrix} \mu & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $G_\mu = \langle S, T_\mu \rangle$. We shall often identify G_μ with the subgroup of the group of Möbius transformations. We define the domain $\mathcal{M}$ in $\mathbb{C}$ as follows: $\mu \in \mathbb{C}$ is contained in $\mathcal{M}$ if and only if $\mathrm{Im}\mu > 0$, and G_μ is a terminal regular b-group of type (1,1). Namely the following hold:

1. G_μ is a free group on two generators.
2. The connected components of the region of discontinuity of G_μ are of two kinds:
   (a) A simply connected G_μ-invariant component of Ω_μ for which the orbit space $S_\mu = \Omega_\mu / G_\mu$ is topologically conjugate to a once punctured torus.
   (b) Non-invariant components $\Omega_{i,\mu}$, $i \geq 1$, that are conjugate to one another under G_μ and for which each orbit space $\Omega_{i,\mu}/\mathrm{stab}(G_{i,\mu})$ is conformally equivalent to the thrice punctured sphere.

Notice that $\mathcal{M}$ coincides with the D.Wright's picture (cf. [16], [2, p.721] and [8, p.180]). Then $\mathcal{M}$ is a simply connected Jordan domain in $\hat{\mathbb{C}}$ (cf. [12, Section 12]) and contains $\{\imath\, t \in \mathbb{C} \mid t > 2\}$. For $\mu \in \mathcal{M}$, G_μ is obtained by applying the Maskit's second combination theorem for T_μ and the level 2 principal congruence subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ (cf. [5, Section 6]).

Next, we recall the holomorphic bijection between $\mathcal{M}$ and $T_{1,1}$. On S_μ, $\mu \in \mathcal{M}$, let α_μ and β_μ be oriented simple closed Poincaré geodesics corresponding to S and T_μ, respectively (it is known that S is an accidental parabolic transformation of G_μ, see [2, p.723]). The pair (α_μ, β_μ) determines the marking on S_μ, and hence $(S_\mu, (\alpha_\mu, \beta_\mu))$ determines a point of $T_{1,1}$. This correspondence is a holomorphic bijection (cf. [4, p.233] and [2, p.722]). We let $\Sigma = S_{4\imath}$, $\alpha = \alpha_{4\imath}$ and $\beta = \beta_{4\imath}$. The fundamental group $\pi_1(\Sigma)$ of Σ with the suitable base point is generated by α and β. For $\mu \in \mathbb{C}$, we define the homomorphism χ_μ from $\pi_1(\Sigma)$ onto G_μ which sends α and β to S and T_μ, respectively. For $\mu \in \overline{\mathcal{M}} \setminus \{\infty\}$, χ_μ is an isomorphism and G_μ is a Kleinian group (cf. [7, Theorem 2.21]). For $\mu \in \mathcal{M}$, χ_μ is induced from a quasi-conformal mapping from Σ to S_μ which maps α and β to α_μ and β_μ, respectively (cf. [2, Section 2.1]).

1.2. In this subsection, we give an inductive procedure for constructing the element $W_{p/q,\mu} \in G_\mu$ corresponding to the p/q-homotopy class.

We recall the formation of rationals by Faray sequences. A pair of rationals $(p/q, r/s)$ are called *neighbors* if $ps - rq = \pm 1$. All rationals are obtained in a unique way by repeated application of the process $(p/q, r/s) \mapsto (p+r)/(q+s)$ to Faray neighbors starting with integer neighbors $(n/1, (n+1)/1)$. Note that if $p/q < r/s$ and if $(p/q, r/s)$ are neighbors then $p/q < (p+r)/(q+s) < r/s$ and both pairs $(p/q, (p+r)/(q+s))$, $((p+r)/(q+s), r/s)$ are again neighbors.

The element $\gamma(p/q) \in \pi_1(\Sigma)$ is defined inductively as follows: if $n \in \mathbb{Z}$, then $\gamma(n/1) = \alpha^{-n}\beta$ and if $(p/q, r/s)$ are neighbors with $p/q < r/s$, then

$$\gamma((p+r)/(q+s)) = \gamma(r/s)\gamma(p/q).$$

Then we can see that $[\gamma(p/q)] = -p[\alpha] + q[\beta]$, where for $\omega \in \pi_1(\Sigma)$, $[\omega]$ is the homology class of a loop in ω (cf. [2, Section 2.4], see also [12, Section 2]). We let for $\mu \in \mathbb{C}$, $W_{p/q,\mu} = \chi_\mu(\gamma(p/q))$ and denote by $\gamma(p/q, \mu)$ the simple closed geodesic on S_μ associated with $W_{p/q,\mu}$. The trace $\mathrm{tr}(W_{p/q,\mu})$ of $W_{p/q,\mu}$ is a polynomial of the form (cf. [2, Proposition 3.1]):

$$\mathrm{tr}(W_{p/q,\mu}) = (-\imath)^q(\mu^q - 2p\mu^{q-1} + b_{q-2}\mu^{q-2} + \cdots + b_0), \quad b_i \in \mathbb{Z}.$$

For example, $\mathrm{tr}(W_{n/1,\mu}) = -\imath(\mu - 2n)$ for $n \in \mathbb{Z}$.

2. The main theorem

2.1. For $p, q \in \mathbb{Z}$ with $(p, q) = 1$ and $q \neq 0$, we denote by $\mu(p/q)$ the end point of the p/q-pleating ray in $\partial\mathcal{M} \setminus \{\infty\}$. For example, $\mu(n/1) = 2n + 2\imath$ for $n \in \mathbb{Z}$. We know that $\mathrm{tr}(W_{p/q,\mu(p/q)})^2 = 4$ and $\{\mu(p/q)\}_{p/q\in\mathbb{Q}}$ is dense in $\partial\mathcal{M} \setminus \{\infty\}$ (for more details, [2, Section 5]). By Theorem 5.1 in [2], we have the following.

Lemma 1. *Unless $p/q = r/s$, $\mu(p/q) \neq \mu(r/s)$ and $W_{p/q,\mu(r/s)}$ is not parabolic.*

2.2. For $p, q \in \mathbb{Z}$ with $(p, q) = 1$ and $q \neq 0$, let $f_{p/q}(\mu) = \mathrm{tr}(W_{p/q,\mu})^2 - 4$. To prove Theorem 1, we shall show the following lemma.

Lemma 2. *There exist a simply connected Jordan domain $\widetilde{\mathcal{M}}$ in $\hat{\mathbb{C}}$ and a holomorphic function $\lambda_{p,q}$ on an open set which contains $\widetilde{\mathcal{M}} \cup \{0\}$ such that*

(a) $0 \in \partial\widetilde{\mathcal{M}}$, $\lambda_{p,q}(0) = 0$, *and* $\mathrm{Re}\{\lambda_{p,q}(t)\} > 0$ *for* $t \in \widetilde{\mathcal{M}}$.

(b) *If* $df_{p/q}/d\mu(\mu(p/q)) \neq 0$ *then* $d\lambda_{p,q}/dt(0) \neq 0$.

(c) *Let* $\rho_1(t) = \mu(p/q) + t^2$ *and* $\rho_2(\zeta) = \zeta^2$. *Then* $\rho_1 \mid_{\widetilde{\mathcal{M}}}$ *is a biholomorphic mapping from* $\widetilde{\mathcal{M}}$ *to* $\mathcal{M}$, *and makes the following diagram commutative:*

$$\begin{tikzcd} \widetilde{\mathcal{M}} \arrow[r, "2\sinh(\frac{\lambda_{p,q}(-)}{2})"] \arrow[d, "\rho_1"'] & \mathbb{C} \arrow[d, "\rho_2"] \\ \mathcal{M} \arrow[r, "f_{p/q}(-)"'] & \mathbb{C} \end{tikzcd}$$

Proof. Since the function $z \mapsto \sinh^2(z)$ $(= \rho_2(\sinh(z)))$ is an even function on $\mathbb{C}$ and $f_{p/q}(\mu(p/q)) = 0$, there exist a neighborhood U_1 of $\mu(p/q)$ and U_2 of $0 \in \mathbb{C}$ and a holomorphic function λ_0 on U_2 such that $\lambda_0(0) = 0$ and that $f_{p/q}(\rho_1(t)) = \rho_2(2\sinh(\lambda_0(t)/2))$ for $t \in U_2$. Since $\mathcal{M}$ is simply connected, and $\mu(p/q) \in \partial\mathcal{M}$, there exists a simply connected domain $\widetilde{\mathcal{M}} \subset \mathbb{C}$ such that $\rho_1 \mid_{\widetilde{\mathcal{M}}}$ is a conformal mapping on $\widetilde{\mathcal{M}}$ onto $\mathcal{M}$. Clearly $0 \in \partial\widetilde{\mathcal{M}}$ and ρ_1 is extended to a homeomorphism from $\overline{\widetilde{\mathcal{M}}}$ onto $\overline{\mathcal{M}}$ with $\rho_1(0) = \mu(p/q)$ and $\rho_1(\infty) = \infty$.

By virtue of Lemma 1, for $\mu \in \overline{\mathcal{M}} \setminus \{\mu(p/q), \infty\}$, $W_{p/q,\mu}$ is either loxodromic or hyperbolic. Hence for $t_1 \in \overline{\widetilde{\mathcal{M}}} \setminus \{0, \infty\}$, there exist a neighborhood U_{t_1} of t_1 and a holomorphic function λ_{t_1} on U_{t_1} such that $f_{p/q}(\rho_1(t)) = \rho_2(2\sinh(\lambda_{t_1}(t)/2))$ for $t \in U_{t_1}$. Since $\mathcal{M}$ is a Jordan domain, so $\widetilde{\mathcal{M}}$ is. By analytic continuation, we find a holomorphic function $\lambda_{p,q}$ on an open set in $\mathbb{C}$ which contains $\widetilde{\mathcal{M}} \cup \{0\}$ such that the diagram above commutes. Since $\mathrm{Re}\{\lambda_{p,q}(t)\} \neq 0$ on $\widetilde{\mathcal{M}}$, it can be taken $\lambda_{p,q}$ so that $\mathrm{Re}\{\lambda_{p,q}(t)\} > 0$ on $\widetilde{\mathcal{M}}$, and hence assertions (a) and (c) are obtained. Finally, (c) implies $\{d\lambda_{p,q}/dt(0)\}^2 = df_{p/q}/d\mu(\mu(p/q))$. Thus, (b) also holds. $\square$

Remark. In general, for a holomorphic function φ from a domain in $\mathbb{C}$ to $\mathrm{SL}_2(\mathbb{C})$, if a value of φ for some point is parabolic, $(\mathrm{tr}(\varphi))^2$ can not be of the form $4\cosh^2(\lambda(t)/2)$ by using some holomorphic function λ on the domain. For instance,

$$\varphi(t) := \begin{pmatrix} \cosh(\sqrt{t}) & \sqrt{t}\sinh(\sqrt{t}) \\ \sinh(\sqrt{t})/\sqrt{t} & \cosh(\sqrt{t}) \end{pmatrix}, \quad \text{for} \quad |t| < 1.$$

2.3. In this subsection, we recall Minsky's theorem, called the "Pivot theorem" (cf. [12, Theorem 4.1]). In this paper, we use the special case of his theorem stated as follows: Let $p, q \in \mathbb{Z}$ be as in the previous subsection. Take $(r, s) \in \mathbb{Z}$ such that a pair $(-p[\alpha] + q[\beta], r[\alpha] + s[\beta])$ is a positively oriented basis of $H_1(\Sigma)$. Then there exist $\nu_+(\mu) \in \mathbb{H}$ and $m, n \in \mathbb{Z}$ with $(m, n) = 1$ and $n \neq 0$ such that $[\alpha] = m(-p[\alpha] + q[\beta]) + n(r[\alpha] + s[\beta])$ and that the following property holds: Let , $= \langle z \mapsto z + 1, z \mapsto z + \nu_+(\mu)\rangle$, $D = \mathbb{C} \setminus$, $((1 + \nu_+(\mu))/2)$, and π be a projection from D onto $D/$, . There exists the conformal mapping h from $D/$, to S_μ such that $h(\pi([0, 1]))$ and $h(\pi([0, \nu_+(\mu)]))$ is freely homotopic to $\gamma(p/q, \mu)$ and $\gamma(-r/s, \mu)$ respectively.

It can be observed that $\nu_+(\mu) + (m/n)$ is not depend on the choice of (r, s) (for more details, see [12, Section 4]). Under notation above, Minsky proved the following.

Theorem (Pivot Theorem) *There exist universal constants $\epsilon_0, C_1 > 0$ such that if $0 < \mathrm{Re}(\lambda_{p/q}(\rho_1^{-1}(\mu))) < \epsilon_0$ and if $|\mathrm{Im}(\lambda_{p/q}(\rho_1^{-1}(\mu)))| < \pi$, then*

$$d_{\mathbb{H}}\left(\frac{2\pi\imath}{\lambda_{p/q}(\rho_1^{-1}(\mu))}, \nu_+(\mu) + (m/n) + \imath\right) < C_1.$$

where $d_{\mathbb{H}}(\cdot,\cdot)$ is the Poincaré hyperbolic metric on $\mathbb{H}$.

2.4. We know that ν_+ is a conformal mapping from $\mathcal{M}$ onto $\mathbb{H}$ (cf. [12, Section 2.3]). Since $\mathcal{M}$ is a Jordan domain, $\Phi := \nu_+^{-1}$ can be extended continuously to $\overline{\mathbb{H}}$ onto $\overline{\mathcal{M}}$. For $s \in \mathbb{R}$, let $\check{\sigma}(s) = \Phi(s - (m/n) + \imath)$ and $\sigma(s) = (\rho_1)^{-1}(\check{\sigma}(s))$.

Proposition 3. *The following hold:*
(a) *$\Phi(\infty) = \mu(p/q)$.*
(b) *There exist disks B_1 and B_2 in $\mathbb{C}$ such that*
  (i) *$0 \in \partial B_j$ and the center of B_j is contained in the real axis for $j = 1, 2$, and $B_1 \cap B_2 = \emptyset$.*
  (ii) *Let $\xi = \{2\sinh(\lambda(\sigma(s))/2) \mid s \in \mathbb{R}\}$. Then $\xi \cap B_j = \emptyset$ for $j = 1, 2$.*
  (iii) *The mapping $s \mapsto 2\sinh(\lambda(\sigma(s))/2)$ tends to 0 from both sides of $\mathbb{C} \setminus B_1 \cup B_2$ as $s \to +\infty$ and $-\infty$ (cf. Figure 1).*

Proof. (a) By Lemma 1, it suffice to show that $\Phi(\infty) \neq \infty$ and that $W_{p/q,\Phi(\infty)}$ is parabolic.

We first prove $\Phi(\infty) \neq \infty$ by contradiction. Assume that $\Phi(\infty) = \infty$. We know that for $\mu \in \mathcal{M}$ with $\mathrm{Im}\mu > 2$, S_μ contains the annulus A_μ with the modulus $(\mathrm{Im}\mu - 2)/2$ such that the central curve of A_μ is freely homotopic to α_μ. (cf. [5, Section 6.3 and Figure 8(with $r = 1$)]). Since $q \neq 0$, the geometric intersection

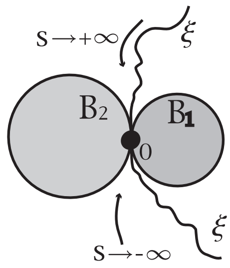

FIGURE 1

number between α_μ and $\gamma(p/q,\mu)$ is not zero. Hence, by the assumption, the extremal length of $\gamma(p/q,\Phi(\nu))$ on $S_{\Phi(\nu)}$ tends to ∞ as $\mathrm{Im}\nu\to+\infty$. On the other hand, by the definition of ν_+, the extremal length of $\gamma(p/q,\Phi(\nu))$ on S_μ is equal to $1/\mathrm{Im}\nu_+(\Phi(\nu))=1/\mathrm{Im}\nu$, which is absord.

By the argument above, $\chi_{\Phi(\infty)}$ is an isomorphism. By Corollary A.6 in [11, p.216], $\mathrm{tr}(W_{p/q,\Phi(\nu)})^2$ tends to 4 as $\mathrm{Im}\nu\to\infty$. Therefore, $W_{p/q,\Phi(\infty)}$ is parabolic.

(b) Since $\Phi(\infty)=\mu(p/q)$, $\sigma(s)\to 0$ as $s\to\pm\infty$. Since $\lambda_{p/q}(0)=0$, there exists $s_0>0$ such that for $|s|>s_0$, $0<\mathrm{Re}\lambda(\sigma(s))<\epsilon_0$ and $|\mathrm{Im}\lambda_{p/q}(\sigma(s))|<\pi$. Since $\nu_+(\rho_1(\sigma(s)))=s-m/n+\imath$, by virtue of the Pivot theorem,

$$d_{\mathbb{H}}\left(\frac{2\pi\imath}{\lambda_{p/q}(\sigma(s))},s+2\imath\right)<C_1. \tag{1}$$

for $|s|>s_0$. Hence there exists $C>0$ such that

$$\frac{1}{C}<\mathrm{Im}\left(\frac{2\pi\imath}{\lambda(\sigma(s))}\right)<C \tag{2}$$

whenever $|s|>s_0$. Since $\mathrm{Re}\lambda_{p/q}(t)>0$ for $t\in\widetilde{\mathcal{M}}$, there exists $C'>0$ such that $|2\sinh(\lambda_{p/2}(\sigma(s))/2)|>C'$ whenever $|s|\le s_0$. Since the map $z\mapsto\sinh(z)$ is conformal at $z=0$, by the equation (2), we obtain the disks B_j, $j=1,2$ which satisfy (i) and (ii). By construction, for $j=1,2$, the center of B_j is contained in the real axis. By (1), $\mathrm{Re}(2\pi\imath/\lambda_{p/q}(\sigma(s)))\to+\infty$ (resp. $-\infty$) as $s\to+\infty$ (resp. $-\infty$). This implies the assertion (iii). □

2.5. In this subsection, we prove Theorem 1. Let $u_{p/q}=df_{p/q}/d\mu(\mu(p/q))$. First, we show the following lemma.

Lemma 4. *Let p and q be as in the previous subsection. If $u_{p/q}\neq 0$, then there exist a neighborhood U of 0 and disks B_j, $j=1,2$ such that*

(a) *$B_j\subset U$, $0\in\partial B_j$ for $j=1,2$ and $B_1\cap B_2=\emptyset$.*
(b) *$B_1\subset\widetilde{\mathcal{M}}$ and $B_2\cap(\overline{\widetilde{\mathcal{M}}}\cap U)=\emptyset$.*
(c) *For $j=1,2$, the center of B_j lies on $\ell(p/q):=\{t(\overline{u_{p/q}})^{1/2}\mid t\in\mathbb{R}\}$.*

Proof. Let U be a neighborhood of 0 such that $|\mathrm{Im}\lambda_{p/q}(t)|<\pi$ for $t\in U$. Let $h(t)=2\sinh(\lambda_{p/q}(t)/2)$. Then $h(U\cap\widetilde{\mathcal{M}})\subset\{\mathrm{Re}\zeta>0\}$ and $h(0)=0$. Since $u_{p/q}\neq 0$, by (b) of Lemma 2, we may suppose that h is univalent on U. Hence there exists a disk B_2 in U so that $0\in\partial B_2$ and that $B_2\cap(\overline{\widetilde{\mathcal{M}}}\cap U)=\emptyset$. Since $h'(0)^2=(\lambda'_{p/q}(0))^2=u_{p/q}$, the center of B_2 lies on $\ell(p/q)$.

By virtue of (i) and (ii) in Proposition 3, there exist disks $\{B'_j\}_{j=1,2}$ such that $0 \in \partial B'_j$ and that these are disjoint from the curve $\hat{\xi}' := \{\sigma(s) \mid s \in \mathbb{R}\}$. Since σ is injective and $\sigma(s) \to 0$ as $|s| \to \infty$, $\hat{\xi} := \hat{\xi}' \cup \{0\}$ is a Jordan curve in $\mathbb{C}$. Since $\hat{\xi} \subset \widetilde{\mathcal{M}} \cup \{0\}$ and h is univalent on U, by (iii) of Proposition 3, either B'_1 and B'_2 is contained in $\widetilde{\mathcal{M}}$. Hence there exists a disk B_1 such that $0 \in \partial B_1$ and $B_1 \subset \widetilde{\mathcal{M}} \cap U$. By the definition of B_2, $B_1 \cap B_2 = \emptyset$ and B_1 is tangent to B_2 at origin. Hence the center of B_1 lies on $\ell(p/q)$. By the above, we conclude the assertion. □

Notice that $u_{n/1} = -4\imath \neq 0$. Hence, to prove Theorem 1, if suffice to show the following.

Theorem 2. *If $u_{p/q} \neq 0$, then $\mu(p/q)$ is an inward-pointing cusp of $\mathcal{M}$.*

Here, in this paper, we say that for a domain $E \subset \mathbb{C}$, $e_0 \in \partial E$ is an inward-pointing cusp at $e_0 \in \partial E$ if there exists a disk $B \subset \mathbb{C}$ such that $0 \in \partial B$ and that $e_0 + t^2 \in E$ for $t \in B$ (cf. [13, p.51] and Figure 2).

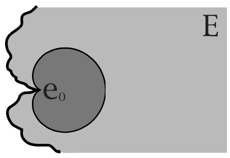

FIGURE 2. An inward-pointing cusp

Proof. By Lemma 4, there exists $\delta \in \mathbb{R}$ so that $B_1 = \{|t - \delta(\overline{u_{p/q}})^{1/2}| < |\delta||u_{p/q}|^{1/2}\}$ and that $B_1 \subset \widetilde{\mathcal{M}}$. Hence $\mathcal{M}$ contains

$$\begin{aligned} \rho_1(B_1) &= \{\mu(p/q) + t^2 \mid |t - \delta(\overline{u_{p/q}})^{1/2}| < |\delta||u_{p/q}|^{1/2}\} \\ &= \{\mu(p/q) + \overline{u_{p/q}}t^2 \mid |t - \delta| < |\delta|\}. \qquad \square \end{aligned}$$

Remark. 1. The idea of the proof of Theorem 2 can be applied to the study of the image of the Bers embedding of $T_{1,1}$.

2. By Corollary 1 in [5, p.558], we can immediately observe that the image of the horocyclic coordinate for the Teichmüller space of four times punctured spheres is also not a quasi-disk (for the definition, see [5, Section 6.1]).

3. The cusp opening

In this section, we study the behavior of the cusp opening for the maximally parabolic groups in the boundary of Maskit slice. (for the definition, see [3, Definition 3.2]). For $p, q \in \mathbb{Z}$ such that $(p, q) = 1$, $q \neq 0$, let $v_{p/q} := -\overline{u_{p/q}}/|u_{p/q}|$. For example, $v_{n/1} = -\imath$ for $n \in \mathbb{Z}$. Recall that $G_{\mu(p/q)}$ is a maximal parabolic group with accidental parabolic transformations $W_{p/q,\mu(p/q)}$ and S.

The following is an immediate consequence of the proof of Theorem 2.

Theorem 3. *Let $p, q \in \mathbb{Z}$ as above. For $v \in \{v \in \mathbb{C} \mid |v| = 1\} \setminus \{v_{p/q}\}$, there exists $d_0 = d_0(p/q, v) > 0$ such that $\mu(p/q) + dv \in \mathcal{M}$ if $0 < d < d_0$, that is, for $0 < d < d_0$, $G_{\mu(p/q)+dv}$ is a terminal regular b-group with an accidental parabolic transformation S.*

Remark. The author makes the figures in Figure 3 by using the computer program "OPTi 3.0" produced by Professor Masaaki Wada [15].

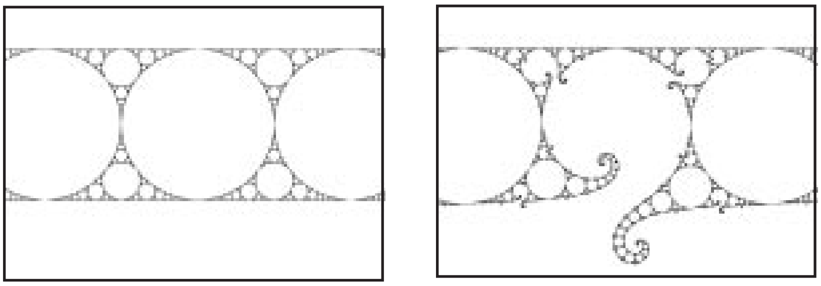

FIGURE 3. The cusp opening

4. THE BOUDARY BEHAVIOR OF THE AUTOMORPHISM OF $\mathcal{M}$

In this section, we study the behavior at boundary of $\mathcal{M}$ of elements of the group of automorphism of $\mathcal{M}$.

Lemma 5. *Let D be a simply connected Jordan domain so that $0 \in \partial D$ and that $\{w \in \mathbb{C} \mid |w - \imath r| < r\} \subset D$ and $\overline{D} \cap \{w \in \mathbb{C} \mid |w + \imath r| < r\} = \emptyset$ for some $r > 0$. Let h be a conformal mapping from D to $\mathbb{H}$ so that $h(0) = 0$. Then for any $0 < \alpha < \pi/2$, the image of $S = \{w \in \mathbb{C} \mid |\arg(w/\imath)| < \alpha, |w| < r\cos\alpha\}$ under h is contained in some Stolz domain in $\mathbb{H}$ whose vertex is at $z = 0$.*

Proof. Let $g = h^{-1}$. By Theorem IX.9 in [14, p.366], there exists $A > 0$ such that $g(z) = Az + o(|z|)$ where z tends to zero inside of any fixed Stolz domain in $\mathbb{H}$ whose vertex is at $z = 0$. Hence there exists $\delta < r\cos\alpha/2A$ so that $|g(\imath y) - \imath Ay| < A\sin\alpha y$, if $0 < y < \delta$. Especially, $g(\imath y) \in S$ for $0 < y < \delta$. Since $\{w \in \mathbb{C} \mid |w - \imath r| < r\} \subset D$, there exists $C > 0$ such that S is contained in $S'' := \cup_{y>0}\{w \in D \mid d_D(g(\imath y), w) < C\}$ where d_D is the hyperbolic distance of D. Let $\hat{S} = \cup_{y>0}\{z \in \mathbb{H} \mid d_{\mathbb{H}}(\imath y, z) < C\}$. Then $\hat{S}$ is a Stolz domain whose vertex is at $z = 0$ and satisfies $h(S) \subset \hat{S}$. In fact, let $w \in S$. Since $S \subset S''$, there exists $y > 0$ so that $d_D(g(\imath y), w) < C$. Hence $d_{\mathbb{H}}(\imath y, h(w)) = d_D(g(\imath y), w) < C$. This implies that $h(w) \in \hat{S}$. $\square$

For $p/q \in \mathbb{Q}$ with $u_{p/q} \neq 0$, $\alpha > 0$ and $\delta > 0$, let

$$S_{p/q}(\alpha, \delta) = \{\mu \in \mathbb{C} \mid |\arg((\mu - \mu(p/q))/\overline{u_{p/q}})| < \alpha, 0 < |\mu - \mu(p/q)| < \delta\}.$$

Notice that by Theorem 3, for $\alpha < \pi$ and a sufficiently small δ, $S_{p/q}(\alpha, \delta) \subset \mathcal{M}$.

Theorem 4. *Let p/q and $r/s \in \mathbb{Q}$ so that $u_{p/q}, u_{r/s} \neq 0$. Then every automorphism τ of $\mathcal{M}$ with $\tau(\mu(p/q)) = \mu(r/s)$ is conformal at $\mu = \mu(p/q)$ in the following sense: There exists $A > 0$ such that*

$$\tau(\mu) = \mu(r/s) + A\left(\frac{\overline{u_{r/s}}}{\overline{u_{p/q}}}\right)(\mu - \mu(p/q)) + o(|\mu - \mu(p/q)|),$$

where $\mu \to \mu(p/q)$ inside of $S_{p/q}(\alpha, \delta)$ for a sufficiently small δ and a fixed $\alpha < \pi$.

Proof. By Lemma 2, for $x = p/q, r/s$, there exists a simply connected Jordan domain $\widetilde{\mathcal{M}}_x$ such that $0 \in \partial\widetilde{\mathcal{M}}_x$ and that $\rho_x(t) := \mu(x) + t^2$ is a conformal mapping from $\widetilde{\mathcal{M}}_x$ to $\mathcal{M}$. Let $\widetilde{\tau} = \rho_{r/s}^{-1} \circ \tau \circ \rho_{p/q}$. Then for $x = p/q$ and r/s, there exists a conformal mapping h_x from $\mathbb{H}$ to $\widetilde{\mathcal{M}}_x$ such that $h_x(0) = 0$ and that $h_{r/s} \circ h_{p/q}^{-1} = \widetilde{\tau}$. Take a neighborhood U_x of 0 and disks $B_{j,x}$, $j = 1, 2$ which satisfy (a), (b) and

(c) in Lemma 4 for $x = p/q$ and r/s. Fix the branch of $(\overline{u_x})^{1/2}$ so that the center of $B_{1,x}$ is of the form $r_x(\overline{u_x})^{1/2}$ for some $r_x > 0$. Then, by Theorem IX.9 in [14, p.366], there exists $\alpha_x > 0$ such that

$$h_x(z) = (\alpha_x \imath / \overline{u_x}) z + o(|z|), \tag{3}$$

where z tends to 0 inside of any fixed Stolz domain whose vertex is at $z = 0$.

Let $0 < \alpha < \pi$ and $0 < \delta < r_{p/q}^2 |u_{p/q}| \cos^2(\alpha/2)$. Then $\widetilde{\mathcal{M}}_{p/q}$ contains $S' = \{t \in \mathbb{C} \mid |\arg(t/(\overline{u_x})^{1/2})| < \alpha/2, |w| < \delta^{1/2}\}$. By Lemma 5, $h_x^{-1}(S')$ is contained in some Stolz domain in $\mathbb{H}$ whose vertex is at $z = 0$. Hence, by (3),

$$\widetilde{\tau}(t) = A^{1/2}(\overline{u_{r/s}}^{1/2}/\overline{u_{p/q}}^{1/2})t + o(|t|),$$

where $t \to 0$ inside of S' and $A = (\alpha_{r/s}/\alpha_{p/q})^2$. Since $\rho_{r/s} \circ \widetilde{\tau} = \tau \circ \rho_{p/q}$ and $\rho_{p/q}(S') = S_{p/q}(\alpha, \delta)$, we conclude the assertion. □

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