The Norm Estimates of Pre-Schwarzian Derivatives of Spiral-like Functions

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Abstract

For a constant $\beta \in (-\pi/2, \pi/2)$, a normalized analytic function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ on the unit disk is said to be β-spiral-like if $\Re(e^{-i\beta} z f'(z)/f(z)) > 0$ for any point z in the unit disk. In this paper, for such a function f, we shall present the optimal estimate of the norm of f''/f'.

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1 Introduction

Let A denote the set of analytic functions f on the unit disk $\mathbb{D}$ normalized so that $f(0) = f'(0) - 1 = 0$. For a constant $\beta \in (-\pi/2, \pi/2)$, a function $f \in A$ is called *β-spiral-like* if f is univalent on $\mathbb{D}$ and for any $z \in \mathbb{D}$, the β-logarithmic spiral $\{f(z) \exp(-e^{i\beta} t); t \geq 0\}$ is contained in $f(\mathbb{D})$. It is equivalent to the condition that $\Re(e^{-i\beta} z f'(z)/f(z)) > 0$ in $\mathbb{D}$. We denote by $SP(\beta)$ the set of β-spiral-like functions. We call $f_\beta(z) := z(1-z)^{-2e^{i\beta}\cos\beta} \in SP(\beta)$ the β-spiral Koebe function. Note that $SP(0)$ is the set of starlike functions and that $f_0(z) = z(1-z)^{-2}$ is the Koebe function. The β-spiral Koebe function conformally maps the unit disk onto the complement of the β-logarithmic spiral $\{f_\beta(-e^{-2i\beta}) \exp(-e^{i\beta} t); t \leq 0\}$ in $\mathbb{C}$. For the known results about these classes of the functions, see, for example, [1].

For a locally univalent holomorphic function f, we define

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = (T_f)' - \frac{1}{2}(T_f)^2,$$

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which are said to be the *pre-Schwarzian derivative* (or nonlinearity) and the *Schwarzian derivative* of f, respectively. For a locally univalent function f in $\mathbb{D}$, we define the norms of T_f and S_f by

$$\|T_f\|_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)|T_f(z)| \quad \text{and} \quad \|S_f\|_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_f(z)|,$$

respectively.

As well as $\|S_f\|_2$, the norm $\|T_f\|_1$ has a significant meaning in the theory of Teichmüller spaces. For example, see [9], [3] and [14]. On the other hand, there is a deep relation between the boundary of the universal Teichmüller space and such selfsimilar quasiarcs as logarithmic spirals ([2], [4], and [7]).

In the present paper, we shall give the best possible estimate of the norms of pre-Schwarzian derivatives for the class $SP(\beta)$.

Main Theorem 1. *For any $f \in SP(\beta)$, where $\beta \in (-\pi/2, \pi/2)$, we have the following.*

I) *In the case $|\beta| \le \pi/3$, we have*

$$\|T_f\|_1 \le \|T_{f_\beta}\|_1 = 2|2+e^{2i\beta}|. \tag{1}$$

II) *In the case $|\beta| > \pi/3$, we have $\|T_f\|_1 \le \|T_{f_\beta}\|_1$, where*

$$\|T_{f_\beta}\|_1 = \max_{0\le m\le \frac{4}{3}\sin|\beta|} 2m\cos\beta\left(1+\sqrt{\frac{m^2+4-4m\sin|\beta|}{m^2+1-2m\sin|\beta|}}\right) \quad \textit{and} \tag{2}$$

$$2|2+e^{2i\beta}| < \|T_{f_\beta}\|_1 < 2\left(1+\frac{4}{3}\sin 2|\beta|\right). \tag{3}$$

In particular, $\|T_{f_\beta}\|_1 \to 2$ as $|\beta| \to \pi/2$.

In both cases, the equality $\|T_f\|_1 = \|T_{f_\beta}\|_1$ holds if and only if f is a rotation of the β-spiral Koebe function, i.e., $f(z) = (1/\varepsilon)f_\beta(\varepsilon z)$ for some $|\varepsilon| = 1$.

From the proof, if $|\beta| \le \pi/3$, the function $(1-|z|^2)|T_{f_\beta}(z)|$ does not attain its supremum in $\mathbb{D}$. However if $|\beta| > \pi/3$, it does since

$$\max_{\partial\mathbb{D}\ni z_0}\limsup_{\mathbb{D}\ni z\to z_0}(1-|z|^2)|T_{f_\beta}(z)| = 2|2+e^{2i\beta}| < \|T_{f_\beta}\|_1.$$

This phenomenon of *phase transition* seems to be quite interesting.

Remark. Clearly, the β-spiral Koebe function f_β converges to $id_{\mathbb{D}}$ (which is bounded) locally uniformly on $\mathbb{D}$ as $|\beta| \to \pi/2$ but does not converge to it with respect to the norm $\|\cdot\|_1$ since $\lim_{|\beta|\to\pi/2}\|T_{f_\beta}\|_1 = 2$. On the other hand, it is known that $f \in A$ is bounded if $\|T_f\|_1 < 2$. Thus the value 2 is the least one of the norms of unbounded $f \in A$.

We would also like to mention the related works about norm estimates of pre-Schwarzian derivatives in other classes of A by Shinji Yamashita [12] and Toshiyuki Sugawa [10].

Theorem 1.1. *Let* $0 \leq \alpha < 1$ *and* $f \in A$.

If f *is starlike of order* α, *i.e.,* $\Re(zf'(z)/f(z)) > \alpha$, *then* $\|T_f\|_1 \leq 6-4\alpha$.

If f *is convex of order* α, *i.e.,* $\Re(1+zf''(z)/f'(z)) > \alpha$, *then* $\|T_f\|_1 \leq 4(1-\alpha)$.

If f *is strongly starlike of order* α, *i.e.,* $\arg(zf'(z)/f(z)) < \pi\alpha/2$, *then* $\|T_f\|_1 \leq M(\alpha)+2\alpha$, *where* $M(\alpha)$ *is a specified constant depending only on* α *satisfying* $2\alpha < M(\alpha) < 2\alpha(1+\alpha)$.

All of the bounds are sharp.

On the other hand, we also obtain the estimate of the norms of Schwarzian derivatives of β-spiral-like functions.

Main Theorem 2. *Assume* $|\beta| < \pi/2$. *For any* $f \in SP(\beta)$, $\|S_f\|_2 \leq \|S_{f_\beta}\|_2 = 6$.

In Theorem 2, the extremality of f_β is trivial since the Kraus–Nehari theorem says that $\|S_f\|_2 \leq 6$ for any univalent $f \in A$.

The proof of Theorems 1 and 2 will be given in Section 2 and 3. Knowing the norm $\|T_f\|_1$ of $f \in A$ enables us to estimate the growth of coefficients of f. For example, the following holds.

Theorem 1.2 (cf. [8]). *Let* $(3/2) < \lambda \leq 3$. *For* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in A$ *such that* $\|T_f\|_1 \leq 2\lambda$, *it holds that* $a_n = O(n^{\lambda-2})$ *as* $n \to +\infty$. *This order estimate is best possible.*

We shall also remark on the sharp order estimate of coefficients of $f \in SP(\beta)$ in Section 4.

2 Proof of Theorem 1

Let $f \in A$ be a β-spiral-like function. We set $p(z) = P_f(z) = zf'(z)/f(z)$. Then, by assumption, p is a holomorphic function on $\mathbb{D}$ satisfying $p(0) = 1$ and $p(\mathbb{D}) \subset \{w \in \mathbb{C}; -\frac{\pi}{2}+\beta < \arg w < \frac{\pi}{2}+\beta\} =: \mathbb{H}_\beta$. The univalent map $q(z) = (1+ze^{2i\beta})/(1-z)$ on $\mathbb{D}$ satisfies $q(0) = 1$ and $q(\mathbb{D}) = \mathbb{H}_\beta$. Then p is subordinate to q, i.e., there exists a holomorphic function $\omega = \omega_f : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that

$$p = q \circ \omega = \frac{1+\omega e^{2i\beta}}{1-\omega}. \tag{4}$$

We note that, for $|\varepsilon| = 1$, $f(z) = (1/\varepsilon) f_\beta(\varepsilon z)$ if and only if $\omega(z) = \varepsilon z$.

By the logarithmic differentiation of (4), we have

$$\begin{aligned} T_f(z) =& \frac{f''(z)}{f'(z)} = c\frac{\frac{\omega}{z}(1+\omega e^{2i\beta}) + \omega'}{(1-\omega)(1+\omega e^{2i\beta})}, \text{ thus} \\ (1-|z|^2)T_f(z) =& c\frac{(1-|z|^2)\{\frac{\omega}{z}(2+\omega e^{2i\beta}) + (\omega' - \frac{\omega}{z})\}}{(1-\omega)(1+\omega e^{2i\beta})}. \end{aligned}$$

Here we set $c := e^{2i\beta} + 1$. Setting $\omega = id_{\mathbb{D}}$, we also have

$$T_{f_\beta}(z) = c\frac{2+ze^{2i\beta}}{(1-z)(1+ze^{2i\beta})} \text{ and} \tag{5}$$

$$(1-|z|^2)T_{f_\beta}(z) = c\frac{(1-|z|^2)(2+ze^{2i\beta})}{(1-z)(1+ze^{2i\beta})}. \tag{6}$$

We can easily see that $\max_{\partial\mathbb{D}\ni z_0} \limsup_{\mathbb{D}\ni z\to z_0}(1-|z|^2)|T_{f_\beta}(z)| = 2|2+e^{2i\beta}|$.

By the Schwarz-Pick lemma for ω/z, we obtain $(1-|z|^2)|z\omega' - \omega| \leq |z|^2 - |\omega|^2$. So we can estimate as

$$\begin{aligned} (1-|z|^2)|T_f(z)| \leq& \frac{|\omega|(1-|z|^2)}{|z|(1-|\omega|^2)} \cdot |c|\frac{(1-|\omega|^2)|2+\omega e^{2i\beta}|}{|1-\omega||1+\omega e^{2i\beta}|} \\ &+ \frac{|z|^2-|\omega|^2}{|z|(1-|\omega|^2)} \cdot |c|\frac{1-|\omega|^2}{|1-\omega||1+\omega e^{2i\beta}|} \\ =& \frac{|2+\omega e^{2i\beta}||\omega|(1-|z|^2) + (|z|^2-|\omega|^2)}{|2+\omega e^{2i\beta}||z|(1-|\omega|^2)}(1-|\omega|^2)|T_{f_\beta}(\omega)|. \end{aligned}$$

To show $\|T_f\|_1 < \|T_{f_\beta}\|_1$ for $SP(\beta) \ni f$ with $|\omega'(0)| < 1$, we show the following.

Lemma 2.1. *Let $\omega : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $\omega(0) = 0$ and $|\omega'(0)| < 1$. For any set $U \subset \mathbb{D}$ with $\overline{\omega(U)} \not\ni -e^{-2i\beta}$, there exists a positive constant $C < 1$ such that*

$$D(z) := \frac{|2+\omega e^{2i\beta}||\omega|(1-|z|^2) + (|z|^2-|\omega|^2)}{|2+\omega e^{2i\beta}||z|(1-|\omega|^2)} \leq C \quad (z \in U).$$

From the above lemma, we can conclude the following immediately.

Corollary 2.1. *Let $f \in SP(\beta)$ not be a rotation of f_β. For any set $U \subset \mathbb{D}$ with $\overline{\omega_f(U)} \not\ni -e^{-2i\beta}$,*

$$\sup_{z\in U}(1-|z|^2)|T_f(z)| < \|T_{f_\beta}\|_1$$

□

In particular, we can show the essentially unique extremality of f_β on some condition.

Corollary 2.2. *Let $f \in SP(\beta)$ not be a rotation of f_β. If $\overline{\omega(\mathbb{D})} \not\ni -e^{-2i\beta}$, then $\|T_f\|_1 < \|T_{f_\beta}\|_1$.*

Proof. We can take $\mathbb{D}$ itself as such U in Corollary 2.1. $\square$

Proof of Lemma 2.1. We take such U as above.

Put

$$c_1 := \inf_{z \in U}(|2 + \omega(z)e^{2i\beta}| - 1) > 0.$$

For $z \in U$,

$$\begin{aligned} 1 - D(z) &= \frac{(|z| - |\omega|)\{(|2 + \omega e^{2i\beta}| - 1)(1 + |z||\omega|) + (1 - |z|)(1 - |\omega|)\}}{|2 + \omega e^{2i\beta}||z|(1 - |\omega|^2)} \\ &\geq \frac{(|z| - |\omega|)\{c_1(1 + |z||\omega|) + (1 - |z|)(1 - |\omega|)\}}{6|z|(1 - |\omega|)} \\ &= \frac{1}{6}\{c_1 \frac{1 + |z||\omega|}{|z|}(1 - \frac{1 - |z|}{1 - |\omega|}) + (1 - \left|\frac{\omega}{z}\right|)(1 - |z|)\}. \end{aligned}$$

In Yamashita [11] (p. 313, (6.8**a)), it is shown that for a holomorphic map $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ which is not a rotation at the origin,

$$|\omega(z)| \leq |z|Q(|z|) < |z| \quad (z \in \mathbb{D}), \tag{7}$$

where

$$Q(x) = \frac{x^2 + Bx + A}{Ax^2 + Bx + 1} \quad (0 \leq x \leq 1),$$
$$A = |\omega'(0)| < 1 \quad \text{and}$$
$$B = \frac{|\omega''(0)|}{2(1 - |\omega'(0)|)} \leq 1 + A < 2.$$

From this, it follows that

$$1 - D(z) \geq \frac{1}{6}\{c_1(1 - \frac{1 - |z|}{1 - |z|Q(|z|)}) + (1 - Q(|z|))(1 - |z|)\} \tag{8}$$

on U. It is easy to see that $Q(x)$ is strictly increasing and $(1-x)/(1-xQ(x))$ is strictly decreasing in $[0, 1]$. Thus for $z \in U$,

$$\begin{aligned} 1 - D(z) &\geq \begin{cases} \frac{1}{6}(1 - Q(|z|))(1 - |z|) \geq \frac{1}{12}\left(1 - Q(\frac{1}{2})\right) & \text{if } |z| < \frac{1}{2}, \\ \frac{1}{6}c_1(1 - \frac{\frac{1}{2}}{1 - \frac{1}{2}Q(\frac{1}{2})}) \geq \frac{1}{12}c_1\left(1 - Q(\frac{1}{2})\right) & \text{if } \frac{1}{2} \leq |z| < 1 \end{cases} \\ &\geq \frac{1}{12}\min(1, c_1)\left(1 - Q(\frac{1}{2})\right) > 0. \end{aligned}$$

Now the proof is completed. $\square$

We consider the case $\overline{\omega(\mathbb{D})} \ni -e^{-2i\beta}$. Then $\inf_{z\in\mathbb{D}}(|2+\omega(z)e^{2i\beta}|-1)=0$, so the inequality similar to (8):

$$1-D(z) \geq \frac{1}{6}(1-Q(|z|))(1-|z|) > 0$$

holds for $z \in \mathbb{D}$. We obtain the following.

Lemma 2.2. *Let $f \in SP(\beta)$ not be a rotation of f_β. For $z \in \mathbb{D}$,*

$$(1-|z|^2)|T_f(z)| < (1-|\omega|^2)|T_{f_\beta}(\omega)| \leq \|T_{f_\beta}\|_1. \tag{9}$$

In particular, $\|T_f\|_1 \leq \|T_{f_\beta}\|_1$. Moreover, if

$$\max_{z_0\in\partial\mathbb{D}} \limsup_{\mathbb{D}\ni z\to z_0}(1-|z|^2)|T_{f_\beta}(z)| = 2|2+e^{2i\beta}| < \|T_{f_\beta}\|_1,$$

this inequality is strict. □

From now on, we turn our attention to the norm of T_{f_β}. If $f \in SP(-\beta)$, then $g(z) := \overline{f(\overline{z})} \in SP(\beta)$ and $\|T_f\|_1 = \|T_g\|_1$, so we can assume $\beta \geq 0$ without any loss of generality.

We consider the conformal automorphism $z \mapsto w = h(z)$ of $\mathbb{D}$ with $h(1) = 1$, $h(-e^{-2i\beta}) = -1$ and $h(ie^{-i\beta}) = i$. This is given by the relation

$$\frac{w-1}{w+1} = e^{-i\beta}\frac{z-1}{z+e^{-2i\beta}}. \tag{10}$$

By the Schwarz-Pick lemma, we have

$$1-|z|^2 = (1-|w|^2)\left|\frac{dz}{dw}\right|.$$

Differentiating (10), we have

$$\left|\frac{dz}{dw}\right| = \frac{2|z+e^{-2i\beta}|^2}{|c||w+1|^2}.$$

From them, it follows that

$$1-|z|^2 = (1-|w|^2)\frac{2|z+e^{-2i\beta}|^2}{|c||w+1|^2}.$$

From (10), we also have

$$|1-z| = |z+e^{-2i\beta}|\left|\frac{w-1}{w+1}\right|.$$

Thus

$$(1-|z|^2)|T_{f_\beta}(z)| = |c|\frac{(1-|z|^2)|z+2e^{-2i\beta}|}{|1-z||1+ze^{2i\beta}|} = 2\frac{1-|w|^2}{|w^2-1|}\cdot|z+2e^{-2i\beta}|.$$

Since $(1-|w|^2)/|w^2-1| \le 1$, we have $(1-|z|^2)|T_{f_\beta}(z)| < 2|2+e^{2i\beta}|$ on $\{z \in \mathbb{D}; |z+2e^{-2i\beta}| < |1+2e^{-2i\beta}|\}$. In the case $\beta = 0$, it coincides the whole $\mathbb{D}$. Therefore it is sufficient to consider the only case $\beta > 0$. For the estimate of $(1-|z|^2)|T_{f_\beta}(z)|$ on $\{z \in \mathbb{D}; |z+2e^{-2i\beta}| \ge |1+2e^{-2i\beta}|\}$, we use some geometric argument.

Noting that $|w^2-1|^2 = (1-|w|^2)^2 + 4(\Im w)^2$, we can see that the circular arc C_1 passing through the three points ± 1 and ki ($|k| < 1$) in the w-plane is the following:

$$\frac{1-|w|^2}{|w^2-1|} = \frac{1-k^2}{1+k^2}.$$

So C_1 is the level curve of $(1-|w|^2)/|w^2-1|$. Put $C_2 = h^{-1}(C_1)$. Since C_2

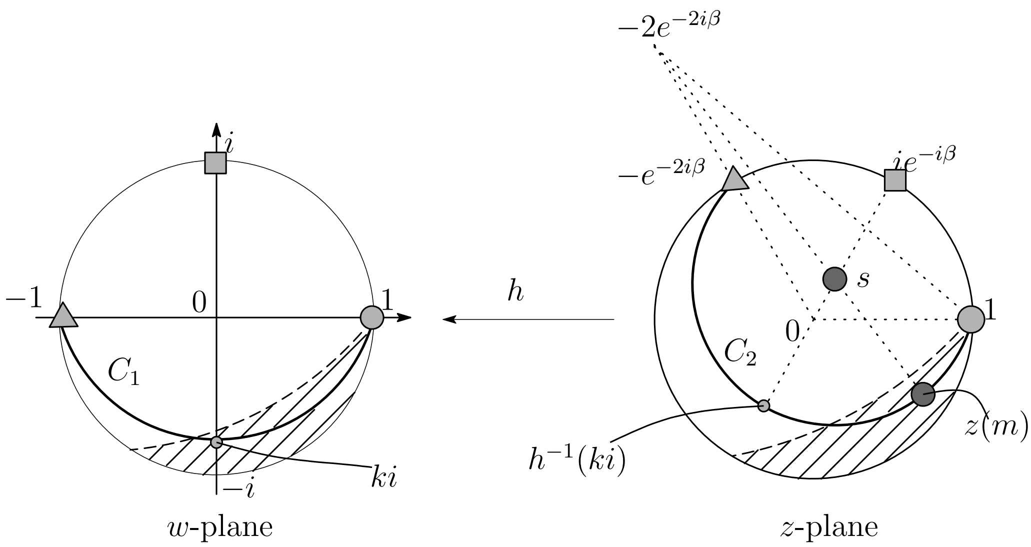

Figure 1: the level curve C_1 and C_2

is the circle passing through the three points

$$1,\ -e^{-2i\beta} \text{ and } h^{-1}(ki) = ie^{-i\beta}\cdot\frac{(1-ki)e^{-i\beta}-(1+ki)}{(k-i)e^{-i\beta}-(k+i)},$$

we can calculate the center s and the radius r of C_2 :

$$s = \frac{ie^{-i\beta}(k^2-1)}{2k\cos\beta + (k^2-1)\sin\beta} \quad \text{and} \tag{11}$$

$$r = \frac{(k^2+1)\cos\beta}{|2k\cos\beta + (k^2-1)\sin\beta|}. \tag{12}$$

Putting $m := \frac{k^2-1}{2k\cos\beta+(k^2-1)\sin\beta}$, we have $s = m \cdot ie^{-i\beta}$. On the level curve C_2, $|z + 2e^{-2i\beta}|$ takes the maximum at the point $z(m)$ in Figure 1, which is the intersection of the circular arc C_2 and the straight line passing through $-2e^{-2i\beta}$ and $s = m \cdot ie^{-i\beta}$. Therefore

$$(1-|z|^2)|T_{f_\beta}(z)| \leq 2\frac{1-k^2}{1+k^2}(|s+2e^{-2i\beta}| + r)$$

on C_2.

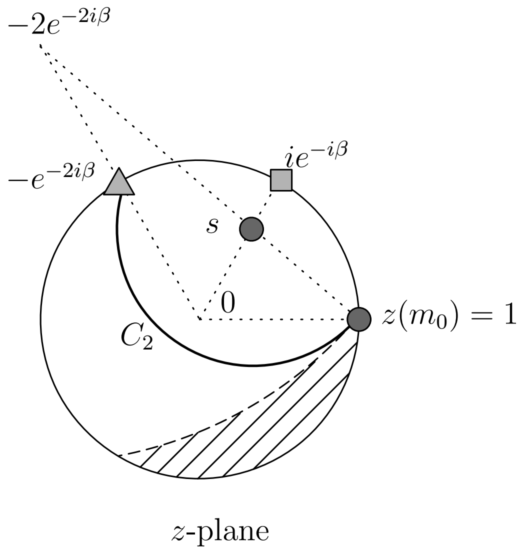

Figure 2: the level curve C_2 in the case $m = m_0$.

Since we are considering the case $|z+2e^{-2i\beta}| \geq |1+2e^{-2i\beta}|$, we can assume $0 \leq m \leq m_0 := \frac{4}{3}\sin\beta$. We note that if $m = m_0$, then C_2 is tangential to the circular arc $\{z \in \mathbb{D}; |z + 2e^{-2i\beta}| = |1 + 2e^{-2i\beta}|\}$ and the tangent point between them is $z(m_0) = 1$, and that if m moves from 0 to m_0, the level curve C_2 sweeps out $\{z \in \mathbb{D}; |z + 2e^{-2i\beta}| \geq |1 + 2e^{-2i\beta}|\}$ (see Figure 2).

Noting that $2k\cos\beta + (k^2-1)\sin\beta < 0$ since $m \geq 0$ and $k^2 \leq 1$, we also have $r = \frac{1+k^2}{1-k^2} m \cos\beta$. It follows that

$$\begin{aligned}
|s + 2e^{-2i\beta}| &= \sqrt{m^2 + 4 - 4m\sin\beta}, \\
\frac{1-k^2}{1+k^2} &= \frac{m\cos\beta}{\sqrt{m^2+1-2m\sin\beta}} \quad \text{and for } z \in C_2, \\
(1-|z|^2)|T_{f_\beta}(z)| \le E(m) &:= 2m\cos\beta\left(1 + \sqrt{\frac{m^2+4-4m\sin\beta}{m^2+1-2m\sin\beta}}\right).
\end{aligned}$$

We also note that $z(m_0) = 1 \in \partial\mathbb{D}$ and that $z(m) \in \{z \in \mathbb{D}; |z + 2e^{-2i\beta}| \ge |1 + 2e^{-2i\beta}|\}$ for $0 \le m < m_0$.

Here we consider the case $0 < \beta \le \pi/3$. Noting that $2|2 + e^{2i\beta}| = 2\sqrt{5 + 4\cos 2\beta}$ and $E(m_0) = 4\sin 2\beta$, we can see that $E(m_0) \le 2|2 + e^{2i\beta}|$ and the equality holds if and only if $\beta = \pi/3$.

The following holds.

Lemma 2.3. *If $0 < \beta \le \pi/3$, then $E(m) \le E(m_0)$ for any $0 \le m \le m_0$, and the equality holds if and only if $m = m_0$.*

Proof. It is easy to see that $E(0) < E(m_0)$ and that for $0 < m \le m_0$, $E(m) \le E(m_0)$ if and only if

$$(m - m_0)\{2m^2\sin\beta + (1 - 8\sin^2\beta)m + 4\sin\beta\} \le 0.$$

Putting

$$g(m) := 2m^2\sin\beta + (1 - 8\sin^2\beta)m + 4\sin\beta, \tag{13}$$

we calculate as

$$\begin{aligned}
g(m) = 2\sin\beta\left(m + \frac{1-8\sin^2\beta}{4\sin\beta}\right)^2 - \frac{64\sin^4\beta - 48\sin^2\beta + 1}{8\sin\beta} \quad \text{and} \\
m_0 - \left(-\frac{1-8\sin^2\beta}{4\sin\beta}\right) = \frac{3-8\sin^2\beta}{12\sin\beta}.
\end{aligned}$$

The following holds:

(i) If $0 < \sin^2\beta \le \frac{1}{8}$, then $g(m) > 0$ for any $0 < m \le m_0$ from (13).

(ii) If $\frac{1}{8} < \sin^2\beta \le \frac{3}{8}$, the same thing as the above holds since $-\frac{64\sin^4\beta - 48\sin^2\beta + 1}{8\sin\beta} > 0$.

(iii) If $\frac{3}{8} < \sin^2\beta \le \frac{3}{4}$, then $g(m) \ge 0$ for any $0 < m \le m_0$ since $g(m)$ is decreasing in $(0, m_0]$ and

$$g(m_0) = g(\frac{4}{3}\sin\beta) = -\frac{64}{9}\sin\beta(\sin^2\beta - \frac{3}{4}) \ge 0.$$

Thus $E(m) \le E(m_0)$ for any $0 \le m \le m_0$ and the equality holds if and only if $m = m_0$. $\square$

Consequently it follows that for $|\beta| \le \pi/3$, $(1-|z|^2)|T_{f_\beta}(z)| < 2|2+e^{2i\beta}|$ on $\mathbb{D}$. Noting that $(1-|z|^2)|T_{f_\beta}(z)|$ tends to $2|2+e^{2i\beta}|$ as z tends to $1-0$ along the real axis, we can conclude that $\|T_{f_\beta}\|_1 = 2|2+e^{2i\beta}|$ and that the function $(1-|z|^2)|T_{f_\beta}(z)|$ does not attain its supremum in $\mathbb{D}$.

Next we consider the case $\beta > \pi/3$. In this case, we can see $\|T_{f_\beta}\|_1$ is strictly larger than $2|2+e^{2i\beta}|$. In fact, we have $0 \le 1/\sin\beta < m_0$ and $E(1/\sin\beta) > 2|2+e^{2i\beta}|$. Therefore from Lemma 2.2, we can also conclude that a rotation of f_β is a unique extremal function.

Moreover, for $0 \le m \le \frac{4}{3}\sin\beta$, we have a uniform estimate:

$$\begin{aligned} E(m) &= 2m\cos\beta\left(1+\left|1-\frac{ie^{-i\beta}}{m-ie^{-i\beta}}\right|\right) \\ &< 2m\cos\beta\left(2+\frac{1}{|m-ie^{-i\beta}|}\right) \\ &\le \frac{8}{3}\sin 2\beta + 2\cos\beta\frac{m}{|m-ie^{-i\beta}|} \\ &\le 2\left(1+\frac{4}{3}\sin 2\beta\right). \end{aligned}$$

Thus $\|T_{f_\beta}\|_1 \to 2$ as $\beta \to \pi/2$.

Finally, we will show that for $|\beta| \le \pi/3$, f_β is also the essentially unique extremal function in $SP(\beta)$.

Let $f \in SP(\beta)$ not be a rotation of f_β. Noting Corollary 2.2, we consider the only case that $\overline{\omega(\mathbb{D})} \ni -e^{-2i\beta}$. Put $\varepsilon := |2+e^{2i\beta}| - 1 > 0$. Noting that

$$\limsup_{\mathbb{D}\ni z\to -e^{-2i\beta}} (1-|z|^2)|T_{f_\beta}(z)| = 2,$$

we can take the constant $r > 0$ such that $(1-|z|^2)|T_{f_\beta}(z)| < 2+\varepsilon$ on $\mathcal{N} := \{z \in \mathbb{D}; |z+e^{-2i\beta}| < r\}$. We note that $2+\varepsilon < 2|2+e^{2i\beta}| = \|T_{f_\beta}\|_1$. Next put $\mathcal{M} := \omega^{-1}(\mathcal{N})$. From (9) in Lemma 2.2, we obtain the following:

$$\sup_{z\in\mathcal{M}}(1-|z|^2)|T_f(z)| \le \sup_{z\in\mathcal{N}}(1-|z|^2)|T_{f_\beta}(z)| \le 2+\varepsilon < \|T_{f_\beta}\|_1.$$

On the other hand, since $\overline{\omega(\mathbb{D} \setminus \mathcal{M})} \not\ni -e^{-2i\beta}$, we have

$$\sup_{z \in \mathbb{D} \setminus \mathcal{M}} (1 - |z|^2)|T_f(z)| < \|T_{f_\beta}\|_1.$$

Combining both estimates, we can conclude $\|T_f\|_1 < \|T_{f_\beta}\|_1$.

Now the proof of Theorem 1 is completed. □

3 Proof of Theorem 2

From (5), it follows that

$$\begin{aligned} S_{f_\beta} =& (T_{f_\beta})' - \frac{1}{2}(T_{f_\beta})^2 \\ =& - c\frac{e^{2i\beta}\{e^{2i\beta}(e^{2i\beta} - 1)z^2 + 4(e^{2i\beta} - 1)z + 6\}}{2(1-z)^2(1+ze^{2i\beta})^2}. \end{aligned}$$

So we also have

$$(1 - |z|^2)^2|S_{f_\beta}(z)| = |c|\frac{(1 - |z|^2)^2|e^{2i\beta}(e^{2i\beta} - 1)z^2 + 4(e^{2i\beta} - 1)z + 6|}{2|1 - z|^2|1 + ze^{2i\beta}|^2}.$$

It follows easily that $(1 - |z|^2)^2|S_{f_\beta}(z)| \to 6$ as $z \to -e^{-2i\beta}$ radially. By the Kraus-Nehari theorem, $\sup_{z\in\mathbb{D}} |S_{f_\beta}(z)|(1 - |z|^2)^2 \leq 6$. Therefore we obtain $\|S_{f_\beta}\|_2 = 6$ for any $|\beta| < \pi/2$. □

4 Order estimate of the coefficients

Knowing the norm $\|T_f\|_1$ enables us to estimate the growth of coefficients of f (cf. [8]). However the sharp estimate of coefficients of $f \in SP(\beta)$ has been already obtained by Zamorski [13] in 1960. We would like to remark that we can derive the sharp growth estimate of coefficients of $f \in SP(\beta)$ from this.

Theorem 4.1 (Zamorski). *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is in $SP(\beta)$ and $|\beta| < \pi/2$, then*

$$|a_n| \leq \prod_{k=1}^{n-1} \left| 1 + \frac{e^{2i\beta}}{k} \right| \tag{14}$$

for any $n \geq 2$. The equality in (14) holds for some $n \geq 2$ if and only if f is a rotation of the β-spiral Koebe function f_β.

Remark. This is also shown in terms of generalized spiral-like functions by C. Burniak, J. Stankiewicz and Z. Stankiewicz [6](1980).

Corollary 4.1. *Let* $|\beta| < \pi/2$ *and* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ *be a* β*-spiral-like function. Then it holds that*

$$a_n = O(n^{\cos 2\beta}) \quad (n \to +\infty). \tag{15}$$

This order estimate is sharp.

Proof. From the inequality (14), we have that for $|\beta| < \pi/2$,

$$\begin{aligned}\log|a_n| \leq & \frac{1}{2}\sum_{k=1}^{n-1} \log\left(1 + \frac{2\cos 2\beta}{k} + \frac{1}{k^2}\right) \\ = & \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{2\cos 2\beta}{k}\right) + O(1) \\ = & \cos 2\beta \log n + O(1)\end{aligned}$$

as $n \to +\infty$. Therefore we obtain the estimate (15). □

Remark. In the case $|\beta| < \pi/4$, this is shown by Basgöze and Keogh in [5](1970).

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