Skew product maps related to finitely generated rational semigroups *

Hiroki Sumi
Department of mathematics,
Tokyo Institute of Technology,
2-12-1, Oh-okayama, Meguro-ku, Tokyo, 152-8550, Japan
e-mail; sumi@math.titech.ac.jp

June 6, 1999

Abstract

We consider skew product maps related to dynamics of semigroups generated by rational maps on the Riemann sphere. The entropy of those maps will be given and we will see there exists the unique maximal entropy measure. We will also show the uniqueness of self-similar measure. We will estimate Hausdorff dimension of the Julia sets of semigorups.

1 Introduction

For a Riemann surface S, let $\text{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of functions. A *rational semigroup* is a subsemigroup of $\text{End}(\overline{\mathbb{C}})$ without any constant elements. We say that a rational semigroup G is a *polynomial semigroup* if each element of G is a polynomial.

Definition 1.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\},\ J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the *Fatou set* for G and $J(G)$ is called the *Julia set* for G.

$J(G)$ is backward invariant under G but not forward invariant in general. If $G = \langle f_1, f_2, \ldots, f_n\rangle$ is a finitely generated rational semigroup , then $J(G)$ has the *backward self-similarity.* That is, we have $J(G) = \cup_{i=1}^{n} f_i^{-1}(J(G))$.

For any rational semigroup G, we set

$$E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp G^{-1}(z) < 2\},$$

*1991 Mathematical Subject Classification: 30D05, 58F23 where $G^{-1}(z) = \cup_{g \in G} g^{-1}(z)$. We have that if there exists an element of G of degree at least two or if there exists an element of G of degree one and the order is infinite, then $E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp G^{-1}(z) < \infty\}$ and $\sharp E(G) \leq 2$.

For any $z \in \overline{\mathbb{C}} \setminus E(G)$, we have $J(G) \subset \overline{G^{-1}(z)}$.

If $\sharp J(G) \geq 3$, then we have $J(G)$ is a perfect set and is equal to the closure of the set of points $z \in \overline{\mathbb{C}}$ which satisfies there exists an element $g \in G$ such that $g(z) = z$ and $|g'(z)| > 1$.

In general, the Julia set of a rational semigroup may have non-empty interior points. For example, $J(\langle z^2, 2z \rangle) = \{|z| \leq 1\}$. In fact, in [HM2] it was shown that if G is a finitely generated rational semigroup, then any super attracting fixed point of any element of G does not belong to $\partial J(G)$. Hence we can easily get many examples that the Julia sets have non-empty interior points. For more detail about these properties, see [HM1], [HM2], [ZR], [GR], [S1] and [S2]. In this paper we use the notations in [HM1] and [S1].

We will define skew product maps related to a generator system of finitely generated rational semigroup and will show fundamental properties of them.

We will investigate the upper esitimate of Hausdorff dimension of the Julia sets of finitely generated rational semigroups applying the methods of thermodynamical formalisms to the skew product maps(Theorem 4.4).

We will define *(backward) self-similar measure* in the Julia sets, that is, a kind of invariant measures whose projection to the base space(space of one-sided infinite words) are some Bernoulli measures. We will show the uniform convergence of orbits of the Perron-Frobenius operator which implies the uniqueness of the measure(Theorem 5.3). Using it, we will see that the backward self-similar measures are exact(Theorem 6.11) and will see the lower estimate of (topological, metric) entropy of the skew product maps.

Using the *Ruelle's inequality* for skew product maps, we generalize some results in [Ma1] and will show the existence of the measure theoritic generator(Lemma 6.9). From this we will see the upper estimate of the entropy of the skew product maps.

Finally, we see the metric entropy of backward self-similar measures with respect to the weight $a = (a_1, \dots, a_m)$ is equal to

$$-\sum_{j=1}^{m} a_j \log a_j + \sum_{j=1}^{m} a_j \log d_j$$

and we will show that the topological entropy of the skew product constructed by the generator system $\{f_1, \dots, f_m\}$ is equal to

$$\log(\Sigma_{j=1}^{m} \deg(f_j))$$

and there exists a unique maximal entropy measure $\tilde{\mu}$, which coincides with the backward self-similar measure corresponding to the weight

$$a_0 := (\frac{\deg(f_1)}{\sum_{j=1}^{m} \deg(f_j)}, \ \ldots, \ \frac{\deg(f_m)}{\sum_{j=1}^{m} \deg(f_j)}).$$

Hence the projection of the maximal entropy measure of the skew product to the base space is equal to the Bernoulli measure corresponding to the above weight a_0(Theorem 6.11).

Applying this result if $\{f_j^{-1}(J(G))\}_{j=1,\ldots,m}$ are mutually disjoint, then we get the following lower estimate of Hausdorff dimension of the Julia set of G(Theorem 6.13),

$$\dim_H(J(G)) \geq \frac{\log(\sum_{j=1}^{m} \deg(f_j))}{\int_{J(G)} \log(\|f'\|) \, d\mu},$$

where $\mu = (\pi_2)_* \tilde{\mu}$ and $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$.

Throughout this paper the generator system $\langle f_1, f_2, \ldots f_m \rangle$ consists of rational maps and the degrees of f_j's may not be the same. The degree of some f_j may be equal to one. After having being written this paper Mattias Jonsson contacted and told the author that in [J] he investigated the entropy of dynamics on "$\overline{\mathbb{C}}$-fibrations"such that the degree of all fiberwise maps are equal to some common $d, \ d \geq 2$. He uses the potential theory and Lyubich's method to show the uniqueness of maximal entropy measure. On the other hand, in this paper we do not use potential theory but we will use Lyubich's and Mañé's. The most different points are to show the uniform convergence of orbits of Perron-Frobenius operator and to show the existence of measure theoritic generator.

Note. This paper belongs to a part of the author's thesis([S4]) which was written under supervision of Prof. Shigehiro Ushiki.

Acknowledgement. The Author would like to express his gratitude to Prof. Shigehiro Ushiki for many valuable discussions and advices.

The Author would like to thank Professor Manfred Denker, Hartje Kriete and Stefan Heinemann for their kind hospitality in Goettingen University from April to September in 1998. This paper was written during this stay.

The Author would like to thank Professor Manfred Denker and David Boyd for many valuable discussions and suggestions.

2 Hyperbolicity

Definition 2.1. Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g \in G} \{ \text{ critical values of } g \}}.$$

We call $P(G)$ the post critical set of G. We say that G is *hyperbolic* if $P(G) \subset F(G)$.

Theorem 2.2 ([S2]). *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated hyperbolic rational semigroup . Assume that G contains an element with the degree at least two and each Möbius transformation in G is neither the identity nor an elliptic element. Let K be a compact subset of $\overline{\mathbb{C}} \setminus P(G)$. Then there are a positive number c, a number $\lambda > 1$ and a Riemannian metric ρ on an open subset V of $\overline{\mathbb{C}} \setminus P(G)$ which contains $K \cup J(G)$ and is backward invariant under G such that for each k*

$$\inf\{\|(f_{i_k} \circ \cdots \circ f_{i_1})'(z)\|_\rho \mid z \in (f_{i_k} \circ \cdots \circ f_{i_1})^{-1}(K), (i_k, \dots, i_1) \in \{1, \dots, n\}^k\}$$

$\geq c\lambda^k$, here we denote by $\| \cdot \|_\rho$ the norm of the derivative measured from the metric ρ to it.

Now we will show the converse of Theorem 2.2.

Theorem 2.3 ([S2]). *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. If there are a positive number c, a number $\lambda > 1$ and a Riemannian metric ρ on an open subset U containing $J(G)$ such that for each k*

$$\inf\{\|(f_{i_k} \circ \cdots \circ f_{i_1})'(z)\|_\rho \mid z \in (f_{i_k} \circ \cdots \circ f_{i_1})^{-1}(J(G)), (i_k, \dots, i_1) \in \{1, \dots, n\}^k\}$$

$\geq c\lambda^k$, where we denote by $\| \cdot \|_\rho$ the norm of the derivative measured from the metric ρ on V to it, then G is hyperbolic and for each $h \in G$ such that $\deg(h)$ *is one the map h is not elliptic.*

Remark. Because of the compactness of $J(G)$, we can show, with an easy argument, which is familiar to us in the iteration theory of rational functions, that even if we exchange the metric ρ to another Riemannian metric ρ_1, the enequality of the assumption holds with the same number λ and a different constant c_1.

Definition 2.4. Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. We say that G is expanding if the assumption in Theorem 2.3 holds.

3 Skew product

Let m be a positive integer. We denote by Σ_m the one-sided word space, that is

$$\Sigma_m = \{1, \dots, m\}^{\mathbb{N}}$$

and denote by $\sigma : \Sigma_m \to \Sigma_m$ the shift map, that is

$$(w_1, \dots) \mapsto (w_2, \dots).$$

Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. We define a map $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ by

$$\tilde{f}((w, x)) = (\sigma w, f_{w_1} x).$$

$\tilde{f}$ is a finite-to-one and open map. We have that a point $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ satisfies $f'_{w_1}(x) \neq 0$ if and only if $\tilde{f}$ is a homeomorphism in a small neighborhood of (w, x). Hence the map $\tilde{f}$ has infinitely many critical points in general.

Definition 3.1. For each $w \in \Sigma_m$ we denote by F_w the set of all the points $x \in \overline{\mathbb{C}}$ which satisfies that there exists an open neighborhood U of x such that the family $\{f_{w_n} \circ \cdots \circ f_{w_1}\}_n$ is normal in U. We set $J_w = \overline{\mathbb{C}} \setminus F_w$ and $\tilde{J}_w = \{w\} \times J_w$. Moreover we set

$$\tilde{J}(\tilde{f}) = \overline{\bigcup_{w \in \Sigma_m} \tilde{J}_w}, \ \tilde{F}(\tilde{f}) = (\Sigma_m \times \overline{\mathbb{C}}) \setminus \tilde{J}(\tilde{f}).$$

We often write $\tilde{F}(\tilde{f})$ as $\tilde{F}$ and $\tilde{J}(\tilde{f})$ as $\tilde{J}$. We call $\tilde{F}(\tilde{f})$ the *Fatou set* for $\tilde{f}$ and $\tilde{J}(\tilde{f})$ the *Julia set* for $\tilde{f}$.

For each $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ and an positive integer n we set

$$(\tilde{f}^n)'((w, x)) = (f_{w_n} \cdots f_{w_1})'(x).$$

Let $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ be a periodic point of $\tilde{f}$ with the period p. We say that (w, x) is *attracting* if $|(\tilde{f}^p)'((w, x))| < 1$, *indifferent* if $|(\tilde{f}^p)'((w, x))| = 1$ and *repelling* if $|(\tilde{f}^p)'((w, x))| > 1$.

Proposition 3.2.
1. *$\tilde{F}$ and $\tilde{J}$ are completely invariant under $\tilde{f}$. $\tilde{F}$ is open and $\tilde{J}$ is compact. $\tilde{f}(\tilde{J}_w) = \tilde{J}_{\sigma w}$. $\tilde{F}(\tilde{f})$ is equal to the set of all the points $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ which satisfies that there exists an open neighborhood U of x and an open neighborhood V of w such that for each $a \in V$ the family of maps $\{f_{a_n} \circ \cdots \circ f_{a_1}\}$ is normal in U.*

2. *$\tilde{J} = \cap_{n=0}^{\infty} \tilde{f}^{-n}(\Sigma_m \times J(G))$. $\pi_2(\tilde{J}) = J(G)$, where we denote by $\pi_2 : \Sigma_m \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the second projection.*

3. *$\tilde{J}$ has no interior points or is equal to $\Sigma_m \times \overline{\mathbb{C}}$.*

4. *If $\sharp(J(G)) \geq 3$, then $\tilde{J}$ is a perfect set.*

5. *If $\sharp(J(G)) \geq 3$, then $\tilde{J}$ is equal to the closure of the set of all repelling period points of $\tilde{f}$.*

6. *Assume $\sharp(J(G)) \geq 3$ and $E(G) \subset F(G)$. Let K be a compact subset of $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$. If U is an open set in $\Sigma_m \times \overline{\mathbb{C}}$ satisfying $U \cap \tilde{J} \neq \emptyset$, then there exists a positive integer N such that for each integer n with $n \geq N$, we have $\tilde{f}^n(U) \supset K$.*

Proof. By definition, it is easy to see 1. and 2. Assume $\tilde{J} \neq \Sigma_m \times \overline{\mathbb{C}}$ and $\tilde{J}$ contains a non-empty open set U. Then $F(G) \neq \emptyset$ and for each positive integer n we have

$$\pi_2 \tilde{f}^n(U) \subset \overline{\mathbb{C}} \setminus F(G).$$

By Montel's theorem, this is a contradiction. Hence 3. holds.

Let $z \in \tilde{J}$ be a point and assume there exists an open neighborhood U of z such that $U \setminus \{z\} \subset \tilde{F}$. There exists a positive integer n such that $\pi_1(\tilde{f}^n(U)) = \Sigma_m$. It follows that $\pi_2(\tilde{f}^n(z)) \in J(G)$ and $\pi_2(\tilde{f}^n(U \setminus \{z\})) \subset F(G)$. Since $\sharp(J(G)) \geq 3$, we have $J(G)$ is perfect and so that is a contradiction. Hence 4. holds.

Now we will show 5. Let $(w, x) \in \tilde{J}$. Let U be a neighborhood of w in Σ_m and V be a neighborhood of x in $\overline{\mathbb{C}}$. There exists a positive integer n such that if we set

$$U_n = \{\alpha \in \Sigma_m \mid \alpha_j = w_j,\ j = 1, \ldots, n, \}$$

then $U_n \subset U$. We set

$$G_n = \{g \in G \mid g = \cdots f_{w_n} \circ \cdots \circ f_{w_1}\}.$$

Then this is a subsemigroup of G. We have

$$J(G_n) = (f_{w_n} \cdots f_{w_1})^{-1} J(G)$$

and since $J(G)$ has infinitely many points, $J(G_n)$ must have at least three points. By Theorem 3.1 in [HM1](Note that if we read the proof of this theorem, we can see that the statement of this theorem holds whenever the Julia set has at least three points), we get that $J(G_n)$ is the closure of the set of repelling fixed point of all elements of G_n. Since $x \in (f_{w_n} \cdots f_{w_1})^{-1} J(G) = J(G_n)$, there exists an element $g \in G_n$ and a point $y \in V$ such that y is a repelling fixed point of g. Hence 5. holds.

Now we will show 6. Let K and U be as in 6. Assume $E(G) \neq \emptyset$. If $E(G)$ has exactly two points, then it is easy to show the statement of 6. Suppose $E(G) = \{x\}$ and let V be a connencted component of $F(G)$ containing x. Let ρ be the hyperbolic metric in V. Since each f_j does not increase the metric and x is a fixed point of it, there exists an open hyperbolic ball A about x included in V such that $f_j(A) \subset A$ for each j. It implies that $\tilde{f}(\pi_2^{-1}(A)) \subset \pi_2^{-1}(A)$. Hence, from the beginning of the proof, we can assume that

$$\tilde{f}(K) \supset K. \tag{1}$$

We will show that for each positive integer k,

$$K \subset \cup_{j=1}^{\infty} \tilde{f}^{kj}(U). \tag{2}$$

There exists a positive integer j such that $\pi_1 \tilde{f}^{kj}(U) = \Sigma_m$. On the other hand, by [HM1], for any rational semigroup G_1 the closure of the backward orbit of each point $x \in \overline{\mathbb{C}} \setminus E(G)$ under G_1 contains the Julia set $J(G_1)$. By [HM1] again, the Julia set of the subsemigroup H_k of G which is generated by :

$$\{f_{\alpha_k} \circ \cdots \circ f_{\alpha_1} \mid (\alpha_k, \ldots, \alpha_1) \in \{1, \ldots, m\}^k\}$$

is equal to $J(G)$. Also we have $E(H_k) = E(G)$ by definition of the exceptional set. Hence (2) holds.

By 5., we obtain that there exists an open set U_0 included in U and a positive integer s such that $\tilde{f}^s(U_0) \supset U_0$. Hence by (2), we get that there exists a positive integer N such that $\tilde{f}^N(U) \supset K$. From (1), it follows that for each positive integer n with $n \geq N$, we have $\tilde{f}^n(U) \supset K$. Hence the statement of 6. holds when $E(G)$ has exactly one point. If $E(G) = \emptyset$, we can show the statement in the same way as the above. $\square$

4 Hausdorff dimension of Julia sets of expanding semigroups

Definition 4.1. Let G be a rational semigroup and δ be a non-negative number. We say that a probability measure μ on $\overline{\mathbb{C}}$ is δ-subconformal if for each $g \in G$ and for each measurable set A,

$$\mu(g(A)) \leq \int_A \|g'(z)\|^{\delta} d\mu.$$

And we set

$$s(G) = \inf\{\delta \mid \exists \mu : \delta\text{-subconformal measure}\}$$

For each $j = 1, \ldots, m$, let φ_j be a Hölder continuous function on $f_j^{-1}(J(G))$. We set for each $(w, x) \in \tilde{J}$, $\varphi((w, x)) = \varphi_{w_1}(x)$. Then φ is a Hölder continuous function on $\tilde{J}$. We define an operater L on $C(\tilde{J}) = \{\psi : \tilde{J} \to \mathbb{C} \mid$ continuous $\}$ by

$$L\psi((w, x)) = \sum_{\tilde{f}((w', y)) = (w, x)} \frac{\exp(\varphi((w', y)))}{\exp(P)} \psi((w', y)),$$

counting multiplicities, where we denote by $P = P(\tilde{f}|_{\tilde{J}}, \varphi)$ the pressure of $(\tilde{f}|_{\tilde{J}}, \varphi)$.

Lemma 4.2. *With the same notations as the above, let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then for each set of Hölder continuous functions $\{\varphi_j\}_{j=1,\dots,m}$, there exists a unique probability measure τ on $\tilde{J}$ such that*

- $L^*\tau = \tau$,
- *for each $\psi \in C(\tilde{J})$, $\|L^n\psi - \tau(\psi)\alpha\|_{\tilde{J}} \to 0, n \to \infty$, where we set $\alpha = \lim_{l\to\infty} L^l(1) \in C(\tilde{J})$ and we denote by $\|\cdot\|_{\tilde{J}}$ the supremum norm on $\tilde{J}$,*
- *$\alpha\tau$ is an equilibrium state for $(\tilde{f}|_{\tilde{J}}, \varphi)$.*

Lemma 4.3. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then there exists a unique number $\delta > 0$ such that if we set $\varphi_j(x) = -\delta \log(\|f_j'(x)\|), j = 1, \dots, m,$ then $P = 0$.*

From Lemma 4.2, for this δ there exists a unique probability measure τ on $\tilde{J}$ such that $L_\delta^*\tau = \tau$ where L_δ is an operator on $C(\tilde{J})$ defined by

$$L_\delta\psi((w,x)) = \sum_{\tilde{f}((w',y))=(w,x)} \frac{\psi((w',y))}{\|(f_{w_1'})'(y)\|^\delta}.$$

Also δ satisfies that

$$\delta = \frac{h_{\alpha\tau}(\tilde{f})}{\int_{\tilde{J}} \tilde{\varphi}\alpha d\tau} \leq \frac{\log(\sum_{j=1}^m \deg(f_j))}{\int_{\tilde{J}} \tilde{\varphi}\alpha d\tau},$$

where $\alpha = \lim_{l\to\infty} L_\delta^l(1)$, we denote by $h_{\alpha\tau}(\tilde{f})$ the metric entropy of $(\tilde{f}, \alpha\tau)$ and $\tilde{\varphi}$ is a function on $\tilde{J}$ defined by $\tilde{\varphi}((w,x)) = \log(\|f_{w_1}'(x)\|)$.

By these argument, we get the following result.

Theorem 4.4. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup and δ the number in the above argument. Then*

$$\dim_H(J(G)) \leq s(G) \leq \delta.$$

Moreover, if the sets $\{f_j^{-1}(J(G))\}$ are mutually disjoint, then $\dim_H(J(G)) = \delta < 2$ and $0 < H_\delta(J(G)) < \infty$, where we denote by H_δ the δ-Hausdorff measure.

Corollary 4.5. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then*

$$\dim_H(J(G)) \leq \frac{\log(\sum_{j=1}^m \deg(f_j))}{\log \lambda},$$

where λ denotes the number in Definition 2.4.

Example 4.6. 1. Let $G = \langle f_1, f_2 \rangle$ where $f_1(z) = z^2$ and $f_2(z) = 2.3(z-3)+3$. Then we can see easily that $\{|z| < 0.9\} \subset F(G)$ and G is hyperbolic. By the corollary 4.5, we get

$$\dim_H(J(G)) \leq \frac{\log 3}{\log 1.8} < 2.$$

In particular, $J(G)$ has no interior points. In [S3], it is shown that if a finitely generated rational semigroup satisfies the open set condition with an open set O, then the Julia set is equal to the closure of the open set O or has no interior points. Note that the fact that the Julia set of the above semigroup G has no interior points was shown by only using analytic quantity. It seems to be true that G does not satisfy the open set condition.

2. Let $G = \langle \frac{z^3}{4},\ z^2 + 8 \rangle$. Then we can see easily that $\{|z| < 2\} \subset F(G)$ and G is hyperbolic. Hence we have

$$\dim_H J(G) \leq \frac{\log 5}{\log 3} < 2.$$

In particular, $J(G)$ has no interior points.

5 self-similar measure

We now consider about invariant measures and self-similar measures on Julia sets. In the cases of iterations of rational functions, Brolin's and Lyubich's studies are well known([Br], [L]). Recently, D.Boyd investigated "invariant measure" (that is, the measure $(\pi_2)_*\tilde{\mu}$ in the notation in Theorem 5.3) in the case that each f_j is of degree at least two and have shown the uniqueness in [Bo]. We introduce some notations and results from [L]. Let A be a bounded operator in the complex Banach space $\mathcal{B}$. The operator A is called *almost periodic* if the orbit $\{A^m\varphi\}_{m=1}^{\infty}$ of any vector $\varphi \in \mathcal{B}$ is strongly conditionally compact. The eigenvalue λ and related eigenvector are called unitary if $|\lambda| = 1$. The set of unitary eigenvectors of the operator A will be denoted by $\mathrm{spec}_u A$. We denote by $\mathcal{B}_u$ the closure of the linear span of the unitary eigenvectors of the operator A. And we set

$$\mathcal{B}_0 = \{\varphi \mid A^m\varphi \to 0\ (m \to \infty)\},$$

here the convergence is assumed to be strong.

Theorem 5.1. *([L]) If $A : \mathcal{B} \to \mathcal{B}$ is an almost periodic operator in the complex Banach space $\mathcal{B}$, then*

$$\mathcal{B} = \mathcal{B}_u \oplus \mathcal{B}_0.$$

Corollary 5.2. *([L]) Let $A : \mathcal{B} \to \mathcal{B}$ be an almost periodic operator in the complex Banach space $\mathcal{B}$. Assume that $spec_u A = \{1\}$ and the point $\lambda = 1$ is a simple eigenvalue. Let $h \neq 0$ be an invariant vector of the operator A. Then there exists an A^* invariant functional $\mu \in \mathcal{B}^*$, $\mu(h) = 1$, such that*

$$A^m \varphi \to \mu(\varphi)h \quad m \to \infty.$$

Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. We set $d_j = \deg(f_j)$ for each $j = 1, \dots, m$ and $d = \sum_{j=1}^m d_j$. For each compact set K of $\overline{\mathbb{C}}$ we denote by $C(K)$ all continuous complex valued functions on K. It is a Banach space with supremum norm on K. Assume that K is backward invariant under G. For each j and for each element φ we set

$$(A_j \varphi)(z) = \frac{1}{d_j} \sum_{\zeta \in f_j^{-1}(z)} \varphi(\zeta),$$

where z is any point of K. Then $A_j \varphi$ is an element of $C(K)$ and A_j is a bounded operator on $C(K)$. We set

$$\mathcal{W} = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid \sum_j a_j = 1, \ a_j \geq 0\}.$$

And for each $a \in \mathcal{W}$ we set

$$(B_a \varphi)(z) = \sum_{j=1}^n a_j (A_j \varphi)(z).$$

Then B_a is a bounded operator on $C(K)$.

Similarly, let $\tilde{K}$ be a compact subset of $\Sigma_m \times \overline{\mathbb{C}}$ which is backward invariant under $\tilde{f}$. We define an operator $\tilde{B}_a$ on $C(\tilde{K})$ as follows. For each element $\tilde{\varphi} \in C(K)$ we set

$$(\tilde{B}_a \tilde{\varphi})(z) = \Sigma_{\zeta \in \tilde{f}^{-1}(z)} \tilde{\varphi}(\zeta) \tilde{\psi}_a(\zeta)$$

where $\tilde{\psi}_a(\zeta) = \frac{a_{w_1}}{d_{w_1}}$ if $\pi_1(\zeta) = (w_1, w_2, \dots)$.

$\tilde{B}_a$ is a bounded operator on $C(\tilde{K})$. Furtheremore, if $\pi_2(\tilde{K}) = K$, then we get

$$\pi_2^* B_a = \tilde{B}_a \pi_2^*$$

and $\pi_2^*; C(K) \to C(\tilde{K})$ is an isometry.

Theorem 5.3. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Assume that there exists an element $g_0 \in G$ of degree at least two, the exceptional set $E(G)$ for G is included in $F(G)$ and $F(H) \supset J(G)$ where H is a rational semigroup defined by $H = \{h^{-1} \mid h \in Aut\,(\overline{\mathbb{C}}) \cap G\}$.(if H*

is empty, put $F(H) = \overline{\mathbb{C}}$.) Then for each $a \in \mathcal{W}$ with $a \neq 0$ there exists a unique regular Borel probability measure $\tilde{\mu}_a$ on $\Sigma_m \times \overline{\mathbb{C}}$ such that for each compact set $\tilde{K}$ which is included in $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$ and backward invariant under $\tilde{f}$,

$$\|\tilde{B}_a^n(\tilde{\varphi}) - \tilde{\mu}_a(\varphi)\mathbf{1}\|_{\tilde{K}} \to 0,$$

as $n \to \infty$, for each $\tilde{\varphi} \in C(\tilde{K})$, where we denote by $\mathbf{1}$ the constant function taking its value 1. Similarly, there exists a unique regular Borel probability measure μ_a on $\overline{\mathbb{C}}$ such that for each compact set K which is included in $\overline{\mathbb{C}} \setminus E(G)$ and backward invariant under G,

$$\|B_a^n(\varphi) - \mu_a(\varphi)\mathbf{1}\|_K \to 0,$$

as $n \to \infty$, for each $\varphi \in C(K)$.

Moreover, $(\pi_2)_*(\tilde{\mu}_a) = \mu_a$. The support of $\tilde{\mu}_a$ is equal to $\tilde{J}$ and the support of μ_a is equal to $J(G)$.

Definition 5.4. We call $\tilde{\mu}_a$ or μ_a the *self-similar measure* with respect to the weight a.

We need some lemmas to prove Theorem 5.3.

Lemma 5.5. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Assume that there exists an element $g_0 \in G$ of degree at least two and $F(H) \supset J(G)$ where H is a rational semigroup defined by $H = \{h^{-1} \mid h \in \mathrm{Aut}(\overline{\mathbb{C}}) \cap G\}$.(if H is empty, put $F(H) = \overline{\mathbb{C}}$.) Then there exists a $\delta > 0$ such that for each $x \in J(G)$, if we denote by $\mathcal{F}_{x,\delta}$ the family of maps satisfying that each element of it is a well-defined inverse branch of some element of G on $B(x, \delta)$ where $B(x, \delta)$ is a ball about x with the radius δ with respect to the spherical metric, then $\mathcal{F}_{x,\delta}$ is a normal family on $B(x, \delta)$.*

Proof. Take $x \in J(G)$. Let $D = B(x, \epsilon)$ where ϵ is a small positive number. Let G_1 be the subsemigroup of G such that for each element $g \in G_1$ there exists an element $g_1 \in G$ and integer j with $1 \leq j \leq m$ and $d_j \geq 2$ satisfying $g = g_1 f_j$. Let $\mathcal{F}_1$ be the family of maps from D to $\overline{\mathbb{C}}$ such that each element of $\mathcal{F}_1$ is an inverse branch of some element of G_1. Then from Theorem 2.1 in [HM3], $\mathcal{F}_1$ is normal in D. Taking ϵ smaller, we can assume that

$$\overline{\cup_{\beta \in \mathcal{F}_1} \beta(D)} \subset F(H). \tag{3}$$

Now let $(g_j)_j$ be any sequence of elements of $\mathcal{F}_{x,\epsilon}$ where g_j is an element of G. For each j there exists an element $\psi_j \in G_1 \cup \{Id\}$ and an $h_j \in H \cup \{Id\}$ such that $g_j = \psi_j h_j$. Then for each j, $\psi_j(h_j g_j^{-1}) = g_j g_j^{-1} = Id_D$. Hence we have $h_j g_j^{-1} \in \mathcal{F}_1$. Since $\mathcal{F}_1$ is normal in D, there exists a map g from D to $\overline{\mathbb{C}}$ and a sequence of positive integers (j_k) such that $h_{j_k} g_{j_k}^{-1} \to g$ locally uniformly on D as $k \to \infty$. By (3), we can assume that there exists a map h from a neighborhood V of $\overline{g(D)}$ to $\overline{\mathbb{C}}$ such that $h_{j_k}^{-1} \to h$ locally uniformly on V as $k \to \infty$. It follows that $g_{j_k}^{-1} = h_{j_k}^{-1} h_{j_k} g_{j_k} \to hg$ locally uniformly on D as $k \to \infty$. Hence we have $\mathcal{F}_{x,\epsilon}$ is normal on D. Since $J(G)$ is compact, the statement of our lemma holds.

$\square$

Lemma 5.6. *Under the same assumption as Theorem 5.3, let $\tilde{K}$ be a compact subset of $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$ which is backward invariant under $\tilde{f}$. If $\tilde{B}_a\varphi = \lambda\varphi$, $|\lambda| = 1$, then $\lambda = 1$ and φ is constant. That is, $(C(\tilde{K}))_u = \mathbb{C} \cdot \mathbf{1}$.*

Proof. Let z be a point of $\tilde{K}$ such that

$$|\varphi(z)| = \sup_{w \in \tilde{K}} |\varphi(w)|.$$

Then

$$\begin{aligned} |\varphi(z)| &= |(\tilde{B}_a\varphi)(z)| \\ &\leq \sum_{\zeta \in \tilde{f}^{-1}(z)} |\psi_a(\zeta)||\varphi(\zeta)| \\ &\leq \sum_j a_j |\varphi(z)| = |\varphi(z)|. \end{aligned}$$

Hence if ζ is a point of $\tilde{f}^{-1}(z)$, then $|\varphi(\zeta)| = |\varphi(z)|$ and it implies $\varphi(\zeta) = \lambda\varphi(z)$. Fix any point $\zeta_0 \in \tilde{J}$. By Proposition 3.2.6, there exists a sequence (ζ_n) such that $\zeta_n \in \tilde{f}^{-n}(z)$ for each positive integer n and $\zeta_n \to \zeta_0$ as $n \to \infty$. Hence we have $\lambda^n\varphi(z) = \varphi(\zeta_n) \to \varphi(\zeta)$ as $n \to \infty$. It implies $\lambda = 1$.

Now we will show that φ is constant. We put $\varphi = \Re\varphi + i\Im\varphi$. Then

$$\tilde{B}_a(\Re\varphi) = \Re\varphi, \ \tilde{B}_a(\Im\varphi) = \Im\varphi.$$

Let z be a point of $\tilde{K}$ such that

$$\Re\varphi(z) = \sup_{w \in K} \Re\varphi(w).$$

By a similar argument we can show that $\Re\varphi(\zeta) = \Re\varphi(z)$ for each $\zeta \in \tilde{f}^{-1}(z)$. Let ζ be any point of $\tilde{J}$. Let $(\zeta_n)_n$ be a sequence such that for each n the point ζ_n belongs to $\tilde{f}^{-n}(z)$ and $\zeta_n \to \zeta$. Then $\Re\varphi(\zeta_n) \to \Re\varphi(\zeta)$ so $\Re\varphi(z) = \Re\varphi(\zeta)$. In the same way we can show that if x is the minimum point of the function $\Re\varphi$, then $\varphi(x) = \varphi(\zeta)$, where ζ is any point of $\tilde{J}$. Hence $\Re\varphi$ is constant and by the same argument $\Im\varphi$ is also constant. Whence φ is constant. $\square$

Lemma 5.7. *Under the same assumption as Theorem 5.3, if K is a compact subset of $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$ which is backward invariant under $\tilde{f}$, then $\tilde{B}_a$ is an almost periodic operator on $C(K)$.*

Proof. We will develop the methods of key lemma about equicontinuity of $\{B_{a_0}^n \phi\}_n$ where $a_0 = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$, $\phi \in C(K)$ in [Bo]. Let $\varphi \in C(K)$ be any element. We have $\|\tilde{B_a}^n \varphi\|_K \leq \|\varphi\|_K$ for each positive integer n. By the Ascoli-Arzela Theorem, we have only to show that the family $\{\tilde{B_a}^n \varphi\}_n$ is equicontinuous on $\tilde{K}$.

For each $t = (t_1, \dots, t_m) \in \mathbb{N}^m$, we set

$$a_{r,t} = \frac{t_r d_r}{\sum_{k=1}^n t_k d_k}, \; r = 1, \dots, m,$$

and $a(t) = (a_{1,t}, \dots, a_{m,t}) \in \mathcal{W}$. Then there exists a sequence $(t^l)_l$ of elements of $\mathbb{N}^m$ such that $a(t^l) \to a$, as $l \to \infty$.

For each $p \in \mathbb{N}$, we set

$$Z_p = \cup_{(i_p, \dots, i_1) \in \{1, \dots, m\}^p} \{\mathrm{cv}(f_{i_p} \cdots f_{i_1})\} = \pi_2(\mathrm{cv}(\tilde{f}^p)),$$

where cv means the critical values.

Let U be any simply connected domain such that $U \subset \overline{\mathbb{C}} \setminus Z_p$. For each $i = 1, \dots, m$ and $l \in \mathbb{N}$, we set

$$g_{i,j}^l = f_i, \; j = 1, \dots, t_i^l.$$

For each $l \in \mathbb{N}$ we consider $\{g_{i,j}^l\}_{i,j}$ as a generator system and let $\tilde{f}_l : \Sigma_{m(l)} \times \overline{\mathbb{C}} \to \Sigma_{m(l)} \times \overline{\mathbb{C}}$ be the skew product map constructed by that generator system in the same way as the beginning of this section where $m(l) = \sum_{i=1}^m t_i^l$. For each $s \in \mathbb{N}$ and $l \in \mathbb{N}$, we denote by $\sigma_{s,l} = \sigma_{s,l}(U)$ the cardinality of the family consisting of well-defined inverse branches of $\tilde{f_l}^s$ on $\Sigma_{m(l)} \times U$. For each finite word $\{1, 2, \dots, m(l)\}^s$, let $\sigma_{s,k,\alpha}$ be the cardinality of the family consisting of well-defined inverse branches of the element in $\langle g_{i,j}^l \rangle_{i,j}$ corresponding to the word α on U. Then by definition, we have

$$\sigma_{s,l} = \sum_{\alpha \in \{1,2,\dots,m(l)\}^s} \sigma_{s,l,\alpha}.$$

For each $k = 1, 2, \dots, m(l)$, let e_k be the degree of k-th element of $\{g_{i,j}^l\}_{i,j}$. Then we have

$$\sigma_{s+1,l,\alpha k} \geq e_k(\sigma_{s,l,\alpha} - (2e_k - 2)).$$

Hence we get

$$\begin{aligned}
\sigma_{s+1,l} &= \sum_{\alpha\in\{1,2,\ldots,m(l)\}^s} \sum_{k\in\{1,2,\ldots,m(l)\}} \sigma_{s+1,l,\alpha k} \\
&\geq \sum_{\alpha\in\{1,2,\ldots,m(l)\}^s} \sum_{k\in\{1,2,\ldots,m(l)\}} e_k(\sigma_{s,l,\alpha} - (2e_k - 2)) \\
&= (\sum_{k\in\{1,2,\ldots,m(l)\}} e_k)\sigma_{s,l} - m(l)^s \sum_{k\in\{1,2,\ldots,m(l)\}} e_k(2e_k - 2) \\
&= (\sum_{j=1}^{m} t_j^l d_j)\sigma_{s,l} - m(l)^s(\sum_{j=1}^{m} 2t_j^l(d_j^2 - d_j)) \\
&= d(l)\sigma_{s,l} - m(l)^s e(l),
\end{aligned}$$

where $d(l) = \sum_{j=1}^m t_j^l d_j$ and $e(l) = \sum_{j=1}^m 2t_j^l(d_j^2 - d_j)$. It follows that $\sigma_{p,l} = d(l)^p$ and for each positive integer n,

$$\sigma_{p+n,l} \geq d(l)^{p+n} - e(l)m(l)^p \sum_{i=0}^{n-1} m(l)^{n-1-i} d(l)^i.$$

Hence we get for each $l \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$\frac{d(l)^{p+l} - \sigma_{p+n,l}}{d(l)^{p+n}} \leq e(l)(\frac{m(l)}{d(l)})^p \frac{1}{d(l)} \sum_{i=1}^{n-1} (\frac{m(l)}{d(l)})^i. \tag{4}$$

We have

$$\frac{m(l)}{d(l)} = \sum_{j=1}^{m} a_{j,t^l} \frac{1}{d_j} \to \sum_{j=1}^{m} a_j \frac{1}{d_j} < 1, \tag{5}$$

$$\frac{e(l)}{d(l)} = \sum_{j=1}^{m} a_{j,t^l} 2(d_j^2 - d_j) \to \sum_{j=1}^{m} a_j 2(d_j^2 - d_j), \tag{6}$$

as $l \to \infty$. By (5), we can assume that there exists a number δ with $0 < \lambda < 1$ such that for each $l \in \mathbb{N}$,

$$\frac{m(l)}{d(l)} < \lambda. \tag{7}$$

Now let $\epsilon > 0$ be arbitrary small positive number. From (4), (6) and (7), we get that there exists a positive integer p such that for each simply connected domain U satisfying $U \subset \overline{\mathbb{C}} \setminus Z_p$, the number $\sigma_{p+n,l} = \sigma_{p+n,l}(U)$ satisfies that

$$\frac{d(l)^{p+n} - \sigma_{p+n,l}}{d(l)^{p+n}} \leq \epsilon, \tag{8}$$

for each $l \in \mathbb{N}$ and $n \in \mathbb{N}$.

Let d_{Σ_m} be a fixed metric in Σ_m and $d_{\overline{\mathbb{C}}}(\ ,\ )$ the spherical metric on $\overline{\mathbb{C}}$. Let $\tilde{d}(\ ,\ )$ be the metric on $\Sigma_m \times \overline{\mathbb{C}}$ defined by: $\tilde{d}((w', y'),\ (w, y)) = \max\{d_{\Sigma_m}(w', w),\ d_{\overline{\mathbb{C}}}(y', y)\}$.

Let δ be a number in Lemma 5.5. Let $K' = \overline{B(J(G),\ \frac{1}{2}\delta)}$. Let $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ be a point such that $x \in \pi_2(K) \cap K' \setminus Z_p$. We can easily see that there exists a positive number δ_1 such that if $\tilde{d}(z, z') < \delta_1,\ z, z' \in K$ and $\pi_2(z) = \pi_2(z'),$ then

$$|\tilde{B}^n_{a(l)}\varphi(z) - \tilde{B}^n_{a(l)}\varphi(z')| < \epsilon, \tag{9}$$

for each $l \in \mathbb{N}$ and $n \in \mathbb{N}$. Hence by Lemma 5.5, (8) and (9), we get that if we take δ_2 so small then for each $(w', x') \in K$ with $\tilde{d}((w, x),\ (w', x')) < \delta_2$, we have

$$\begin{aligned} & |\tilde{B}^n_{a(l)}\varphi((w, x)) - \tilde{B}^n_{a(l)}\varphi((w', x'))| \\ \leq\ & |\tilde{B}^n_{a(l)}\varphi((w, x)) - \tilde{B}^n_{a(l)}\varphi((w, x'))| + |\tilde{B}^n_{a(l)}\varphi((w, x')) - \tilde{B}^n_{a(l)}\varphi((w', x'))| \\ \leq\ & \epsilon + 2M\epsilon + \epsilon = \epsilon(2 + 2M), \end{aligned} \tag{10}$$

where $M = \sup_{z \in K} |\varphi(z)|$, for each $l \in \mathbb{N}$ and $n \in \mathbb{N}$.

Now, let $z \in K$ be any point. By Proposition 3.2.6, there exists a positive integer τ such that for each $y \in K$, we have

$$\pi_2(\tilde{f}^{-\tau}(y)) \cap (\pi_2(K) \cap K' \setminus Z_p) \neq \emptyset. \tag{11}$$

For each $l \in \mathbb{N}$, we set

$$\beta(l) = \min_{(w_1, \dots, w_\tau) \in \{1, \dots, m\}^\tau} t^l_{w_1} \cdots t^l_{w_\tau}. \tag{12}$$

Then we have

$$0 < (\min_{j=1,\dots,m} \frac{a_{w_j}}{d_{w_j}})^\tau \leq \liminf_{l \to \infty} \frac{\beta(l)}{d(l)^\tau}, \tag{13}$$

$$\limsup_{l \to \infty} \frac{\beta(l)}{d(l)^\tau} \leq (\max_{j=1,\dots,m} \frac{a_{w_j}}{d_{w_j}})^\tau < 1. \tag{14}$$

Hence we can assume that there exist constants c_1 and c_2 such that for each $l \in \mathbb{N}$,

$$0 < c_1 \leq \frac{\beta(l)}{d(l)^\tau} \leq c_2 < 1. \tag{15}$$

For each $l \in \mathbb{N}$, let $\iota_l : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_{m(l)} \times \overline{\mathbb{C}}$ be an natural embedding and $\pi^l : \Sigma_{m(l)} \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ the natural projection. For each $l \in \mathbb{N}$ and $n \in \mathbb{N}$, let $S_{n,l}$ be the set of solution of $\tilde{f}_l^{\tau n}(z') = \iota_l(z)$ and $\sharp S_{n,l}$ the cardinality counting multiplicity. Let $S_{1,l,1}$ be a subset of $S_{1,l}$ such that the second projection of each point of the set belongs to $\pi_2(K) \cap K' \setminus Z_p$ and $\sharp S_{1,l,1} = \beta(l)$. And let $S_{1,l,2} = S_{1,l} \setminus S_{1,l,1}$. Inductively, for each $n \geq 1$, let $S_{n+1,l,1}$ be a set of backward images of $S_{n,l,2}$ by $\tilde{f}_l^{\tau}$ such that the second projection of each point of the set belongs to $\pi_2(K) \cap K' \setminus Z_p$ and $\sharp S_{n+1,l,1} = \beta(l) \sharp S_{n,l,2}$ where the cardinalities are counted considering multiplicity. And let $S_{n+1,l,2} = \tilde{f}^{-\tau}(S_{n,l}) \setminus S_{n+1,l,1}$. Then inductively we can see that for each $n \in \mathbb{N}$,

$$\sharp S_{n,l,2} = (d(l)^{\tau} - \beta(l))^{n-1} \tag{16}$$

and

$$\sharp S_{n,l,1} = (d(l)^{\tau} - \beta(l))^{n-2} \beta(l). \tag{17}$$

By (15), there exists a positive integer N such that for each $l \in \mathbb{N}$,

$$\left(\frac{d(l)^{\tau} - \beta(l)}{d(l)^{\tau}}\right)^N < \epsilon. \tag{18}$$

By (9), there exists a number $\eta > 0$ such that for each $n \in \mathbb{N}, l \in \mathbb{N}$ and $j = 1, \ldots N$, if $z' \in \pi^l(S_{j,l,1})$ and $\tilde{d}(z', x)$, $\tilde{d}(z', y) < \eta$,

$$|\tilde{B}^n_{a(l)}(\varphi)(x) - \tilde{B}^n_{a(l)}(\varphi)(y)| < 2\epsilon(2 + 2M). \tag{19}$$

By (16), (17), (18) and (19), we can see that if we take $\delta_2 > 0$ small enough then $\tilde{d}(z, z') < \delta_2$, $z' \in K$ implies that for each $n \in \mathbb{N}$ and $l \in \mathbb{N}$,

$$\begin{aligned}
& |\tilde{B}^{n+\tau N}_{a(l)}(\varphi)(z) - \tilde{B}^{n+\tau N}_{a(l)}(\varphi)(z')| \\
\leq \; & d(l)^{-N\tau} \sum_{i=1}^{d(l)^{\tau N}} |\tilde{B}^n_{a(l)}(\varphi)(\pi^l(z_i)) - \tilde{B}^n_{a(l)}(\varphi)(\pi^l(z'_i))| \\
\leq \; & \frac{1}{d(l)^{N\tau}} \left(\sum_{j=1}^{N} \frac{\sharp S_{j,l,1}}{d(l)^{N\tau - j\tau}} 2\epsilon(2 + 2M) + \sharp S_{N,l,2} 2M\right) \\
= \; & \left(\sum_{j=1}^{N} \left(\frac{d(l)^{\tau} - \beta(l)}{d(l)^{\tau}}\right)^{j-1} \frac{\beta(l)}{d(l)^{\tau}} \cdot 2\epsilon(2 + 2M)\right) + 2M\epsilon,
\end{aligned}$$

where on the above we set $\{z_1, \ldots z_b\} = S_{\tau N,l}$, $b = d(l)^{\tau N}$ and we denoted by z'_i the point of $\tilde{f}_l^{-\tau N}(\iota_l(z'))$ corresponding to z_i. By (15), there exists a constant $C > 0$, not depending on N, such that for each $l \in \mathbb{N}$,

$$\sum_{j=1}^{N} \left(\frac{d(l)^{\tau} - \beta(l)}{d(l)^{\tau}}\right)^{j-1} \frac{\beta(l)}{d(l)^{\tau}} \leq C. \tag{20}$$

Hence we get

$$|\tilde{B}_{a(l)}^{n+\tau N}(\varphi)(z) - \tilde{B}_{a(l)}^{n+\tau N}(\varphi)(z')| \le \epsilon(4C + 4MC + 2M). \tag{21}$$

Letting $l \to \infty$, we get that for each $n \in \mathbb{N}$,

$$|\tilde{B}_{a}^{n+\tau N}(\varphi)(z) - \tilde{B}_{a}^{n+\tau N}(\varphi)(z')| \le \epsilon(4C + 4MC + 2C). \tag{22}$$

Thus we have proved the lemma. □

Proof. of Theorem 5.3. By Corollary 5.2, Lemma 5.6 and Lemma 5.7 we can show the statement about convergence of the operator and that the support of $\tilde{\mu}_a$ is included in $\tilde{J}$ in the same way as that in [L]. Since $\tilde{\mu}_a$ is $\tilde{B}_a^*$-invariant and $\inf_{z \in \tilde{J}} \tilde{\psi}_a(z) > 0$, by Proposition 3.2.6, we can show that the support of $\tilde{\mu}_a$ is equal to $\tilde{J}$ immediately. It implies that the support of μ_a is equal to $J(G)$. □

Lemma 5.8. *Under the same assumption as Theorem 5.3, for any $a \in \mathcal{W}$ with $a \neq 0$, we have μ_a is non-atomic.*

Proof. We set for each $n \in \mathbb{N}, l \in \mathbb{N}$ and $z \in J(G)$,

$$c(n,l)(z) = \sum_{\alpha \in \{1,\dots,m(l)\}^n,\ g_{\alpha_1}\circ\cdots\circ g_{\alpha_n}(z) \in J(G)} (\text{mul } (g_{\alpha_1}\circ\cdots\circ g_{\alpha_n})\text{at } z), \tag{23}$$

where we denote by g_{α_j} any element of $\{g_{i,j}^l\}$ and mul denotes the multiplicity. We will show the following claim.

Claim 1. for any $z \in J(G)$, there exists an open neighborhood $U(z)$ of z and a word $(w_1(z), \dots, w_2(z)) \in \{1, \dots, m\}^2$ such that for each $y \in U(z)$,

$$(\text{mul } (f_{w_2(z)} \circ f_{w_1(z)})\text{at } y) < d_{w_2(z)} d_{w_1(z)}.$$

Suppose there exists a point $z \in J(G)$ such that for each $(w_2, w_1) \in \{1, \dots, m\}^2$,

$$\text{mul } (f_{w_2} \circ f_{w_1})\text{at } z = d_{w_2} d_{w_1}.$$

For each $j = 1, \dots m$, we set $z_j = f_j(z)$. We can assume that there exists a positive integer t with $1 \le t \le m$ such that $d_1, \dots, d_t \ge 2$ and $d_{t+1} = \cdots = d_m = 1$.

If there exists an integer i such that $z \neq z_i$ then for each integer s with $1 \le s \le t$, mul f_s at z and at z_i are equal to d_s. Hence, conjugating G by some Möbius transformation, we can assume that $z = 0$, $z_s = \infty$, $f_s(z) = \frac{1}{z^{d_s}}$ for each s with $1 \le s \le t$ and $z_{t+1}, \dots, z_m \in \{0, \infty\}$. It implies $z \in E(G)$ but this contradicts to the assumption $E(G) \subset F(G)$.

If $z = z_i$ for each $i = 1, \dots, m$, then conjugating G by some Möbius transformation, we can assume that $z = \infty$ and $f_1, \dots, f_m$ are polynomials. It contradicts to $E(G) \subset F(G)$. Hence the claim 1. holds.

From claim 1, there exists a finite collection $U(x_1), \ldots, U(x_k)$ with $\cup_{j=1}^{k} U(x_j) \supset J(G)$ where $x_1, \ldots x_k \in J(G)$ such that for each $j = 1, \ldots, k$, there exists a word $(w_2(x_j), w_1(x_j)) \in \{1, \ldots m\}^2$ satisfying that for each $y \in U(x_j)$,

$$(\text{mul } (f_{w_2(x_j)} \circ f_{w_1(x_j)}) \text{at } y) < d_{w_2(x_j)} d_{w_1(x_j)}.$$

We set

$$c = \min_{j=1,\ldots,k} \min_{y \in U(x_k)} (d_{w_2(x_j)} d_{w_1(x_j)} - (\text{mul } f_{w_2(x_j)} \circ f_{w_1(x_j)} \text{at } y)) > 0.$$

We get for each $z \in J(G)$ and $l \in \mathbb{N}$,

$$c(2,l)(z) \leq d(l)^2 - (\min_{j=1,\ldots,m} t_j^l)^2 c.$$

Hence for each $n \in \mathbb{N}, l \in \mathbb{N}$ and $z \in J(G)$,

$$\frac{c(2n,l)(z)}{d(l)^{2n}} \leq (\frac{d(l)^2 - (\min_{j=1,\ldots m} t_j^l)^2 c}{d(l)^2})^n. \tag{24}$$

Let $\epsilon > 0$ be any small number. And fix $z \in J(G)$. By (24), there exists a positive integer n_0 such that for each $l \in \mathbb{N}$,

$$\frac{c(2n_0,l)(z)}{d(l)^{2n_0}} \leq \epsilon. \tag{25}$$

Take $\zeta \in J(G)$. For each $l \in \mathbb{N}$ and $n \in \mathbb{N}$, we set

$$\mu_{l,n}^{\zeta} = \frac{1}{d(l)^n} \sum_{\alpha \in \{1,\ldots,m(l)\}^n} \sum_{y \in (g_{\alpha_1} \circ \cdots \circ g_{\alpha_n})^{-1}(\zeta)} \delta_y,$$

where δ_y denotes the dirac measure concentrated at y and g_k denotes the k-th element of $\{g_{i,j}^l\}_{i,j}$. Note that by Theorem 5.3, $\mu_{l,n}^{\zeta} \to \mu_{a(t^l)}$ weakly as $n \to \infty$. There exists an open neighborhood U of z such that if we set

$$c'(2n_0,l)(U) = \sum_{\alpha \in \{1,\ldots,m(l)\}^{n_0},\ g_{\alpha_1} \circ \cdots \circ g_{\alpha_{n_0}}(z) \in J(G)} \deg(g_{\alpha_1} \circ \cdots \circ g_{\alpha_{n_0}}|_U),$$

then we have $c'(2n_0,l)(U) = c(2n_0,l)(z)$. Hence by (25), we get that for each $n \in \mathbb{N}$ and $l \in \mathbb{N}$,

$$\mu_{l,2n_0+n}^{\zeta}(U) \leq \frac{d(l)^n c'(2n_0,l)(U)}{d(l)^{2n_0+n}} \leq \epsilon,$$

Letting $n \to \infty$, since we can assume that $\mu_{a(t^l)}(\partial U) = 0$ for each $l \in \mathbb{N}$, we get for each $l \in \mathbb{N}$,

$$\mu_{a(t^l)}(U) \leq \epsilon. \tag{26}$$

By the uniqueness of the self-similar measure with respect to the weight a, we have $\mu_{a(t^l)} \to \mu_a$ weakly as $l \to \infty$. Since we can assume $\mu_a(\partial U) = 0$, by (26), we get

$$\mu_a(U) \leq \epsilon.$$

Since ϵ can be taken arbitrary small, we get $\mu_a(\{z\}) = 0$. Hence μ_a is non-atomic. $\square$

6 entropy

Lemma 6.1. *Under the same assumption as Theorem 5.3, let $\tilde{\mu}_a$ be the self-similar measure with respect to the weight $a \in \mathcal{W}$. Then $\tilde{\mu}_a$ is $\tilde{f}$-invariant and*

1. *$(\tilde{f}, \tilde{\mu}_a)$ is exact.*
2. *$h_{\tilde{\mu}_a}(\tilde{f}) \geq H(\epsilon|(\tilde{f})^{-1}\epsilon) = -\sum_{j=1}^{m} a_j \log a_j + \sum_{j=1}^{m} a_j \log d_j$, where we denote by ϵ the partition of $\Sigma_m \times \overline{\mathbb{C}}$ into one point subsets.*

Proof. By Theorem 5.3, the measure $\tilde{\mu}_a$ is $\tilde{B}_a^*$-invariant. Hence for each $\varphi \in C(\Sigma_m \times \overline{\mathbb{C}})$,

$$\int \varphi \circ \tilde{f} d\tilde{\mu} = \int \tilde{B}_a(\varphi \circ \tilde{f}) d\tilde{\mu} = \int \varphi d\tilde{\mu}.$$

Hence $\tilde{\mu}_a$ is $\tilde{f}$-invariant.

Let ν_z denote the conditional measure on the element of partition $\tilde{f}^{-1}\epsilon$ containing $z \in \Sigma_m \times \overline{\mathbb{C}}$with respect to the measure $\tilde{mu}_a$. Then by Theorem 5.3 and using the same argument as that in p366-367 in [L], we can show that

$$\nu_z = \sum_{j=1}^{m} \frac{a_j}{d_j} \sum_{\zeta \in \tilde{f}^{-1}\tilde{f}(z) \cap \Sigma_{m,j}} \delta_\zeta, \tag{27}$$

where $\Sigma_{m,j} = \{w \in \Sigma_m \mid w_1 = j\}$. By Theorem 5.3 and (27), using the same argument as that in P367 in [L] again, we can show that $(\tilde{f},\ \tilde{\mu}_a)$ is exact.

By Lemma 5.8, we have $\pi_{2*}\tilde{\mu}_a$ is non-atomic. In particular,

$$\tilde{\mu}_a(\text{cv }(\tilde{f})) = 0. \tag{28}$$

By (27) and (28), we get that

$$I(\epsilon|\tilde{f}^{-1}\epsilon)(z) = -\sum_{j=1}^{m} d_j \cdot \frac{a_j}{d_j} \log \frac{a_j}{d_j} = -\sum_{j=1}^{m} a_j \log \frac{a_j}{d_j}, \tag{29}$$

for $\tilde{\mu}$-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Hence

$$H(\epsilon|\tilde{f}^{-1}\epsilon) = \int I(\epsilon|\tilde{f}^{-1}\epsilon)(z)d\tilde{\mu}(z) = -\sum_{j=1}^{m} a_j \log \frac{a_j}{d_j}.$$

$\square$

Now we will estimate the topological entropy of $\tilde{f}$ from above.

Theorem 6.2. *Let $G = \langle f_1, \ldots, f_m \rangle$ be a rational semigroup and $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ the skew product map constructed by the generator system $\{f_1, \ldots, f_m\}$. Then the toplogical entropy $h(\tilde{f})$ on $\Sigma_m \times \overline{\mathbb{C}}$ satifies that*

$$h(\tilde{f}) \leq \log(\sum_{j=1}^{m} \deg f_j).$$

To prove this theorem, we need several lemmas.

The first one is the *Ruelle's inequality* for skew product maps. Let X be a compact metric space and M a compact C^∞ manifold. Let $f : X \times M \to X \times M$ be a continuous map such that $f(x,y) = (\sigma(x), g_x(y))$ where $\sigma : X \to X$ is a continuous map, $g_x : M \to M$ is a differential map for each $x \in X$. Let $D_y g_x : T_y M \to T_{g_x(y)} M$ be the linear map induced by g_x. Assume that $(x,y) \mapsto D_y g_x$ is continuous. For each positive integer n and $(x,y) \in X \times M$, we define $D_{(x,y)} f^n : T_y M \to T_{\pi_2(f^n(x,y))} M$ as $v \mapsto D(g_{\sigma^n(x)} \circ \cdots \circ g_x)(v)$. Then we get the following result by a slight modification of Theorem 2. in [Ru].

Lemma 6.3. *Under the above, let ρ be an f-invariant probability measure on $X \times M$. Then,*

1. *there exists a Borel set Ω in $X \times M$ such that $\rho(\Omega) = 1$ and for each $(x,y) \in \Omega$ the following holds. There is a strictly increasing sequence of subspaces:*

$$0 = V_{x,y}^{(0)} \subset V_{x,y}(1) \subset \cdots \subset V_{x,y}^{(s(x,y))} = T_y M$$

such that, for $r = 1, \ldots, s(x,y)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_{(x,y)} f^n u\| = \lambda_{x,y}^{(r)} \text{ if } u \in V_{x,y}^{(r)} \setminus V_{x,y}^{(r-1)}$$

and $\lambda_{x,y}^{(1)} < \lambda_{x,y}^{(2)} < \cdots < \lambda_{x,y}^{(s(x,y))}$: here we may have $\lambda_{x,y}^{(1)} = -\infty$. The $V_{x,y}^{(r)}$ and $\lambda_{x,y}^{(r)}$ are uniquely defined with these properties and independent of the choice of the Riemannian metric on M. The maps $(x,y) \mapsto s(x,y)$, $(V_{x,y}^{(1)}, \ldots, V_{x,y}^{(s(x,y))})$, $(\lambda_{x,y}^{(1)}, \ldots, \lambda_{x,y}^{(s(x,y))})$ are Borel.

2. Let $m_{x,y}^{(r)} = \dim V_{x,y}^{(r)} - \dim V_{x,y}^{(r-1)}$ *for* $r = 1, \dots, s(x,y)$ *and define*

$$\lambda_+(x,y) = \sum_{r:\lambda_{x,y}^{(r)}>0} m_{x,y}^{(r)} \lambda_{x,y}^{(r)}.$$

Then, the metric entropy $h_\rho(f)$ *of* (f,ρ) *satisfies that*

$$h_\rho(f) \le \chi_\rho(f) + h_{(\pi_1)_*\rho}(\sigma),$$

where $\chi_\rho(f) = \int \lambda_+(x,y)d\rho(x,y)$.

Corollary 6.4. *Let* $G = \langle f_1, \dots, f_m \rangle$ *be a finitely generated rational semigroup and* $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ *the skew product map constructed by the generator system* $\{f_1, \dots, f_m.\}$ *Let* ρ *be an* $\tilde{f}$*-invariant probability measure on* $\Sigma_m \times \overline{\mathbb{C}}$. *Then we have*

$$h_\rho(\tilde{f}) \le 2\max\{0, \int_{\Sigma_m \times \overline{\mathbb{C}}} \lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{f}^n)'(z)\| d\rho(z)\} + h_{(\pi_1)_*\rho}(\sigma).$$

Let ρ be an $\tilde{f}$-invariant probability measure on $\Sigma_m \times \overline{\mathbb{C}}$. As in p108 in [P], there exists a ρ-integrable function $J_\rho : \Sigma_m \times \overline{\mathbb{C}} \to [1, \infty)$ such that

$$\rho(\tilde{f}(A)) = \int_A J_\rho(z) d\rho(z),$$

for any Borel set A in $\Sigma_m \times \overline{\mathbb{C}}$ such that $\tilde{f}_{|A}$ is injective. Now we will generalize some Mañé's results([Ma1]), using the methods in [Ma1] and Corollary 6.4.

Lemma 6.5. *Let* ρ *be an* $\tilde{f}$*-invariant ergodic probability measure on* $\Sigma_m \times \overline{\mathbb{C}}$ *with* $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. *Then the function* $z \mapsto \log \|\tilde{f}'(z)\|$ *is* ρ*-integrable and*

$$\int_{\Sigma_m \times \overline{\mathbb{C}}} log \|\tilde{f}'(z)\| d\rho(z) \ge \frac{1}{2}(h_\rho(\tilde{f}) - h_{(\pi_1)_*\rho}(\sigma)). \tag{30}$$

Moreover,

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{f}^n)'(z)\| = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log \|\tilde{f}'(z)\| d\rho(z), \tag{31}$$

for ρ*-almost all* $z \in \Sigma_m \times \overline{\mathbb{C}}$.

Proof. $\log \|\tilde{f}'(z)\|$ is upper bounded. Since ρ is ergodic, we have either $\log \|\tilde{f}'(z)\|$ is not ρ-integrable and then

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{f}^n)'(z)\| = -\infty \tag{32}$$

for ρ-a.e. $z \in \Sigma_m \times \overline{\mathbb{C}}$, or $\log \|\tilde{f}'\|$ is μ-integrable and :

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{f}^n)'(z)\| = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log \|\tilde{f}'(z)\| d\rho(z) \tag{33}$$

for ρ-a.e. $z \in \Sigma_m \times \overline{\mathbb{C}}$. By Corollary 6.4, we have (32) contradicts to our assumption. Hence (33) holds. Using again Corollary 6.4, we get that $\int_{\Sigma_m \times \overline{\mathbb{C}}} \log \|\tilde{f}'(z)\| d\rho(z) > 0$ and

$$h_\rho(\tilde{f}) \le 2 \int_{\Sigma_m \times \overline{\mathbb{C}}} \log \|\tilde{f}'(z)\| d\rho(z) + h_{(\pi_1)_*\rho}(\sigma).$$

$\square$

Corollary 6.6. *Let $x \in \overline{\mathbb{C}}$ be a critical point of some f_j, $j = 1, \dots, m$. We set $A = \{(w, x) \in \Sigma_m \times \overline{\mathbb{C}} \mid w_1 = j\}$. Then the function $z \mapsto \tilde{d}(z, A)$ is ρ-integrable for each ergodic $\tilde{f}$-invariant probability measure ρ with $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$.*

We set

$$\{x_1, \dots, x_b\} = \cup_{j=1}^m \mathrm{cp}(f_j),$$

where cp means the critical points. For each $j = 1, \dots, m$, we set

$$X_j = \{(w, x_j) \in \Sigma_m \times \overline{\mathbb{C}} \mid f'_{w_1}(x_j) = 0\}.$$

Then the following lemma holds.

Lemma 6.7. *For each k with $0 < k < 1$, there exists a continuous function τ on $\Sigma_m \times \overline{\mathbb{C}}$, a constant $C > 0$ and a constant $\alpha > 0$ such that*

1. *$\tau(z) \ge C \prod_{j=1}^b \tilde{d}(z, X_j)^\alpha$, (if $d_j = 1$ for each $j = 1, \dots, m$, then $\tau(z) \ge C$)*
2. *if $z \in (\Sigma_m \times \overline{\mathbb{C}}) \setminus \cup_{j=1}^b X_j$ and $\tilde{d}(z_1, z)$, $\tilde{d}(z_2, z) < \tau(z)$, then*

$$d_{\overline{\mathbb{C}}}(\pi_2(\tilde{f}(z_1)), \pi_2(\tilde{f}(z_2))) \ge k \|\tilde{f}'(z)\| d_{\overline{\mathbb{C}}}(\pi_2(z_1), \pi_2(z_2)).$$

Proof. By Lemma II.5 in [Ma1] and the proof of it, for each $i = 1, \dots, m$, there exists a continuous function τ_i, a constant $C_i > 0$ and a constant $\alpha_i > 0$ such that

1. $\tau_i(x) \ge C_i \prod_{k=1}^{b_i} d_{\overline{\mathbb{C}}}(x, y_k)^{\alpha_i}$, where $y_1, \dots, y_{b_i}$ are critical points of f_i. (if $d_i = 1$, then $\tau_i(x) \ge C_i$.)
2. if $x \in \overline{\mathbb{C}}$ is not a critical point of f_i and $d_{\overline{\mathbb{C}}}(a_1, x), d_{\overline{\mathbb{C}}}(a_2, x) < \tau(x)$, then

$$d_{\overline{\mathbb{C}}}(f_i(a_1), f_i(a_2)) \ge k \|f'_i(x)\| d_{\overline{\mathbb{C}}}(a_1, a_2).$$

We set $\tau(w,x) = \tau_{w_1}(x)$ for each $(w,x) \in \Sigma_m \times \overline{\mathbb{C}}$. Then there exists a constant $C > 0$ such that 1. of our lemma holds. We can assume $\sup_{z\in\Sigma_m\times\overline{\mathbb{C}}} \tau(z) < 1$. Then we can assume that if $\tilde{d}(z_1,z_2) < \sup_{z\in\Sigma_m\times\overline{\mathbb{C}}} \tau(z)$, then $\pi_1(z_1) = \pi_1(z_2)$. By the property of $\tau_i,\ i = 1,\dots m,$ we have 2. of our lemma holds. $\square$

We can show the following lemma using the same proof as that of Lemma 13.3 in [Ma2](with a slight modification).

Lemma 6.8. *Let ρ be an $\tilde{f}$-invariant probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ and $\tau : \Sigma_m\times\overline{\mathbb{C}} \to [0,1)$ a function such that $\log\tau$ is a ρ-integrable function. Then there exists a measurable partition $\mathcal{P}$ of $\Sigma_m \times \overline{\mathbb{C}}$ such that $h_\rho(\tilde{f},\mathcal{P}) < \infty$ and diam$\mathcal{P}(z) \le \tau(z)$ for ρ-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$, where $\mathcal{P}(z)$ denotes the atom of $\mathcal{P}$ containing z.*

Lemma 6.9. *Let ρ be an $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m\times\overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. Then there exists a measurable partition $\mathcal{P}$ of $\Sigma_m \times \overline{\mathbb{C}}$ such that $h_\rho(\tilde{f},\mathcal{P}) < \infty$ and $\mathcal{P}$ is a generator for $(\tilde{f},\rho)$ i.e. $\vee_{i=1}^{\infty}\tilde{f}^{-n}(\mathcal{P}) = \epsilon$ (mod 0) where ϵ denotes the partition of $\Sigma_m \times \overline{\mathbb{C}}$ into one point subsets.*

Proof. By Lemma 6.5, there exists a constant k with $0 < k < 1$ such that for ρ-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$,

$$\lim_{n\to\infty} \frac{1}{k^n}\|(\tilde{f}^n)'(z)\|^{-1} = 0. \tag{34}$$

For this k, take $\tau : \Sigma_m \times \overline{\mathbb{C}} \to [0,1)$ in Lemma 6.7. By Lemma 6.6 and Lemma 6.7, we have $\log\tau$ is ρ-integrable. By Lemma 6.8, we get that there exists a measurable partition $\mathcal{P}$ on $\Sigma_m \times \overline{\mathbb{C}}$ such that $h_\rho(\tilde{f},\mathcal{P}) < \infty$ and diam$\mathcal{P}(z) \le \tau(z)$ for ρ-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. We will show that $\mathcal{P}$ is a generator for $(\tilde{f},\rho)$. For each $n \in \mathbb{N}$, let $\mathcal{P}_n = \vee_{i=0}^{n}\tilde{f}^{-n}(\mathcal{P})$. It is sufficient to show that

$$\lim_{n\to\infty} \text{diam } \mathcal{P}_n(z) = 0 \tag{35}$$

for ρ-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Let $z_i \in \mathcal{P}_n(z),\ i = 1,2$. Then $\tilde{f}^j(z_i) \in \mathcal{P}(\tilde{f}^j(z)),\ i = 1,2,$ for all $j = 1,\dots n$. Since diam $\mathcal{P}(\tilde{f}^j(z)) \le \tau(\tilde{f}^j(z)),\ j = 1,\dots n,$ we have

$$d_{\overline{\mathbb{C}}}(\pi_2\tilde{f}^j(z_1),\ \pi_2\tilde{f}^j(z_2)) \ge k\|\tilde{f}'(\tilde{f}^{j-1}(z))\|d_{\overline{\mathbb{C}}}(\pi_2\tilde{f}^{j-1}(z_1),\ \pi_2\tilde{f}^{j-1}(z_2)),$$

for each $j = 1,\dots,n$. Hence we get

$$d_{\overline{\mathbb{C}}}(\pi_2\tilde{f}^n(z_1),\ \pi_2\tilde{f}^n(z_2)) \ge k^n\|(\tilde{f}^n)'(z)\|d_{\overline{\mathbb{C}}}(\pi_2(z_1),\ \pi_2(z_2)).$$

Let C be the diameter of $\overline{\mathbb{C}}$. We get

$$d_{\overline{\mathbb{C}}}(\pi_2(z_1),\ \pi_2(z_2)) \le C\cdot\frac{1}{k^n}\|(\tilde{f}^n)'(z)\|^{-1}. \tag{36}$$

Hence

$$\text{diam } \pi_2(\mathcal{P}_n(z)) \leq C \cdot \frac{1}{k^n} \|(\tilde{f}^n)'(z)\|^{-1}. \tag{37}$$

We can assume that for each $i = 1, \ldots m$, the set $Y_i = \{(w, x) \in \Sigma_m \times \overline{\mathbb{C}} \mid w_1 = i\}$ is a union of atoms of $\mathcal{P}$. Hence by (34) and (37), we get that (35) holds. Thus we have proved the lemma. $\square$

Lemma 6.10. *Let ρ be an $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. Then*

$$h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z) = \int_{\Sigma_m \times \overline{\mathbb{C}}} I(\epsilon | \tilde{f}^{-1}(\epsilon))(z) d\rho(z).$$

Proof. By Lemma 6.9, there exists a generator $\mathcal{P}$ with $h_\rho(\tilde{f}, \mathcal{P}) < \infty$. By Remark 8.10 and Lemma 10.5 in [P], we get $h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z)$. $\square$

Proof. of Theorem 6.2 Suppose $h(\tilde{f}) \leq \log m$. Then we have nothing to do. Suppose $h(\tilde{f}) > \log m$. Let ρ be any $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > \log m$. Then since $h(\sigma) = \log m$, by variational principle we get

$$h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma).$$

By Lemma 10.5 in [P] and Lemma 6.10, we have $I(\epsilon | \tilde{f}^{-1}\epsilon)(z) = \log J_\rho(z)$ and $h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z)$. Since $\tilde{f}$ is a $d : 1$ map where $d = \sum_{j=1}^m \deg(f_j)$, we have $I(\epsilon | \tilde{f}^{-1}\epsilon)(z) \leq \log(\sum_{j=1}^m \deg(f_j))$. Hence we get

$$h_\rho(\tilde{f}) \leq \log(\sum_{j=1}^m \deg(f_j)).$$

By the variational principle, we get

$$h(\tilde{f}) \leq \log(\sum_{j=1}^m \deg(f_j)).$$

$\square$

Theorem 6.11. *Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that there exists an element $g_0 \in G$ of degree at least two, the exceptional set $E(G)$ for G is included in $F(G)$ and $F(H) \supset J(G)$ where H is a rational semigroup defined by $H = \{h^{-1} \mid h \in Aut\,(\overline{\mathbb{C}}) \cap G\}$.(if H*

is empty, put $F(H) = \overline{\mathbb{C}}$.) Let $\tilde{\mu}_a$ be the self-similar measure with respect to the weight $a \in \mathcal{W}$(See Theorem 5.3). Then it is $\tilde{f}$-invariant and

$$h_{\tilde{\mu}_a}(\tilde{f}) = -\sum_{j=1}^{m} a_j \log a_j + \sum_{j=1}^{m} a_j \log d_j.$$

Also we have that $(\pi_1)_* \tilde{\mu}_a$ is the Bernoulli measure on Σ_m corresponding to the weight a. Moreover, let $\tilde{\mu}$ be the self-similar measure with respect to the weight $(\frac{d_1}{d}, \dots, \frac{d_m}{d})$. Then $\tilde{\mu}$ is the unique maximizing measure for $\tilde{f}$ and we have

$$h(\tilde{f}) = h_{\tilde{\mu}}(\tilde{f}) = \log(\sum_{j=1}^{m} \deg(f_j)).$$

Also we have $(\tilde{f}, \tilde{\mu}_a)$ is exact.

Proof. By Lemma 6.1 and Theorem 6.2, we have

$$h(\tilde{f}) = h_{\tilde{\mu}}(\tilde{f}) = \log(\sum_{j=1}^{m} \deg(f_j)).$$

Now assume there exists an $\tilde{f}$-invariant probability measure ρ on $\Sigma_m \times \overline{\mathbb{C}}$ with $\tilde{\mu} \neq \rho$ and $h_\rho(\tilde{f}) = \log d$ where $d = \sum_{j=1}^{m} \deg(f_j)$. We will show it causes a contradiction. We can assume ρ is ergodic. Since there exists an element $g \in G$ with the degree at least two, we have $\log d > \log m$. Hence $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. By Lemma 6.10, we have

$$h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z).$$

By Lemma 10.5 in [P], we have $I(\epsilon | \tilde{f}^{-1}\epsilon)(z) = \log J_\rho(z)$. Since $\tilde{f}$ is a $d:1$ map, we have $\log J_\rho(z) \le \log d$ for ρ almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Hence we get $\log J_\rho(z) = \log d$ for ρ almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. By Proposition 2.2 in [DU], we get that $\tilde{B}_a^*(\rho) = \rho$ where $a = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$ and $\tilde{B}_a$ denotes the operator on $C(\Sigma_m \times \overline{\mathbb{C}})$ defined in section 5. If $E(G) = \emptyset$, then by Theorem 5.3, we get $\rho = \tilde{\mu}$ and this is a contradiction. Assume $E(G) \neq \emptyset$. Let V be the union of connected components of $F(G)$ having non-empty intersection with $E(G)$. Let $\varphi \in C(\Sigma_m \times \overline{\mathbb{C}})$ be any element with $\varphi(z) \ge 0$ for all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Let $\epsilon > 0$ be any number. Let A_ϵ be the ϵ-open hyperbolic neighborhood in V. Then $K_\epsilon = \pi_2^{-1}(\overline{\mathbb{C}} \setminus A_\epsilon)$ is compact and backward invariant under $\tilde{f}$. Then by Theorem 5.3,

$$\begin{aligned}
\int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\rho(z) &= \int_{\Sigma_m \times \overline{\mathbb{C}}} (\tilde{B}_a^n \varphi)(z) d\rho(z) \\
&\ge \int_{K_\epsilon} (\tilde{B}_a^n \varphi)(z) d\rho(z) \\
&\to \rho(K_\epsilon) \cdot \int_{K_\epsilon} \varphi(z) d\tilde{\mu}(z),
\end{aligned}$$

as $n \to \infty$. Hence we have for each $\epsilon > 0$,

$$\int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\rho(z) \geq \rho(K_\epsilon) \cdot \int_{K_\epsilon} \varphi(z) d\tilde{\mu}(z).$$

Since $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$ and ρ is ergodic, we have $\rho(\pi_2^{-1}(E(G))) = 0$. Letting $\epsilon \to 0$, we get

$$\int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\rho(z) \geq \int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\tilde{\mu}(z).$$

It implies that $\rho \geq \tilde{\mu}$. Since ρ and $\tilde{\mu}$ are probability measures, it follows that $\rho = \tilde{\mu}$ but it is a contradiction. $\square$

Now we consider a generalization of Mañé's result([Ma3]).

Theorem 6.12. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. We define a map $f : J(G) \to J(G)$ by $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$. If μ is an ergodic invariant probability measure for $f : J(G) \to J(G)$ with $h_\mu(f) > 0$, then*

$$\int_{J(G)} \log(\|f'\|)\ d\mu > 0$$

and

$$HD(\mu) = \frac{h_\mu(f)}{\int_{J(G)} \log(\|f'\|)\ d\mu},$$

where we set

$$HD(\mu) = \inf\{\dim_H(Y) \mid Y \subset J(G),\ \mu(Y) = 1\}.$$

Proof. We can show the statement in the same way as [Ma3]. Note that the Ruelle's inequality([Ru]) also holds for the map $f : J(G) \to J(G)$. $\square$

By Theorem 6.11 and Theorem 6.12, we get the following result.

Theorem 6.13. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(H) \supset J(G)$ where $H = \{h^{-1} \mid h \in Aut(\overline{\mathbb{C}}) \cap G\}$ (if $H = \emptyset$, put $F(H) = \overline{\mathbb{C}}$.) Also assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. Then*

$$\dim_H(J(G)) \geq \frac{\log(\sum_{j=1}^m \deg(f_j))}{\int_{J(G)} \log(\|f'\|)\ d\mu},$$

where $\mu = (\pi_2)_*\tilde{\mu}_a$, $a = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$ and $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$.

References

[Bo] D.Boyd, *Invariant measure for finitely generated rational semigroups,* to appear in Complex Variables.

[Br] H.Brolin, *Invariant sets under iteration of rational functions,* Ark.Mat.**6**(1965), 103-144.

[DU] M.Denker, M.Urbański, *On the existence of conformal measure,* Trans.Ame.Math.Soc. Vol.328, No.2, 1991.

[HM1] A.Hinkkanen, G.J.Martin, *The Dynamics of Semigroups of Rational Functions I*, Proc.London Math.Soc. (3)**73**(1996), 358-384.

[HM2] A.Hinkkanen, G.J.Martin, *Julia Sets of Rational Semigroups* , Math.Z. **222**, 1996, no.2, 161-169.

[HM3] A.Hinkkanen, G.J.Martin, *Some Properties of Semigroups of Rational Functions,* XVIth Rolf Nevanlinna Colloquium(Joensuu,1995), 53-58, de Gruyter, Berlin, 1996.

[J] M.Jonsson, *Ergodic properties of fibered rational maps and applications,* preprint, http://www.math.lsa.umich.edu/~mattiasj/

[L] M.J.Lyubich, *Entropy Properties of Rational Endomorphisms of the Riemann Sphere,* Ergod.th.&Dynam.Sys. (1983), **3**, 351-385

[Ma1] R.Mañé, *On the uniqueness of the maximizing measure for rational maps,* Bol.Soc.Bras.Mat.Vol.14 No.1 (1983), 27-43.

[Ma2] R.Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, 1987.

[Ma3] R.Mañé, *The Hausdorff dimension of invariant probabilities of rational maps,* Dynamical systems, Valparaiso 1986, Lecture Notes in Mathematics 1331(Springer, Berlin, 1988)86-117.

[MTU] S.Morosawa, M. Taniguchi and T. Ueda *A Primer on Complex Dynamics,* (Japanese version; Baihuukan, 1995) English version, in preparation.

[P] W.Parry, *Entropy and generators in ergodic theory,* W.A.Benjamin Inc.1969.

[Re] F.Ren, Advances and problems in random dynamical systems, preprint.

[ZR] W.Zhou, F.Ren, The Julia sets of the random iteration of rational functions, Chinese Bulletin, **37**(12), 1992, 969-971.

[GR] Z.Gong, F.Ren, A random dynamical system formed by infinitely many functions, Journal of Fudan University, **35**, 1996, 387-392.

[Ru] D.Ruelle, *An inequality for the entropy,* Bol. Soc.Bras.Mat.vol.9, No.1(1978), 83-87.

[Sul] D.Sullivan, *Conformal Dynamical System, in Geometric Dynamics,* Springer Lecture Notes **1007**(1983), 725-752.

[S1] H.Sumi, *On Dynamics of Hyperbolic Rational Semigroups,* Journal of Mathematics of Kyoto University, Vol.37, No.4, 1997, pp.717-733.

[S2] H.Sumi, *On Hausdorff Dimension of Julia Sets of Hyperbolic Rational Semigroups,* Kodai Mathematical Journal, Vol.21, No.1, pp.10-28, 1998.

[S3] H.Sumi, *Dynamics of Sub-hyperbolic and Semi-hyperbolic Rational Semigroups and Skew Products* , preprint 1998.

[S4] H.Sumi, *Dynamics of rational semigroups and Hausdorff dimension of the Julia sets*, Thesis, Graduate School of Human and Environmental Studies, Kyoto University, 1999.