UNIFORM PERFECTNESS OF THE JULIA SETS OF QUADRATIC POLYNOMIALS

TOSHIYUKI SUGAWA

Abstract. The author gave in [12] an explicit estimate of uniform perfectness of the Julia sets of general rational maps of degree $d \geq 2$. In this note, we will present some exapmles of such an estimate for quadratic polynomials.

1. Uniform perfectness of Julia sets

A compact set C with $\#C > 1$ in the Riemann sphere $\widehat{\mathbb{C}}$ is said to be *uniformly perfect* if $C \cap \{z \in \mathbb{C}; cr < |z-a| < r\} \neq \emptyset$ for any $a \in C \setminus \{\infty\}$ and $0 < r < \operatorname{diam} C$, where $0 < c < 1$ is a constant and diam stands for the diameter with respect to the Euclidean metric. It is easy to see that $C = \widehat{\mathbb{C}} \setminus D$ is uniformly perfect if and only if finite is the supremum M_D° of the moduli of essential round annuli in D separating C, where the modulus of the round annulus $\{z; r_1 < |z-a| < r_2\}$ is defined by $\log r_2/r_1$ and an annulus in D is said to separate C if both of two components of the complement of the annulus intersect C. Teichmüller's theorem tells us that the condition $M_D^\circ < \infty$ is equivalent to finiteness of the supremum M_D of the moduli of essential (not necessarily round) annuli in D separating C. We remark that the above equivalences are quantitative (for more information, see [13]).

The notion of uniform perfectness first appeared in [1] and then was intensively investigated by Pommerenke in [8] and [9]. In [9] Pommerenke proved that the Julia set of a hyperbolic rational map of degree≥ 2 is uniformly perfect. Afterwards, Mañé-da Rocha [7] and Hinkkanen [4] independently proved the uniform perfectness for general rational maps of degree≥ 2. But their proofs are made by contradiction argument, thus no explicit bounds for uniform perfectness were given so far. Recently, the author gave in [12] another proof for uniform perfectness of the Julia sets of general rational maps with explicit bounds by using the hyperbolic geometry. We shall present this estimate in the following.

We begin with several definitions and notation needed later. Let C be a compact set in $\widehat{\mathbb{C}}$ containing at least three points and D its complement. Then each component D_0

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of D has a holomorphic universal covering map $p : \mathbb{D} \to D_0$ from the unit disk $\mathbb{D}$. Since the hyperbolic metric $\rho_{\mathbb{D}} = (1-|z|^2)^{-1}|dz|$ is invariant under the action of the cover transformation group $\varGamma <$ Möb, this metric induces a Riemannian metric $\rho_{D_0}(z)|dz|$, which is also called the hyperbolic metric, that is, $\rho_{D_0}(p(z))|p'(z)| = (1-|z|^2)^{-1}|dz|$. We define ρ_D by $\rho_D = \rho_{D_0}$ on each component D_0. For a piecewise smooth curve α in D the hyperbolic length $\ell_D(\alpha)$ of α is defined as $\int_\alpha \rho_D(z)|dz|$. We set

$$d_D(z,w) = \inf_\alpha \ell_D(\alpha), \quad \iota_D(z) = \frac{1}{2}\inf_\beta \ell_D(\beta),$$

where α runs over all curves joining z and w in D and β runs over all nontrivial loops passing through z in D, which are called the hyperbolic distance and the injectivity radius of D, respectively. Let L_D be the infimum of hyperbolic lengths of nontrivial loops in D. (If D is simply connected we define $L_D = +\infty$.) In other words, $L_D = 2\inf_{z\in D}\iota_D(z)$. The following result is essential for our argument.

Theorem 1.1 ([13]). *The following inequalities hold for any hyperbolic open set D of $\widehat{\mathbb{C}}$ with the hyperbolic metric.*

$$\frac{1}{2}M_D - 1.7332\cdots \le M_D^\circ \le M_D, \ \text{and}$$

$$L_D \le \frac{\pi^2}{M_D} \le \min\{L_D e^{L_D}, \frac{L_D^2}{2}\coth^2(L_D/2)\}.$$

In particular, the complement $\widehat{\mathbb{C}} \setminus D$ is uniformly perfect if and only if $L_D > 0$.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \ge 2$ and denote by J_f and $\varOmega_f$ the Julia set and the Fatou set of f. In other words, $\varOmega_f$ is the maximal open set where the iterates $f^n \quad (n = 1, 2, \cdots)$ form a normal family and J_f is its complement. Since $d \ge 2$ the Julia set is non-empty and perfect, hence J_f is uncountable. In particular, $\varOmega_f$ is hyperbolic. Let $U_1, \cdots, U_s$ be a complete list of the components of $\varOmega_f$ containing a critical point of f. The number of critical points of f with counting multiplicities is known to be $2d-2$, thus $s \le 2d-2$. We denote by C_j the set of critical points of f contained in U_j for $j = 1, \cdots, s$. Then we consider the following two families of loops in $W_j = f(U_j)$. For $v_1, v_2, v \in f(C_j)$ with $v_1 \ne v_2$ we set

$$\begin{aligned}\mathcal{S}(v_1,v_2) =& \{\beta; \beta \text{ is a trivial loop in } W_j \text{ through } v_1, v_2 \text{ with a nontrivial lift in } U_j\},\\ \mathcal{T}(v) =& \{\beta; \beta \text{ is a trivial loop in } W_j \text{ through } v \text{ essentially}\\ & \quad \text{at least two times with a nontrivial lift in } U_j\}.\end{aligned}$$

More precisely, the statement that a loop $\beta : \mathrm{S}^1 \to W_j$ passes through v at least two times means that there exist distinct points ζ_0 and ζ_1 in S^1 with $\beta(\zeta_0) = \beta(\zeta_1) = v$ such that the restrictions $\beta|_{\overline{I_1}}$ and $\beta|_{\overline{I_2}}$ of the loop β are both nontrivial closed curves in W_j, where I_1 and I_2 are the connected component of $\mathrm{S}^1 \setminus \{\zeta_0, \zeta_1\}$.

And we set

$$a_j(v_1, v_2) = \inf_{\beta \in \mathcal{S}(v_1,v_2)} \ell_\Omega(\beta), \quad b_j(v) = \inf_{\beta \in \mathcal{T}(v)} \ell_\Omega(\beta) \quad \text{and}$$

$$a_j = \min_{v_1,v_2 \in f(C_j), v_1 \neq v_2} a_j(v_1, v_2), \quad b_j = \min_{v \in f(C_j)} b_j(v),$$

where we set $a_j = +\infty$ if $\# f(C_j) = 1$.

Next, let $A_1, \cdots, A_t$ be the complete system of representatives of the cycles of Herman rings of f. Remark that Shishikura's theorem says that $0 \leq t \leq d-2$, in particular, if $d = 2$ there are no Herman rings. And, since the Julia set has no isolated points, the Herman rings have finite moduli, so $L_{A_k} > 0$ for all k.

Now we are in a position to state the above-mentioned theorem.

Theorem 1.2 ([12]).

$$L_{\Omega_f} \geq \min\{a_1, \cdots, a_s, b_1, \cdots, b_s, L_{A_1}, \cdots, L_{A_t}\}.$$

Since $\beta \in \mathcal{S}(v_1, v_2)$ satisfies $\ell_{W_j}(\beta) \geq 2d_{W_j}(v_1, v_2)$, we have $a_j(v_1, v_2) \geq 2d_{W_j}(v_1, v_2)$. Similarly we obtain $b_j(v) \geq 4\iota_{W_j}(v)$. Hence, we have the following

Corollary 1.3. *Let V be the set of critical values of f contained in the Fatou set Ω_f. Then we have*

$$L_{\Omega_f} \geq \min\{K_1, K_2, K_3\},$$

where

$$K_1 = \min_{v_1 \neq v_2 \in V} 2d_{\Omega_f}(v_1, v_2), \quad K_2 = \min_{v \in V} 4\iota_{\Omega_f}(v)$$

and $K_3 = \min\{L_{A_1}, \cdots, L_{A_t}\}$. In particular, $L_{\Omega_f} > 0$, equivalently, J_f is uniformly perfect.

In order to prove the above theorem, first we remark that the Schwarz-Pick lemma implies $\ell_{\Omega_f}(\alpha) \geq \ell_{\Omega_f}(f_*\alpha)$ for any loop α in Ω_f. For any nontrivial loop α in Ω_f, by virtue of Sullivan's No Wandering Domains Theorem, the image curve $\alpha_n = (f^n)_*\alpha$ becomes contractible for sufficiently large n unless α_n lands on some Herman ring (in this case, $\ell_{\Omega_f}(\alpha) \geq \ell_{\Omega_f}(\alpha_n) \geq K_3$), thus the following lemma completes the proof.

Lemma 1.4. *Let $f : U \to W$ be a branched holomorphic covering map between hyperbolic Riemann surfaces. We define two families of curves $\mathcal{S}(v_1, v_2)$ and $\mathcal{T}(v)$ for critical values v_1, v_2, v with $v_1 \neq v_2$ of f as in the same way as above. And define $a(v_1, v_2), b(v)$ as in the above and set $a = \inf_{v_1 \neq v_2} a(v_1, v_2)$ and $b = \inf_v b(v)$. Then, for any nontrivial loop α such that $f_*\alpha$ is trivial, it holds that*

$$\ell_U(\alpha) \geq \ell_W(f_*\alpha) \geq \min\{a, b\}.$$

For a rigorous proof of this lemma, see [12]. In this note, let us be content to exhibit only a simplified example illustrating an essence of the proof.

For an $r > 1$ we set $U = U(r) = \{z \in \mathbb{C}; 1/r < |z| < r\}$. And, we consider the Jeukowsky transformation $f(z) = z + 1/z$ and put

$$f(U(r)) = W(r) = \left\{ w = u + iv \in \mathbb{C}; \left(\frac{u}{r+1/r}\right)^2 + \left(\frac{v}{r-1/r}\right)^2 < 1 \right\}.$$

Then $f : U \to W$ is a two-sheeted branched analytic covering with critical points ± 1. Let α be a nontrivial loop in U, then one can observe that the image loop $\beta = f_*\alpha$ "surrounds" both critical values ± 2. Thus it is not difficult to see that $\ell_W(\beta) \geq a(2,-2) = 2d_W(2,-2)$. Since W is simply connected, it holds that $1/4 \leq \delta_W(z)\rho_W(z) \leq 1$, where $\delta_W(z) = \inf_{a\in\partial W} |z-a|$. Letting $A = r + 1/r$, then we have

$$\begin{aligned} d_W(2,-2) &\asymp 2\int_0^2 \frac{dx}{\delta_W(x)} = 2\int_0^{4/A} \frac{dx}{(r-1/r)\sqrt{1-x^2/4}} + 2\int_{4/A}^2 \frac{dx}{A-x} \\ &= \frac{2}{r-1/r} \arcsin\left(\frac{2}{r+1/r}\right) + 2\log\left(1 + \frac{2}{r+1/r}\right) \end{aligned}$$

In particular, we note that $d_{W(r)}(2,-2) \sim 1/(r-1)$ as $r \to 1+$ and $d_{W(r)}(2,-2) \sim 1/r$ as $r \to +\infty$. In this case, in fact, we can calculate $L_{U(r)}$ explicitly as $\pi^2/\log r$. Thus, the estimate $L_U \geq 2d_W(2,-2)$ is not so good when r tends to $+\infty$.

Next, let φ be a conformal map from W onto the unit disk $\mathbb{D}$ so that $\xi := \varphi(2) > 0$ and $\eta := \varphi(-2)$ is very close but not equal to $-\xi$. Now we consider the branched covering map $g : U \to \mathbb{D}$ of degree 4 defined by $g(z) = \varphi(f(z))^2$. Then the critical values of g is ξ^2, η^2 and 0. In this case, $d_{\mathbb{D}}(\xi^2, \eta^2)$ is very close to zero but $a(\xi^2, \eta^2)$ is not so small because any element of $\mathcal{S}(\xi^2, \eta^2)$ goes a long way round another critical value 0. Hence we cannot expect that the estimate in Corollary 1.3 would be always sufficiently good.

We conclude this section by giving some applications of the estimate of uniform perfectness. The first is due to Pommerenke.

Theorem 1.5 (Pommerenke [9]). *A compact set C with $\#C > 1$ in $\widehat{\mathbb{C}}$ is uniformly perfect if and only if there exists a positive constant c such that*

$$\mathrm{Cap}(C \cap B(a,r)) \geq cr$$

for any $a \in C$ and $0 < r < \mathrm{diam} C$, where Cap *denotes the logarithmic capacity and $B(a,r) = \{z; |z-a| \leq r\}$. In particular, a uniformly perfect set is regular in the sense of Dirichlet.*

In fact, the above constant c is explicitly estimated by the uniform perfectness constant and vice versa (see [8]). The next result is essentially due to Järvi-Vuorinen [5], however the following quantitative form appeared in [13].

Theorem 1.6. *The Hausdorff dimension of a uniformly perfect set $C = \widehat{\mathbb{C}} \setminus D$ can be estimated as*

$$\text{H-dim} C \geq \frac{\log 2}{\log(2e^{M_D^\circ}+1)} \geq \frac{\log 2}{M_D^\circ + \log 3} \geq \frac{\log 2}{\pi^2/L_D + \log 3}.$$

2. Explicit estimate in quadratic polynomial case

In this section, we shall present a typical example of estimating the constant L_{Ω_f}. We consider here the quadratic polynomial $f_c(z) = z^2 + c$ where c is a parameter in $\mathbb{C}$. For brevity, we set $J_c = J_{f_c}$ and $\Omega_c = \Omega_{f_c}$ etc. Our aim here is to estimate $L_c = L_{\Omega_c}$ from below. If c is in the Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C}; \{f^n(0); n \in \mathbb{N}\}$ is bounded $\}$, then it is known that J_c is connected, so $L_c = +\infty$. Therefore we have nothing to do in this case. In the case $c \notin M$ the Julia set J_c is known to be a totally disconnected (Cantor type) set and $\Omega_c = \{z \in \widehat{\mathbb{C}}; f^n(z) \to \infty \quad \text{as } n \to \infty\}$ is connected. The critical values of f are c and ∞, so what we should do is estimate the hyperbolic distance $d_c(c, \infty)$ and the injectivity radii $\iota_c(c)$ and $\iota_c(\infty)$ by Corollary 1.3.

For simplicity, we shall make an additional assumption that $c < -2$. Then J_c is contained in the interval $[-\alpha, \alpha]$, where $\alpha = (1 + \sqrt{1-4c})/2$ is a fixed point of f_c with $\alpha > 2$. We denote another fixed point of f_c by β. Note here that $\alpha + \beta = 1$, and $\pm\alpha, \beta \in J_c$. In particular, the hyperbolic domain $D_0 = \widehat{\mathbb{C}} \setminus \{\alpha, -\alpha, \beta\}$ contains Ω_c, thus the Schwarz-Pick lemma yields that $\rho_{D_0} \leq \rho_c = \rho_{\Omega_c}$ on Ω_c. Let T be the Möbius transformation taking $\alpha, -\alpha$ and β to $\infty, 0$ and 1, respectively. Then T can be expressed by $T(z) = (3+t)(\alpha+z)/(\alpha - z)$, thus $T(\infty) = -(3+t)$ and $T(c) = -t(3+t)/(4+t)$, where $t = \sqrt{1-4c} - 3 > 0$. We notice that $T(J_c) \subset [0, +\infty]$. The bihaviour of the hyperbolic metric $\sigma(z)|dz|$ of the canonical domain $D = \widehat{\mathbb{C}} \setminus \{\infty, 0, 1\}$ is well understood. For instance, the precise version of Landau's theorem due to Hempel [3] says that

$$\sigma(z) \geq \frac{1}{2|z|(|\log|z|| + H)}, \tag{2.1}$$

where $H = \Gamma(\frac{1}{4})^4/4\pi^2 = 4.3768796\cdots$ and the equality occurs when $z = -1$. Note that this inequality is actually efficient only on the half plane $\text{Re} z \leq \frac{1}{2}$, otherwise we have only to note that $\sigma(1-z) = \sigma(z)$. We also note that Solynin and Vuorinen [11] gave a recursive procedure for computing $\sigma(z)$.

In order to estimate the metric σ, we shall analyze the universal covering map of D. Let Δ be the domain $\{\tau \in \mathbb{H}; 0 < \text{Im}\tau < 1, |\tau - \frac{1}{2}| > \frac{1}{2}\}$, and $\lambda : \Delta \to \mathbb{H}$ the conformal homeomorphism from Δ onto the upper half plane $\mathbb{H}$ taking $0, 1, \infty$ to $1, \infty, 0$, respectively. By the reflection principle, λ is analytically continued to the holomorphic universal covering map of D from the upper half plane, which will be also denoted by λ, in particular, we see $1/2\mathrm{Im}\tau = \sigma(\lambda(\tau))|\lambda'(\tau)|$. The map $\lambda : \mathbb{H} \to D$ is nothing more than the classical elliptic modular function. Noting that the hyperbolic line $\{z \in \mathbb{H}; \mathrm{Re} z = 1\}$ is mapped onto $(-\infty, 0)$ by λ, one can show that the segment $\gamma = [T(\infty), T(c)]$ is the shortest hyperbolic geodesic joining $T(\infty)$ and $T(c)$ in D. Thus we can estimate as

$$
\begin{aligned}
d_W(\infty, c) &\geq d_{\widehat{\mathbb{C}}\setminus\{\alpha,-\alpha,\beta\}}(\infty, c) = d_D(T(\infty), T(c)) = \int_\gamma \sigma(z)|dz| \\
&\geq \int_\gamma \frac{|dz|}{2|z|(|\log|z|| + H)} = \frac{1}{2}\log\frac{\log(3+t)+H}{H} + \frac{1}{2}\left|\log\frac{\log\frac{4+t}{t(3+t)}+H}{H}\right|.
\end{aligned}
$$

Next, we shall explain the estimation of the injectivity radii. We remark that $\Omega_f \subset \Omega$ does not necessarily imply $\iota_{\Omega_f}(z) \geq \iota_\Omega(z)$, however $\rho_{\Omega_f} \geq \rho_\Omega$ implies that $\iota_{\Omega_f}(z) \geq \inf_{w\in\partial V} d_{\Omega_f}(z,w) \geq \inf_{w\in\partial V} d_\Omega(z,w)$ for any simply connected subdomain V of Ω_f containing z. We shall take $\widehat{\mathbb{C}} \setminus [-\alpha, \alpha]$ as V. In this case, for any $x \in [-\infty, -\alpha) \cup (\alpha, +\infty]$, we have

(2.2)
$$
\iota_{\Omega_f}(x) \geq \inf_{y\in[-\alpha,\alpha]} d_{\Omega_f}(x,y) \geq \inf_{y\in[-\alpha,\alpha]} d_{\widehat{\mathbb{C}}\setminus\{\alpha,-\alpha,\beta\}}(x,y) = \inf_{s>0} d_D(T(x), s) = \iota_D(T(x)).
$$

Now we estimate $I(a) = \iota_D(-a) = \iota_D(1+a)$ for $a > 0$. Let $\varphi : \mathbb{H} \to \Delta$ be the inverse map of $\lambda : \Delta \to \mathbb{H}$. Since Δ is a Jordan domain, Carathéodory's theorem implies that φ extends to a homeomorphism from $\overline{\Delta}$ onto $\overline{\mathbb{H}}$. First we assume that $0 < a \leq 1$. Then $\tau_0 = \varphi(1+a)$ can be expressed by $(e^{i\theta}+1)/2$ with $\pi/2 \leq \theta < \pi$. Thus $d_{\mathbb{H}}(\tau_0, \varphi((0,1))) \leq d_{\mathbb{H}}(\tau_0, \varphi((-\infty,0)))$ and the shortest hyperbolic segment γ joining τ_0 and $\varphi((0,1)) = \{yi; y > 0\}$ is contained in $\{\tau \in \overline{\Delta}; \mathrm{Re}\tau \leq \frac{1}{2}\}$. Because $\lambda(\{\tau \in \Delta; \mathrm{Re}\tau = \frac{1}{2}\}) = \{z \in \mathbb{H}; |z-1| = 1\}$, it follows that $\iota_D(1+a) = \int_{\lambda_*\gamma} \sigma(z)|dz|$ and $\lambda_*\gamma$ is contained in $\{|z-1| \leq 1\}$. Let ν the loop in D obtained as the union of $1 - \lambda_*\gamma$ and its complex conjugate. Then $|\nu| \leq 1$ and $2I(a) = \int_\nu \sigma(z)|dz|$. Set $a_0 = \min|\nu|$. Noting that $|dz| \geq (|dr| + r|d\theta|)/\sqrt{2}$, where $z = re^{i\theta}$, we have

$$
\begin{aligned}
2I(a) &\geq \int_\nu \frac{|dz|}{2|z|(-\log|z| + H)} \geq \int_\nu \frac{|dr| + r|d\theta|}{2r\sqrt{2}(-\log r + H)} \\
&\geq \frac{2}{2\sqrt{2}}\log\left(\frac{-\log a_0 + H}{-\log a + H}\right) + \frac{1}{2\sqrt{2}}\frac{2\pi}{-\log a_0 + H} \\
&\geq \frac{\pi/\sqrt{2}}{-\log a + H},
\end{aligned}
$$

since the function $h(x) = \log x + \pi/x$ is increasing in $x > 0$ thus $h(x) > h(H) > 0$ for $x > H$. Hence $I(a) \geq \pi/2\sqrt{2}(-\log a + H)$ for $0 < a \leq 1$.

In the case $1 < a$, by $I(a) = I(1/a)$, we have $I(a) \geq \pi/2\sqrt{2}(\log a + H)$. In any case, $I(a) \geq \pi/2\sqrt{2}(|\log a| + H)$. Combining this with (2.2), we obtain

$$\begin{aligned} K &:= \min\{2d_c(c,\infty), 4\iota_c(c), 4\iota_c(\infty)\} \\ &\geq \min\left\{\log\frac{\log(3+t)+K}{K} + \left|\log\frac{\log\frac{4+t}{t(3+t)}+K}{K}\right|, \frac{\sqrt{2}\pi}{\log(3+t)+K}, \frac{\sqrt{2}\pi}{\log\frac{4+t}{t(3+t)}+K}\right\} \\ &= \frac{\sqrt{2}\pi}{\log m + K}, \end{aligned}$$

where $m = \max\{3+t, \frac{4+t}{t(3+t)}\}$. By Corollary 1.3 we have the next

Theorem 2.1. *For $c < -2$, the Fatou set Ω_c of $f_c(z) = z^2 + c$ satisfies*

$$L_{\Omega_c} \geq \frac{\sqrt{2}\pi}{\log m + H},$$

where $m = \max\{\sqrt{1-4c}, \frac{\sqrt{1-4c}+1}{\sqrt{1-4c}(\sqrt{1-4c}-3)}\}$ and $H = \Gamma(\frac{1}{4})^4/4\pi^2 = 4.3768796\cdots$.

On the other hand, it is relatively easy to obtain an upper bound for L_c. For simplicity, we assume that $c < -2$ again and use the same notation as above. Let $\gamma = \sqrt{-c-\alpha} > 0$, then one can observe that

$$f_c(x) = x^2 + c < \gamma^2 + c = -\alpha$$

for $x \in (-\gamma, \gamma)$, thus $(-\gamma, \gamma) \subset \Omega_c$. This says that the annulus $A = \widehat{\mathbb{C}} \setminus ([\gamma, \alpha] \cup [-\alpha, -\gamma])$ separates the Julia set J_c. By definition, we have $M_{\Omega_c} \geq m(A)$. Here we note that A is conformally mapped to Teichmüller's extremal domain $\widehat{\mathbb{C}} \setminus ([-r_1, 0] \cup [r_2, +\infty])$ by the Möbius transformation $T(z) = \frac{\gamma+z}{\alpha-z}$, where $r_1 = (\alpha-\gamma)/2\alpha$ and $r_2 = 2\gamma/(\alpha-\gamma)$. It is known that $m(A) = 2\mu(\sqrt{r_1/(r_1+r_2)})$, where $\mu(r)$ denotes the modulus of Grötzsch's extremal domain $\mathbb{D} \setminus [0, r]$ for $0 < r < 1$ and this quantity satisfies the following (cf. [6]):

$$\log\frac{(1+\sqrt{1-r^2})^2}{r} < \mu(r) < \log\frac{2(1+\sqrt{1-r^2})}{r} < \log\frac{4}{r} \quad \text{and}$$

$$\mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2}.$$

In particular, we obtain

$$m(A) = 2\mu\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right) = \frac{\pi^2}{\mu(\gamma/\alpha)}.$$

Combining these with Thoerem 1.1, we get

Theorem 2.2. *For $c < -2$, the Fatou set Ω_c of $f(z) = z^2 + c$ satisfies*

$$L_{\Omega_c} \leq \frac{\pi^2}{M_{\Omega_c}} \leq \frac{\pi^2}{2\mu\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right)} = \mu\left(\frac{\gamma}{\alpha}\right),$$

where $\alpha = (1+\sqrt{1-4c})/2 > 2$ and $\gamma = \sqrt{-c-\alpha} > 0$.

Since $(\alpha-\gamma)/(\alpha+\gamma) = 1/2\sqrt{-c}+O(|c|^{-1})$ as $c \to -\infty$ and $\gamma/\alpha = \sqrt{\varepsilon/6}(1+O(\varepsilon))$ as $\varepsilon := -c-2 \to +0$, we can see that $L_{\Omega_c} \leq \frac{\pi^2}{\log|c|}(1+o(1))$ as $c \to -\infty$ and $L_{\Omega_c} \leq \log\frac{1}{\varepsilon}(1+o(1))$ as $\varepsilon = -c-2 \to +0$.

3. Estimate by Green's function

In this section, we explain an estimate of the uniform perfectness constant by Green's function, which is weaker than one in the previous section but easier to make in most cases. Let $f(z) = c_0 z^d + \cdots + c_{d-1}z + c_d$ be a polynomial of degree $d \geq 2$ and Ω the immediate basin of ∞. (More generally, f may be a rational map with a super attractive fixed point z_0 such that $f^{-1}(z_0) \cap \Omega = \{z_0\}$, where Ω denotes the immediate basin of z_0.)

As is well-known (cf. [2]), Green's function $g(z) = G(z,\infty)$ of Ω with pole at ∞ can be expressed by

$$g(z) = \lim_{n\to\infty} d^{-n}\log|f^n(z)|.$$

Since $g(z) = \log|z| + (d-1)^{-1}\log|c_0| + o(1)$ as $z \to \infty$, we know that $\mathrm{Cap}(J_f) = 1/\sqrt[d-1]{|c_0|}$. We also note that the following functional equation:

$$g(f(z)) = d \cdot g(z). \tag{3.1}$$

On the other hand, by Myrberg's theorem (cf. [14]), Green's function $G(z,a)$ of the domain Ω with pole at a can be written by

$$G(p(z),a) = \sum_{\gamma\in\Gamma} \log\left|\frac{1-\overline{\gamma(0)}z}{z-\gamma(0)}\right|,$$

where $p : \mathbb{D} \to \Omega$ is a holomorphic universal covering of Ω with $p(0) = a$ and Γ its cover transformation group. For a $w = p(z) \in \Omega$ with $z \in \mathbb{D}$ there exists a $\gamma \in \Gamma$ such that $d_\Omega(w,a) = d_{\mathbb{D}}(z,\gamma(0)) = \operatorname{arctanh}\left|\frac{z-\gamma(0)}{1-\overline{\gamma(0)}z}\right|$, therefore

$$G(w,a) \geq \log\left|\frac{1-\overline{\gamma(0)}z}{z-\gamma(0)}\right| = -\log\tanh(d_\Omega(w,a)),$$

equivalently,

$$d_\Omega(w,a) \geq -\frac{1}{2}\log\tanh(G(w,a)/2). \tag{3.2}$$

For simplicity, we assume that $f(z) = f_c(z) = z^2 + c$ with c outside the Mandelbrot set in the sequel. In this case, 0 and its backward orbit under f form the set of critical points for Green's function $g_c(z) = G_c(z, \infty)$, thus we can see that the subdomain $\Omega' = \{z \in \Omega; g_c(z) > g_c(0)\}$ is simply connected.

Now we consider the Böttcher coordinate ψ of f at ∞, i.e., $\psi(z) = \lim_{n\to\infty}(f^n(z))^{-d^{-n}}$ satisfying the functional equation $\psi(f(z)) = \psi(z)^2$, by which ψ can be analytically continued near any point z_0 so far as ψ is already defined near $f(z_0)$. Therefore, $\varphi = \psi \circ p$ can be analytically continued to a holomorphic map from $\mathbb{D}$ to itself, where $p : \mathbb{D} \to \Omega$ is a universal covering map of Ω with $p(0) = \infty$. Let V be the connected component of $p^{-1}(\Omega')$ containing 0 and $s : \Omega' \to V$ the inverse map of $p|_V$. We note that $\varphi(V) = \{|z| < e^{-g_c(0)}\}$.

Since Ω' is simply connected, any nontrivial loop γ passing through c must escape from Ω', thus it contains two parts γ_1 and γ_2 both of which start from c and end at some points in $\partial\Omega'$ and entirely contained in $\overline{\Omega}'$. Since $|\varphi(s(c))| = e^{-g_c(c)}$ and $g_c(c) = 2g_c(0)$ by (3.1), we can estimate as

$$\begin{aligned}\ell_c(\gamma_j) = \ell_{\mathbb{D}}(s_*\gamma_j) \geq \ell_{\mathbb{D}}(\varphi_*(s_*\gamma_j)) &\geq d_{\mathbb{D}}(e^{-g_c(c)}, e^{-g_c(0)}) \\ = \frac{1}{2}\log\frac{\tanh g_c(c)/2}{\tanh g_c(0)/2} &= \frac{1}{2}\log\frac{\tanh g_c(0)}{\tanh g_c(0)/2}.\end{aligned}$$

Thus we have $\iota_c(c) \geq (\ell_c(\gamma_1) + \ell_c(\gamma_2))/2 \geq \frac{1}{2}\log\frac{\tanh g_c(0)}{\tanh g_c(0)/2}$. In the same fashon, we obtain $\iota_c(\infty) \geq -\frac{1}{2}\log\tanh g_c(0)/2$. Summing up these esitimates, we obtain the following result by Corollary 1.3.

Theorem 3.1. *Let Ω_c be the Fatou set of a quadratic polynomial $f_c(z) = z^2 + c$ with c outside the Mandelbrot set. If we denote Green's function of Ω_c with pole at ∞ by $g_c(z)$, we have the following estimate:*

$$L_{\Omega_c} \geq \frac{1}{2}\log\frac{\tanh g_c(0)}{\tanh g_c(0)/2} = \frac{1}{2}\log\frac{2}{1 + \tanh^2(g_c(0)/2)}.$$

Finally, we make some comparison between the above estimates and known results. By Theorem 1.6, Theorems 2.1 and 3.1 produces the following inequalities.

$$\text{H-dim}\,J_c \geq \frac{\sqrt{2}\log 2}{\pi(\log m + H) + \sqrt{2}\log 3}, \quad \text{and} \tag{3.3}$$

$$\text{H-dim}\,J_c \geq \frac{\log 2}{2\pi^2/\log\frac{\tanh g_c(0)}{\tanh g_c(0)/2} + \log 3}, \tag{3.4}$$

where $m = \max\{\sqrt{1-4c}, \frac{\sqrt{1-4c}+1}{\sqrt{1-4c}(\sqrt{1-4c}-3)}\}$ and $c < -2$ in (3.3) and $c \notin \mathcal{M}$ in (3.4).

On the other hand, Ransford [10] proved the following estimate for any $c \in \mathbb{C} \setminus \mathcal{M}$:

$$\frac{\log 2}{g_c(0) + \log 2} \le \text{H-dim} J_c \le \frac{\log 2}{g_c(0) + (e^{-g_c(0)} + 1)^{-1} \log 2}.$$

Note that $G_{\mathcal{M}}(c, \infty) = g_c(c) = 2g_c(0)$, where $G_{\mathcal{M}}$ is Green's function of the exterior of the Mandelbrot set with pole at ∞. Since $g_c(0) = \frac{1}{2} G_{\mathcal{M}}(c, \infty) = \frac{1}{2} \log |c| + o(1)$ as $c \to \infty$ (cf. [2]), we see that $\text{H-dim} J_c = \frac{2 \log 2}{\log |c|}(1 + o(1))$ at ∞. On the other hand, (3.3) yields that $\text{H-dim} J_c \ge \frac{2\sqrt{2}\pi}{\log |c|}$ as $c \to -\infty$, which is rather good estimate. But (3.4) yields only that $\text{H-dim} J_c \ge \log 2 / \pi^2 \sqrt{|c|}$ as $c \to \infty$.

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Department of Mathematics, Kyoto University, 606-01 Sakyo-ku, Kyoto, Japan
E-mail address: sugawa@kusm.kyoto-u.ac.jp