

# Stability of Kleinian groups

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We survey stability of Kleinian groups. Several results in this note are also contained in the forthcoming monograph [7].

Let  $\Gamma$  be a finitely generated non-elementary Kleinian group, which is identified with a discrete subgroup of  $\mathrm{PSL}_2(\mathbf{C})$ , with a fixed system of generators  $\Gamma = \langle \gamma_1, \dots, \gamma_N \rangle$ . We consider the set of  $\mathrm{PSL}_2(\mathbf{C})$ -representations

$$\mathrm{Hom}(\Gamma) = \{ \rho \mid \rho : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{C}) \text{ is a homomorphism} \}.$$

This is regarded as an analytic subset of  $\mathrm{PSL}_2(\mathbf{C})^N$  by the correspondence

$$\rho \mapsto (\rho(\gamma_1), \dots, \rho(\gamma_N)) \in \mathrm{PSL}_2(\mathbf{C})^N.$$

We say that  $\Gamma$  is *structurally stable* if there is a neighborhood  $U$  of the identity representation  $id$  in  $\mathrm{Hom}(\Gamma)$  such that any  $\rho \in U$  is a faithful representation.

However, a weaker condition than structural stability is more interesting in deformation theories of Kleinian groups, where we treat an analytic set of all representations sending any parabolic element to a parabolic one or the identity. Letting

$$\mathrm{PHom}(\Gamma) = \{ \rho \in \mathrm{Hom}(\Gamma) \mid \mathrm{tr}^2 \rho(\gamma) = 4 \text{ for any parabolic } \gamma \in \Gamma \},$$

we call  $\Gamma$  *weakly structurally stable* if the condition of structural stability is satisfied after replacing  $\mathrm{Hom}(\Gamma)$  with  $\mathrm{PHom}(\Gamma)$ .

Here, the property that any  $\rho \in U \subset \mathrm{PHom}(\Gamma)$  is faithful is actually equivalent to that  $\rho$  is a quasiconformal deformation. Indeed, we can apply

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the  $\lambda$ -lemma to a holomorphic family of isomorphisms defined over a complex disk holomorphically embedded in  $U$  that passes a given point of  $U$ . Thus the weakly structural stability is nothing but the following quasiconformal stability:

Let  $T(\Omega(\Gamma)/\Gamma)$  be the Teichmüller space of the union of orbifolds  $\Omega(\Gamma)/\Gamma$ . For every  $[\mu] \in T(\Omega(\Gamma)/\Gamma)$ , we denote by  $f_\mu$  a quasiconformal automorphism of  $\hat{\mathbf{C}}$  that gives the deformation  $[\mu]$  of the complex structure of  $\Omega(\Gamma)/\Gamma$  and that satisfies a suitable normalization. Then a holomorphic map

$$\tilde{\Psi} : T(\Omega(\Gamma)/\Gamma) \times \mathrm{PSL}_2(\mathbf{C}) \rightarrow \mathrm{PHom}(\Gamma)$$

is defined by the conjugation of  $A \circ f_\mu$ , for any pair  $([\mu], A) \in T(\Omega(\Gamma)/\Gamma) \times \mathrm{PSL}_2(\mathbf{C})$ . It is known that  $\tilde{\Psi}$  is well-defined. Moreover, the Sullivan rigidity theorem implies that the image of  $\tilde{\Psi}$  coincides with the set of the whole representations induced by quasiconformal automorphisms of  $\hat{\mathbf{C}}$  (cf. [7, Chapter 5]). This set is called the quasiconformal deformation space and denoted by  $\mathrm{QHom}(\Gamma)$  ( $\subset \mathrm{Hom}(\Gamma)$ ). We say that  $\Gamma$  is *quasiconformally stable* if there is a neighborhood  $U$  of the identity representation  $id$  such that

$$\mathrm{PHom}(\Gamma) \cap U \subset \mathrm{QHom}(\Gamma).$$

Gardiner and Kra [3] investigated the derivative  $d\tilde{\Psi}|_{id}$  of  $\tilde{\Psi}$  at  $id$ , representing the Zariski tangent space of the analytic set  $\mathrm{PHom}(\Gamma)$  as Eichler cohomology. This is called *the Bers map*. They showed that  $d\tilde{\Psi}|_{id}$  is injective and decomposed the tangent space into subspaces caused by  $\mathrm{PSL}_2(\mathbf{C})$ -conjugations, quasiconformal deformations and deformations of projective structure. In particular, they proved that surjectivity of  $d\tilde{\Psi}|_{id}$  implies quasiconformal stability.

For a torsion-free Kleinian group, Marden [5] proved that if  $\Gamma$  is geometrically finite then it is quasiconformally stable, by investigation of fundamental polyhedra. Later Sullivan [10] proved the converse, namely, that quasiconformal stability implies geometric finiteness, by 3-dimensional topological arguments to compare the dimension of the Teichmüller space  $T(\Omega(\Gamma)/\Gamma)$  with the dimension of  $\mathrm{PHom}(\Gamma)$ .

In [6], we extend this equivalence to Kleinian groups with torsion. One direction is easy if we pass  $\Gamma$  to a torsion-free subgroup of finite index by the Selberg lemma, but the other not. We consider a core in the 3-dimensional hyperbolic orbifold  $\mathbf{H}^3/\Gamma$ , where a core means a compact suborbifold such

that the inclusion induces an isomorphism between the orbifold fundamental groups. Moreover, we require that the core is relative to the boundary at infinity  $\Omega(\Gamma)/\Gamma$ . If we can construct such a core, then we can obtain a relation between  $\dim T(\Omega(\Gamma)/\Gamma)$  and  $\dim \text{PHom}(\Gamma)$  at  $id$  because both are topological quantities determined only by the topology of the relative core.

We can prove the existence of an orbifold relative core in the case that  $\Gamma$  is indecomposable as a free product in a certain sense. For the general case, we decompose  $\Gamma$  into indecomposable ones and use induction arguments to compare the dimensions.

**Theorem 1** *The following conditions are equivalent for any finitely generated non-elementary Kleinian group  $\Gamma$ :*

1.  $\Gamma$  is geometrically finite;
2.  $\Gamma$  is quasiconformally stable;
3. the Bers map  $d\tilde{\Psi}|_{id}$  is an isomorphism.

In particular:

**Corollary 2** *If  $\Gamma$  is geometrically finite, then  $\text{QHom}(\Gamma)$  is a complex regular submanifold of  $\text{PSL}_2(\mathbf{C})^N$ .*

Next we will prove that Corollary 2 is actually satisfied for any finitely generated Kleinian group. Since  $\tilde{\Psi}$  is a holomorphic immersion onto  $\text{QHom}(\Gamma)$ , the only problem is compatibility of the Teichmüller topology of  $T(\Omega(\Gamma)/\Gamma) \times \text{PSL}_2(\mathbf{C})$  and the topology of  $\text{QHom}(\Gamma)$ , which is the algebraic topology for  $\text{PSL}_2(\mathbf{C})$ -representations.

**Problem** If  $\rho_n$  converge to  $id$  in  $\text{QHom}(\Gamma)$ , do there always exist  $t_n \in \tilde{\Psi}^{-1}(\rho_n)$  such that  $t_n$  converge to the base point  $0 \in T(\Omega(\Gamma)/\Gamma)$  as  $n \rightarrow \infty$ ?

This problem is originated in Bers [1, p.578], where it was announced that the proof would appear elsewhere, however it has not appeared as far as the author knows. See also [2]. Later Krushkal published a series of papers (cf. [4]) concerning this problem. A finitely generated Kleinian group is called *conditionally stable* or *quasi-stable* if it satisfies the property in the problem above.

We will show that a result on geometric convergence of Kleinian groups yields the affirmative answer to this problem.

**Theorem 3** *Any finitely generated Kleinian group  $\Gamma$  is conditionally stable. Hence  $\text{QHom}(\Gamma)$  is a complex regular submanifold of  $\text{PSL}_2(\mathbf{C})^N$ .*

We first remark the following two facts.

**Lemma 4**  *$\Gamma$  is conditionally stable if any component subgroup of  $\Gamma$  is conditionally stable.*

*Proof.* This follows easily from the definition of conditional stability. ■

**Lemma 5** *Let  $\Gamma$  be a finitely generated Kleinian group and  $\Gamma'$  a subgroup of  $\Gamma$  of finite index. If  $\Gamma'$  is conditionally stable, then so is  $\Gamma$ .*

*Proof.* Suppose that  $\Gamma$  is not conditionally stable. Then there is a sequence  $\rho_n \in \text{QHom}(\Gamma)$  converging to  $id$  such that the maximal dilatation of the extremal quasiconformal automorphism  $f_n$  inducing  $\rho_n$  does not tend to 1 as  $n \rightarrow \infty$ . Here the extremal quasiconformal map is the one with the smallest maximal dilatation among quasiconformal maps with the required property.

We restrict  $\rho_n$  to the subgroup  $\Gamma'$  and have  $\rho'_n \in \text{QHom}(\Gamma')$ . Then  $\rho'_n$  converges to  $id$  and  $f_n$  induces  $\rho'_n$ . Since  $\Gamma'$  is of finite index in  $\Gamma$ ,  $f_n$  is also the extremal quasiconformal automorphism that induces  $\rho'_n$  (cf. Ohtake [9]). But this contradicts the assumption that  $\Gamma'$  is conditionally stable. Thus we see that  $\Gamma$  is also conditionally stable. ■

By these facts, it suffice to consider torsion-free function groups  $\Gamma$  for proving Theorem 3. It is known that such  $\Gamma$  is constructed from elementary groups, quasifuchsian groups and totally degenerate groups without APT by a finite number of applications of the Maskit combination theorem. Moreover we can see that conditional stability is preserved under the Maskit combination theorem:

**Lemma 6** *Assume that a torsion-free function group  $\Gamma$  is constructed from  $\Gamma_1$  and  $\Gamma_2$  (as the amalgamated free product or the HNN-extension) by the Maskit combination theorem. If both  $\Gamma_1$  and  $\Gamma_2$  are conditionally stable, then so is  $\Gamma$ .*

*Proof.* See [7, Section 7.3]. ■

Therefore Theorem 3 will complete if it is solved for totally degenerate groups without torsion nor APT. The crucial fact for this step is the following result due to Thurston (cf. Ohshika [9] and [7, Section 7.2]).

**Proposition 7** *Let  $\Gamma_0$  be a finitely generated torsion-free Fuchsian group and  $\theta_n : \Gamma_0 \rightarrow \Gamma_n$  a sequence of type-preserving isomorphisms onto Kleinian groups, which converges algebraically to a type-preserving isomorphism  $\theta : \Gamma_0 \rightarrow \Gamma$ . If  $\Gamma$  is a totally degenerate group, then  $\Gamma_n$  also converge geometrically to  $\Gamma$ .*

Applying this proposition, we can assert:

**Lemma 8** *Under the same circumstances as in Proposition 7, if  $\Gamma_n$  and  $\Gamma$  are totally degenerate groups, then the marked complex structures  $t_n$  of  $\Omega(\Gamma_n)/\Gamma_n$  converge to  $t$  of  $\Omega(\Gamma)/\Gamma$ . In particular, any torsion-free, totally degenerate group without APT is conditionally stable.*

*Proof.* Let  $C_n$  be the convex core of the hyperbolic manifold  $\mathbf{H}^3/\Gamma_n$  and  $\partial C_n$  the relative boundary of  $C_n$ , which is regarded as a pleated surface with a marked hyperbolic structure  $s_n$ . By Proposition 7, we can see that  $s_n$  converge to the marked hyperbolic structure  $s$  of the boundary surface of the convex core of  $\mathbf{H}^3/\Gamma$ . By Sullivan's theorem (cf. [7, Section 7.1]),  $s_n$  and  $t_n$  are in a bounded Teichmüller distance independent of  $n$ . Hence  $\{t_n\}$  is a bounded sequence in the Teichmüller space and there is a subsequence  $\{t_{n'}\}$  which converges to some  $t'$ . Then  $\Gamma_{n'}$  converge algebraically to a  $b$ -group  $\Gamma'$  such that the marked complex structure of  $D'/\Gamma'$  is  $t'$ , where  $D'$  is the invariant component of  $\Omega(\Gamma')$ . However,  $\Gamma'$  should coincide with  $\Gamma$ , and hence  $t' = t$ . ■

Thus we obtain Theorem 3.

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