VARIOUS DOMAIN CONSTANTS RELATED TO UNIFORM PERFECTNESS

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ABSTRACT. This is a survey article on domain constants related to uniform perfectness. We gather comparison theorems for various domain constants, most of which are, more or less, known or elementary, but not stated quantitatively in literature, and some are new or improved results. Among these theorems, our main result is a comparison of the modulus and the injectivity radius of a hyperbolic Riemann surface. Its proof relies upon a comparison of extremal and hyperbolic lengths, which seems to be interesting in itself. And we include a lower estimate of the Hausdorff dimension of a compact set in the Riemann sphere by the modulus of its complement. We also discuss the variance of these domain constants under conformal, quasiconformal or Möbius maps.

1. INTRODUCTION

A simply connected plane domain $D \subset \mathbb{C}$ with the hyperbolic metric $\rho_D = \rho_D(z)|dz|$ of constant curvature -4 is known to have many remarkable geometric or analytic properties. For instance, the hyperbolic density $\rho_D(z)$ is uniformly estimated by $\delta_D(z) = \mathrm{dist}(z, \partial D)$:

$$\frac{1}{4} \leq \rho_D(z)\delta_D(z) \leq 1, \tag{1.1}$$

and the Schwarzian derivative S_f of the Riemann map $f : \mathbb{H} \to D$ is hyperbolically bounded (by a uniform bound):

$$\|S_f\|_{\mathbb{H}} = \sup_{z\in\mathbb{H}} |S_f(z)|(2\mathrm{Im}z)^2 \leq 6. \tag{1.2}$$

In [4], it was first recognized that the similar property as (1.1) is also valid for a wider class of domains, i.e. hyperbolic domains of bounded geometry. While the simple connectedness of a plane domain is characterized by the connectedness of

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its complement, Pommerenke had an insight into the fact that the above class of domains is also characterized by a way of condensation of its boundary points, and such a set was called a *uniformly perfect set* by him. Independently, Tukia and Väisälä introduced in [55] an equivalent notion for subsets of metric spaces, under the name *homogeneously dense sets*. Pommerenke thoroughy investigated uniformly perfect sets in [45] and [46], and found so many equivalent definitions for uniform perfectness. Moreover he and other authors observed that the limit set of any non-elementary finitely generated Kleinian group and the Julia set of any rational map with degree greater than one are uniformly perfect ([46] and [31], [20], respectively. See also [52] and [51]). After the works of Pommerenke, many authors persued investigations of uniform perfectness or equivalent notions. Osgood [43] found a relationship to BMO (see also Gotoh [13]). Ancona provided in his paper [2] several potential-theoretic conditions that is equivalent to uniform perfectness ones. This result is recently applied to perturbations of Green's function by Aikawa (e.g., [1]). González [12] gave another characterizations of uniform perfectness in terms of Green's functions and fundamental domains of the Fuchsian group uniformizing the domain. And Fernández and Rodríguez pointed out that plane hyperbolic domains of bounded geometry satisfy the isoperimetric inequality in the hyperbolic sense, thus the bottoms of spectra of Laplace-Beltrami operators on them are always positive and the critical exponents of convergence of these surfaces are less than 1 ([8] and [9]). Recently, Järvi and Vuorinen [21] made a great contribution to the case of higher dimensions. Also, Ma, Maitani and Minda [28] has discovered an interesting characterization of the uniform perfectness in terms of two-point comparisons between hyperbolic and Euclidean geometry. For more investigations, we refer to [11], [16], [27], [29], [30], [59], [60], and so on.

These facts tell the universality and richness of the notion of uniform perfectness (or equivalently, boundedness of geometry, or modulatedness). But, in spite of many studies, explicit comparisons have not been completely made for those domain constants relating to uniform perfectness (or, modulatedness), at least in literature with some exceptions ([13], [16], [30], [38], [60], etc.).

In this article, we try to gather explicit estimates for domain constants, some are known or elementary, some are less but well-known in a quite different context, and some are new. It is meaningful to know the amount of uniform perfectness for a plane compact set because some important quantity (for instance, capacity, Hausdorff dimension, and so on) is estimated by it. We also refer to invariance or quasi-invariance of those constants under Möbius, conformal, or quasiconformal mappings.

This article is organized as follows. Section 2 is devoted to give a precise definition for various domain constants related to uniform perfectness, and to present explicit estimates for these constants. Especially, we describe injectivity radius and modulus of a given Riemann surface in terms of hyperbolic and extremal lengths, respectively, and from this we deduce a comparison theorem (Theorem 2.3) for the injectivity radius and the modulus as an immediate corollary of the results in sections 5 and 6.

We discuss in Section 3 the invariance (or, variance) of those domain constants under some class of functions (Möbius, conformal, or quasiconformal mappings). In Section 4, as a special case, we observe the geometry of round annuli, which is fundamental for estimation of the domain constants and gives examples with exactly computable domain constants.

Recently, Riemann surfaces are studied by using extremal lengths of (homotopy class of) simple closed curves, as well as hyperbolic lengths (see, for instance, [6], [10], [23], [26], [39], and [53]). We establish, in Section 5, a comparison theorem for hyperbolic and extremal lengths, which seems to be useful in itself for the geometry of Riemann surfaces as above. We make essential use of the collar lemma there.

In Section 6, we explain that integrable holomorphic quadratic differentials are also (hyperbolically) bounded for modulated Riemann surfaces, and as a corollary of this observation we obtain a quite different estimate (Corollary 6.3) from the one obtained by the collar lemma. The ideas in this section are heavily indebted to Matsuzaki [33].

The final section 7 is used to present another characterization of the uniform perfectness in terms of the Hausdorff contents, from which we can deduce a lower estimate of the Hausdorff dimension with a bound depending only on the uniform perfectness. This result is essentially due to Järvi and Vuorinen [21].

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2. Various domain constants related to uniform perfectness

Throughout this article, we consider only hyperbolic (connected) Riemann surfaces. (But, we remark that the definitions of those domain constants below are naturally extended to disconnected Riemann surfaces, in particular, open subsets in $\widehat{\mathbb{C}}$ excluding at least three points.)

Let R be a hyperbolic Riemann surface, that is, there exists a holomorphic universal covering map $p : \mathbb{H} \to R$ of R from the upper half plane $\mathbb{H} = \{z \in \mathbb{C}; \mathrm{Im} z > 0\}$.

Let Γ be the covering transformation group of $p : \mathbb{H} \to R$, then Γ is a torsion-free discrete subgroup of $Aut\mathbb{H} = PSL(2, \mathbb{R}) = \mathrm{Möb}(\mathbb{H})$. Since the Poincaré (or, hyperbolic) metric $\rho_{\mathbb{H}}(z)|dz| = \frac{|dz|}{2\mathrm{Im} z}$ on $\mathbb{H}$ is Möbius invariant, there exists the unique Riemannian metric $\rho_R = \rho_R(z)|dz|$ such that $\rho_{\mathbb{H}} = p^*\rho_R := \rho_R(p(z))|p'(z)||dz|$. The metric ρ_R is called also the hyperbolic metric of R. Some authors prefer to use $\frac{|dz|}{\mathrm{Im} z}$ as the Poincaré metric instead of $\frac{|dz|}{2\mathrm{Im} z}$, so we should be careful of difference caused by the factor 2 when refering to the papers of such authors.

Injectivity radius. Let $D_R(q,r)$ be the hyperbolic disk $\{x \in R; d_R(x,q) < r\}$ centered at $q \in R$ with radius $r > 0$, where d_R denotes the hyperbolic distance naturally defined by ρ_R, i.e.

$$d_R(q_1,q_2) = \inf_{\alpha} \int_{\alpha} \rho_R(z)|dz|,$$

where the infimum is taken over all piecewise smooth paths α in R joining q_1 and q_2. Let $\iota_R(q)$ denote the injectivity radius of R at $q \in R$, i.e.,

$$\begin{aligned}\iota_R(q) &= \sup\{r > 0; D_R(q,r) \text{ is simply connected }\} \\ &= \sup\{r > 0; p \text{ is injective in } D_{\mathbb{H}}(\tilde{q},r)\},\end{aligned}$$

where $\tilde{q}$ is a point of $\mathbb{H}$ satisfying $p(\tilde{q}) = q$. Further, we call $I_R := \inf_{q \in R} \iota_R(q)$ the *injectivity radius* of R. Since R is locally isometric to the hyperbolic plane $\mathbb{D}$ on each injective disk, the condition $I_R > 0$ is equivalent to one that R is *of bounded geometry* (for the definition in general setting, see [47]).

Hyperbolic length. Here we explain that I_R has several geometric meanings. We denote by $\mathcal{C} = \mathcal{C}_R$ the set of all free homotopy classes $[\alpha]$ of nontrivial closed curves α on R, and let $\mathcal{S} = \mathcal{S}_R$ the subset of $\mathcal{C}$ consisting of homotopy classes represented by simple closed curves.

For $[\alpha] \in \mathcal{C}$, we define the hyperbolic length $\ell[\alpha] = \ell_R[\alpha]$ of $[\alpha]$ by

$$\ell[\alpha] = \inf_{\alpha' \simeq \alpha} \int_{\alpha'} \rho_R(z)|dz|. \tag{2.1}$$

Then, as is easily seen (cf. [27] Theorem 3), it holds that

$$I_R = \inf_{[\alpha] \in \mathcal{C}} \frac{\ell[\alpha]}{2} = \inf_{[\alpha] \in \mathcal{S}} \frac{\ell[\alpha]}{2}. \tag{2.2}$$

On the other hand, when $\gamma \in \Gamma$ covers a closed curve α on R, we directly see that $\ell[\alpha]$ coincides with the *translation length* l_γ of γ, that is

$$|\mathrm{tr}\gamma| = 2\cosh \ell[\alpha],$$

where $|\mathrm{tr}\gamma| = |a+d|$ if γ is represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. In particular, we have

$$\inf_{\gamma \in \Gamma \setminus \{1\}} |\mathrm{tr}\gamma| = 2\cosh 2I_R.$$

Modulus. An annulus (= doubly connected planar domain) A contained in R is called *essential* if the natural homomorphism $\iota_* : \pi_1(A,*) \to \pi_1(R,*)$ induced by the inclusion map $\iota : A \hookrightarrow R$ is injective. We denote by $\mathcal{A}_R$ the totality of essential annuli in R, and set

$$M_R := \sup_{A \in \mathcal{A}_R} m(A),$$

where $m(A)$ is the modulus of A which is defined in this article by the number m such that A is conformally equivalent to the round annulus $\{z \in \mathbb{C}; 1 < |z| < e^m\}$. (Conventionally, we define $M_R = 0$ if $\mathcal{A}_R = \emptyset$, i.e. R is simply connected.) We call M_R the *modulus* of R, and we say that R is *modulated* if $M_R < +\infty$ (cf. [9]). (While Yamashita called a plane domain with an equivalent property as this a domain *of finite type* in [59], we adopt in this article the terms "of bounded geometry" or "modulated".)

Remark. Here again, we should be cautious about the definition of the moduli (or, modules) of annuli. For instance, in literature, the modulus of the annulus $\{z \in \mathbb{C}; 1 < |z| < e^m\}$ is sometimes defined by $\frac{m}{2\pi}$ instead of m.

The domain constant M_R also has a significant expression by extremal lengths similar to (2.2). Now we explain this. For $[\alpha] \in \mathcal{C}$, we define the *extremal length* $E[\alpha] = E_R[\alpha]$ of $[\alpha]$ by

$$E[\alpha] = \sup_{\tau} \frac{\left(\inf_{\alpha' \in [\alpha]} \int_{\alpha'} \tau(z)|dz|\right)^2}{\iint_R \tau(z)^2 dxdy}$$

where the supremum is taken over all Borel measurable conformal metrics $\tau = \tau(z)|dz|$ on R.

The next theorem, which claims the existence of an extremal conformal metric τ_0 attaining the above supremum, is fundamental for our method.

Theorem 2.1 (Jenkins and Strebel, cf. [22] **and** [49]**).** *For any $[\alpha] \in \mathcal{S}_R$, there exists an integrable holomorphic quadratic differential φ_0 (called the Jenkins-Strebel differential for $[\alpha]$ with closed horizontal trajectries homotopic to α) and its characteristic ring domain $R_0 \in \mathcal{A}_R$ satisfies the following conditions.*

(1)
$$E[\alpha] = \frac{\left(\inf_{\alpha' \in [\alpha]} \int_{\alpha'} |\varphi_0|^{1/2}|dz|\right)^2}{\iint_R |\varphi_0| dxdy},$$
i.e. $\tau_0 = |\varphi_0|^{1/2}$ is an extremal metric,

(2) $E[\alpha] = \frac{2\pi}{m(R_0)}$,

(3) $m(R_1) \leq m(R_0)$ *for all $R_1 \in \mathcal{A}_R$ such that the core curve is homotopic to α.*

In particular, by observing this theorem, we can readily see the following

Corollary 2.2.

$$\inf_{[\alpha]\in\mathcal{S}_R} E[\alpha] = \frac{2\pi}{M_R}.$$

One of our main results in this article is the next theorem, which connects the quantities I_R and M_R.

Theorem 2.3. *Let R be a hyperbolic Riemann surface with the hyperbolic metric ρ_R of constant curvature -4. Then, we have the following estimate.*

$$2I_R \leq \frac{\pi^2}{M_R} \leq \min\{2I_R e^{2I_R}, 2{I_R}^2 \coth^2 I_R\}. \tag{2.3}$$

Here, the equality occurs in the left-hand side if and only if R is a doubly connected planar Riemann surface or $I_R = 0$.

Remarks. By a numerical calculation, we see that $2I_R e^{2I_R} > 2I_R^2 \coth^2 I_R$ if and only if $I_R > 0.45752\cdots$.

For a triply connected (planar) Riemann surface, we have a better estimate (see Corollary 5.5).

We shall prove this theorem by dividing it into two parts as follows:

$$2I_R \leq \frac{\pi^2}{M_R} \leq 2I_R e^{2I_R}, \tag{2.4}$$

$$\frac{\pi^2}{M_R} \leq 2{I_R}^2 \coth^2 I_R. \tag{2.5}$$

The estimate (2.4) is obtained by Corollary 5.3 in Section 5 combined with (2.2) and Corollary 2.2. In fact, by Theorem 5.2, we can show a slightly sharper result than the right-hand side inequality in (2.4):

$$\frac{\pi^2}{M_R} \leq \frac{\pi I_R}{\arctan\left(\frac{1}{\sinh 2I_R}\right)}.$$

A proof of (2.5) is given in Section 6. By virture of (2.5), we observe that

$$\frac{1}{M_R} = O({I_R}^2) \quad \text{as} \quad I_R \to \infty. \tag{2.6}$$

Thus the estimate (2.5) is much better than the right-hand side in (2.4) when I_R is sufficiently large, while (2.4) is very sharp when I_R is sufficiently small. The author does not know whether the exponent 2 in (2.6) is best possible or not.

Hahn metric. Now we introduce the Hahn metric $\hat{\rho}_R$ on R. For a point $q \in R$ and a complex coordinate z of R around q, $\hat{\rho}_R = \hat{\rho}_R(z)|dz|$ is defined by

$$\hat{\rho}_R(z)|dz| = \inf_G \rho_G(z)|dz|,$$

where G ranges over all simply connected domain in R containing q. Then, $\hat{\rho}_R$ is a continuous Riemannian metric on R and called the *Hahn metric* on R (for more informations, see [14], [36] and [13]). By definition, the Hahn metric is conformally invariant, and since $\rho_G \geq \rho_R$ on $G \subset R$ by the Schwarz-Pick lemma, it follows that $\hat{\rho}_R \geq \rho_R$. Noting that $\rho_R/\hat{\rho}_R$ is a (well-defined) continuous function on R and is not greater than 1, we set

$$K_R := \inf_{q \in R} \frac{\rho_R}{\hat{\rho}_R}(q).$$

The function $\rho_R/\hat{\rho}_R$ is known to be comparable to ι_R.

Theorem 2.4 (Gotoh [13]**).**

$$\tanh \iota_R(q) \leq \frac{\rho_R}{\hat{\rho}_R}(q) \leq 4 \tanh \iota_R(q).$$

Corollary 2.5.

$$\tanh I_R \leq K_R \leq 4 \tanh I_R.$$

In the rest of the present section, we shall consider only hyperbolic plane domains D, i.e. subdomains of $\widehat{\mathbb{C}}$ excluding more than two points. Therefore the representation $\rho_D(z)|dz|$ of the hyperbolic metric of D by the natural global coordinate z gives a global function $\rho_D(z)$ on D which is called the *hyperbolic* (or, *Poincaré) density* of D. For $z \in D \setminus \{\infty\}$, we denote by $\delta_D(z)$ the distance from z to the boundary ∂D, that is, $\delta_D(z) = \inf_{w \in \partial D} |z - w|$.

Schwarzian derivative. Let $p : \mathbb{H} \to D$ a holomorphic universal covering, then the Schwarzian derivative of p can be defined to be $S_p = (\frac{p''}{p'})' - \frac{1}{2}(\frac{p''}{p'})^2$. Similarly as the case that D is simply connected, we consider the hyperbolic sup norm of the Schwarzian derivative:

$$N_D := \|S_p\|_{\mathbb{H}} = \sup_{z \in \mathbb{H}} |S_p(z)| \rho_{\mathbb{H}}(z)^{-2} = \sup_{z \in \mathbb{H}} |S_p(z)| (2\mathrm{Im}\, z)^2.$$

This quantity has a close connection with the injectivity radius. In fact, for a non-simply connected domain D, an estimate:

$$2 \coth^2 I_D \leq N_D \leq 6 \coth^2 I_D$$

has been noted essentially by [48], [24], [58], [25] (see also [50]). Later, Minda and Ma improved the estimate as the following.

Theorem 2.6 ([38] **and** [29]). *Let D be a hyperbolic subdomain of $\widehat{\mathbb{C}}$. If D is not simply connected, we have*

$$2+\frac{\pi^2}{2I_D^2}\le N_D\le 6(1-1/M)\coth^2 I_D, \tag{2.7}$$

where $M>1$ is the number satisfying

$$\exp\left(2I_D\sqrt{1+3(1-1/M)\coth^2 I_D}\right)-2\sqrt{(\tanh^2 I_D+3)M-3}-1=0.$$

Remark. As is stated in the introduction, if D is simply connected, a sharp estimate $N_D\le 6$, known as the Nehari-Kraus theorem, is valid. Moreover, Nehari [41] showed that if $N_D\le 2$ then D must be simply connected.

Round modulus. A subset A of the form $\{z\in\mathbb{C}; r_1<|z-a|<r_2\}$ $(0\le r_1<r_2\le\infty)$ is called a *round annulus* with center a, denoted by $\mathrm{cent}A$, and of modulus $m(A)=\log r_2/r_1$. Moreover, if $m(A)<\infty$ the curve $\{z;|z-a|=\sqrt{r_1r_2}\}$ is the unique simple closed geodesic of A, called the *core curve* and denoted by $\mathrm{Core}A$. For a hyperbolic plane domain (or, more generally, a hyperbolic open set) $D\subset\widehat{\mathbb{C}}$, we denote by $\mathcal{A}_D^\circ$ the set of all essential round annuli in D, and set

$$M_D^\circ:=\sup_{A\in\mathcal{A}_D^\circ} m(A).$$

We call M_D° the *round modulus* of D. And, a closed set $C\subset\widehat{\mathbb{C}}$ containing more than two points is said to be *uniformly perfect* if $M_D^\circ<\infty$, where $D=\widehat{\mathbb{C}}\setminus C$. This constant M_D° seems to be important by two reasons. The first is that M_D° is easier to compute or estimate, and the second is an application to estimate of Hausdorff dimension (see Theorem 7.2). Since $\mathcal{A}_D^\circ\subset\mathcal{A}_D$, we have $M_D^\circ\le M_D$. On the other hand, there exists a domain D such that $M_D>0$ and $M_D^\circ=0$. For example, consider a domain $D=\{x+iy\in\mathbb{C};|x|<2 \text{ and } |y|<\varepsilon\}\setminus[-1,1]$ for sufficiently small $\varepsilon>0$. Nevertheless, some kind of reverse inequality holds. In fact, as a corollary of the proof of Theorem 3.12 in [19], we obtain the following inequality.

Theorem 2.7 (Herron-Liu-Minda [19]**).** *For any hyperbolic subdomain D of $\mathbb{C}$ with $M_D>3\log 2=2.0794\cdots$, it holds that*

$$M_D^\circ\ge M_D-\log h(e^{-M_D}),$$

where $h(t)=\frac{16\sqrt{1-16t^2}}{\sqrt{3}-16t\sqrt{1-16t^2}}$ is a monotonically increasing function of $t\in(0,\frac{1}{8})$.

Since $M_0=\frac{1}{2}\log\frac{128}{4-\sqrt{3}}=2.8911\cdots$ satisfies the equation $\log h(e^{-M_0})=M_0$, we have the next corollary.

Corollary 2.8 (Theorem 3.12 in [19]). *For any hyperbolic subdomain D of $\mathbb{C}$, it follows that*

$$M_D^\circ \geq M_D - \frac{1}{2}\log\frac{128}{4-\sqrt{3}} = M_D - 2.8911\cdots . \tag{2.8}$$

Proof. If $M_D \leq M_0 = \frac{1}{2}\log\frac{128}{4-\sqrt{3}}$, we have nothing to do. Otherwise, the above theorem says that

$$M_D^\circ \geq M_D - \log h(e^{-M_D}) \geq M_D - \log h(e^{-M_0}) = M_D - M_0.$$

□

Remarks. McMullen presented in [34] a weaker estimate

$$M_D^\circ \geq M_D - 5\log 2 = M_D - 3.4657\cdots$$

with a simpler proof.

We do not know the best possible constant $C > 0$ satisfying

$$M_D^\circ \geq M_D - C$$

for any hyperbolic domain $D \subset \mathbb{C}$. Concerning C, the last corollary says that $C \leq 2.8911\cdots$. But, at least, Herron-Liu-Minda [19] proved that $M_D^\circ > 0$ if $M_D > \frac{\pi}{2}$ and $\frac{\pi}{2}$ is optimal. In fact, Mori's extremal ring domain $D = \mathbb{C}\setminus((-\infty,0]\cup\{e^{i\theta}; |\theta| \leq \frac{\pi}{2}\})$ satisfies that $M_D^\circ = 0$ and $M_D = \frac{\pi}{2}$. Furthermore, the authors of [19] conjectured that for any $t \in (0,2)$ Mori's extremal ring domain

$$D_t = \mathbb{C}\setminus\Big((-\infty,0]\cup\{e^{i\theta}; |\theta| \leq \arcsin\frac{t}{2}\}\Big)$$

maximizes the difference $M_D - M_D^\circ$ among hyperbolic domains $D \subset \mathbb{C}$ with $M_D = M_{D_t}$.

If this conjecture is true, we can show that $M_D^\circ \geq M_D - 3\log 2 = M_D - 2.0794\cdots$ for any hyperbolic domain $D \subset \mathbb{C}$ and $3\log 2$ is best possible. The number $3\log 2$ already appeared in Example 3.14 in [19].

For the case $\infty \in D$, we also have a similar result, which is obtained by the above corollary combined with Theorem 3.4.

Theorem 2.9. *For any hyperbolic plane domain $D \subset \widehat{\mathbb{C}}$, we have the following estimate.*

$$2M_D^\circ \geq M_D - \frac{1}{2}\log\frac{128}{4-\sqrt{3}} - 2\log\frac{4}{3} = M_D - 3.4665\cdots .$$

Capacity density. First we recall a definition of the logarithmic capacity $\mathrm{Cap}K$ for a compact set $K \subset \mathbb{C}$. Let $G(z)$ be the Green function of D with pole at ∞, where D is the connected component of $\widehat{\mathbb{C}} \setminus K$ containing ∞. Then, $G(z)$ can be written as

$$G(z) = \log |z| + h(z)$$

near ∞, where $h(z)$ is a harmonic function in a neiborhood of ∞. The number $h(\infty)$ is called the Robin constant for K and the logarithmic capacity $\mathrm{Cap}K$ is defined by $\mathrm{Cap}K = e^{-h(\infty)}$. When the Green function of D does not exist, we define $\mathrm{Cap}K = 0$.

We note that $\mathrm{Cap}\{|z| \le r\} = r$ and $\mathrm{Cap}(r \cdot K) = r\mathrm{Cap}K$ for $r > 0$. And we define the *infimum of the capacity density* F_C of a closed set $C \subset \widehat{\mathbb{C}}$ by

$$F_C = \inf \left\{ \frac{\mathrm{Cap}\{z \in C; |z-a| \le r\}}{r}; a \in C, r \in (0, \mathrm{diam}C) \right\},$$

where $\mathrm{diam}C$ denotes the Euclidean diameter of C.

Now we introduce an auxiliary domain constant. For an open set D in $\widehat{\mathbb{C}}$, we set

$$E_D = \sup\{E > 0; \{z \in \partial D; Er < |z-a| < r\} \ne \emptyset \text{ for all } a \in \partial D \text{ and } 0 < r < \mathrm{diam}\partial D\}.$$

Then the following remarkable result was proved by Pommerenke.

Theorem 2.10 (Pommerenke [45]**).** *Let C be a closed set in $\widehat{\mathbb{C}}$ and D its complement, then it holds that*

$$\frac{{E_D}^2}{32} \le F_C \le E_D.$$

As a corollary, we obtain the following result.

Corollary 2.11.

$$\frac{1}{2}\log\frac{1}{F_C} - \frac{7}{2}\log 2 \le M_D^\circ \le \log\frac{1}{F_C} + \log 3.$$

In particular, if C contains ∞, in the left-hand side inequality we can replace the constant $\frac{7}{2}$ by $\frac{5}{2}$.

In fact, elementary calculations yield that

$$\log\frac{1}{2E_D} \le M_D^\circ \le \log\frac{E_D+2}{E_D} \le \log\frac{1}{E_D} + \log 3, \tag{2.9}$$

and if $\infty \in C$ then further it holds that $\log 1/E_D \le M_D^\circ$. Combining these inequalities with Pommerenke's theorem, we can easily obtain the desired estimate.

We should note that, by Wiener's criterion, the assertion $F_C > 0$ implies the regularity of C in the sense of Dirichlet (cf. Tsuji [54] p.104). In other words, every continuous function on C extends to a continuous function on $\widehat{\mathbb{C}}$ that is harmonic off C.

Quasi-hyperbolic metric. For a proper subdomain D of $\mathbb{C}$, the continuous Riemannian metric $\frac{|dz|}{\delta_D(z)}$ is called the quasi-hyperbolic metric. If D is simply connected, the following estimate easily follows from the Koebe one-quarter theorem and Schwarz's lemma (see [44]): for each $z \in D$,

$$\frac{1}{4\delta_D(z)} \leq \rho_D(z) \leq \frac{1}{\delta_D(z)}.$$

For a general hyperbolic domain $D \subset \mathbb{C}$, it also holds that $\rho_D(z) \leq 1/\delta_D(z)$, but $\delta_D(z)\rho_D(z)$ need not to be bounded away from 0. On the other hand, the quasi-hyperbolic metric is always comparable to the Hahn metric $\hat{\rho}_D = \hat{\rho}_D(z)|dz|$.

Theorem 2.12 (Minda [36]). *For any proper subdomain D of $\mathbb{C}$, we have*

$$\frac{1}{4\delta_D(z)} \leq \hat{\rho}_D(z) \leq \frac{1}{\delta_D(z)}.$$

Now we introduce a domain constant

$$C_D := \inf_{z \in D} \delta_D(z)\rho_D(z).$$

Then, it is obvious that $C_D \leq 1$, and that $\frac{1}{4} \leq C_D$ if moreover D is simply connected. More strongly, Hilditch and Harmelin-Minda [16] proved the following theorem, independently.

Theorem 2.13 ([16]. See also [35] and [37]). *It is always true that $C_D \leq \frac{1}{2}$. The equality occurs if and only if D is convex.*

By Theorem 2.12, we also have $\delta_D(z)\rho_D(z) \leq \frac{\rho_D(z)}{\hat{\rho}_D(z)} \leq 4\delta_D(z)\rho_D(z)$, therefore we obtain the next

Corollary 2.14. *For a hyperbolic subdomain of $\mathbb{C}$, it holds that*

$$C_D \leq K_D \leq 4C_D.$$

As for C_D, further we can state the following theorem, essentially indebted to Yamashita [60].

Theorem 2.15. *Let D be a hyperbolic subdomain of $\mathbb{C}$, then we have the following estimates:*

$$\frac{\tanh I_D}{4} \leq C_D \leq \frac{2\sqrt{3}I_D}{\sqrt{\pi^2 + 16{I_D}^2}}, \tag{2.10}$$

$$\min\left\{\frac{1}{\sqrt{8N_D}}, \frac{1}{4}\right\} \leq C_D \leq \sqrt{\frac{6}{N_D + 6}}. \tag{2.11}$$

Proof. The left-hand side inequality in (2.10) easily follows from the Koebe one-quarter theorem (see [38] or [60]), and combining this with the well-known inequality $2\coth^2 I_D \le N_D$, we obtain the left-hand side in (2.11) in case that D is not simply connected (here, we remark that $1/\sqrt{8N_D} < \frac{1}{4}$ by Nehari's theorem). If D is simply connected, we know that $1/4 \le C_D$, so the left-hand side inequality in (2.11) holds, too.

Yamashita [60] showed that for a universal covering map $p : \Delta \to D$ from the unit disk Δ, the inequality

$$\frac{1}{2}(1-|z|^2)\,|S_f(z)| \le 3\left(\frac{1}{\delta_D(w)^2\rho_D(w)^2} - 1\right)$$

holds for any $z \in \Delta$ with $w = p(z)$. From this inequality, we immediately obtain $C_D \le \sqrt{6/(N_D+6)}$, and from this and Minda's estimate (2.7) we also have the right-hand side inequality in (2.10). $\square$

Remarks. In the non-simply connected case, by utilizing Minda's estimate (2.7), we have an inequality $\frac{1}{4}\tanh(\pi/\sqrt{2(N_D-2)}) \le C_D$ better but difficult to use than $1/\sqrt{8N_D} \le C_D$. Yamashita [60] showed also an inequality: $\delta_D(w)\rho_D(w) \le \frac{4r}{(1+r)^2} = 1 - e^{-4\iota_D(w)}$, where $r = \tanh \iota_D(w)$, and from this we have $C_D \le 1 - e^{-4I_D}$ but this is always worse than the right-hand side inequality in (2.10).

Finally, we present the result connecting C_D with the round modulus M_D°.

Theorem 2.16. *For a hyperbolic subdomain D of $\mathbb{C}$, if $M_D^\circ > 0$ we have the following estimate.*

$$\begin{aligned}(2.12)\qquad \frac{1}{M_D^\circ + 2c_H} \le C_D &\le \inf_{0<k\le 1/2} \frac{\pi}{2}\frac{1-e^{-kM_D^\circ}}{M_D^\circ \sin k\pi} \\ &\le \frac{\pi}{2}\frac{1-e^{-M_D^\circ/2}}{M_D^\circ} < \frac{\pi}{2M_D^\circ},\end{aligned}$$

where c_H is a universal constant $[2\rho_{\mathbb{C}\setminus\{0,1\}}(-1)]^{-1} = \frac{\Gamma(1/4)^4}{4\pi^2} = 4.3768\cdots$ due to Hempel [18].

Remark. The assumption that $\infty \notin D$ is essential in this theorem. In fact, if let $D = \Delta^* = \{z \in \widehat{\mathbb{C}}; 1 < |z| \le \infty\}$, we have $\rho_{\Delta^*}(z) = \frac{1}{|z|^2-1}$ and $\delta_{\Delta^*}(z) = |z| - 1$, hence

$$\delta_{\Delta^*}(z)\rho_{\Delta^*}(z) = \frac{1}{|z|+1} \to 0$$

as $z \to \infty$, whereas $\tanh I_{\Delta^*} = \tanh I_\Delta = 1$ and $M_\Delta^\circ = 0$.

Before step into the proof, concerning a technical quantity

$$\beta_D(z) = \inf\left\{\left|\log\frac{|z-a|}{|b-a|}\right|; a, b \in \partial D, |z-a| = \delta_D(z)\right\},$$

we state an important result first shown by Beardon-Pommerenke [4]. The following form of the theorem appeared in Yamashita' paper [60].

Theorem 2.17 (Sharp version of Beardon-Pommerenke's theorem). *If D is a hyperbolic subdomain of $\mathbb{C}$, then we have an estimate*

$$\frac{1}{2(\beta_D(z)+c_H)\delta_D(z)} \le \rho_D(z) \le \frac{\pi}{4\beta_D(z)\delta_D(z)}.$$

The equality in the left-hand side occurs when $D = \mathbb{C}\setminus\{0,1\}$ and $z = -1$.

Proof of Theorem 2.16. First, we show that $2\beta_D(z) \le M_D^\circ$ for any $z \in D$. In fact, the annulus $A = \{w \in \mathbb{C}; \delta e^{-\beta} < |w-a| < \delta e^{\beta}\}$, where $\delta = \delta_D(z), \beta = \beta_D(z)$ and $a \in \partial D$ is a point satisfying $|z-a| = \delta$, separates ∂D hence $A \in \mathcal{A}_D^\circ$. Thus, we have $2\beta = m(A) \le M_D^\circ$.

Then, the inequality $1/C_D \le M_D^\circ + 2c_H$ directly follows from Theorem 2.17 and this observation. Now we prove the other inequality. For this, we fix any number $k \in (0, 1/2]$. Let $A = \{z \in \mathbb{C}; r_1 < |z-c_0| < r_2\} \in \mathcal{A}_D^\circ$ and $z_0 \in A$ so that $|z_0 - c_0| = (\frac{r_2}{r_1})^k r_1 = r_1^{1-k} r_2^k$, then by (4.5) we see that

$$\rho_A(z_0) = \frac{\pi}{2|z_0-c_0|m(A)\sin\pi k}.$$

Here, we may assume that A is maximal in the class $\{A' \in \mathcal{A}_D^\circ; \mathrm{cent}A' = c_0\}$, thus we can select z_0 so that $\delta_D(z_0) = |z_0-c_0| - r_1$. Hence,

$$C_D \le \delta_D(z_0)\rho_D(z_0) \le \delta_D(z_0)\rho_A(z_0) = \frac{\pi}{2m(A)}\frac{1-\left(\frac{r_2}{r_1}\right)^{-k}}{\sin\pi k} = \frac{\pi(1-e^{-km(A)})}{2m(A)\sin\pi k},$$

and tending $m(A)$ to M_D° yields that

$$C_D \le \frac{\pi(1-e^{-kM_D^\circ})}{2M_D^\circ\sin\pi k}$$

for any $k \in (0, 1/2]$, thus the proof is now completed. □

By summing up the results in this section, we have the following thorem, which is of course well-known.

Theorem 2.18. *The following conditions are mutually equivalent for a hyperbolic subdomain D of $\widehat{\mathbb{C}}$.*

(1) *D is modulated, i.e. $M_D < \infty$.*
(2) *$\widehat{\mathbb{C}} \setminus D$ is uniformly perfect, i.e. $M_D^\circ < \infty$.*
(3) *The injectivity radius I_D is positive.*
(4) *The Hahn metric is comparable to the Poincaré metric, i.e. $K_D > 0$.*
(5) *The norm N_D of the Schwarzian derivative of a universal covering is finite.*
(6) *The infimum of capacity density is positive, i.e. $F_{\widehat{\mathbb{C}} \setminus D} > 0$.*

Furthermore, if $\infty \notin D$, we can add the following conditions to the above list.

(7) *The quasi-hyperbolic metric is comparable to the hyperbolic metric, i.e. $C_D > 0$.*

Remark. Generally, for a hyperbolic open subset D of $\widehat{\mathbb{C}}$, the above theorem is still valid under suitable interpretations for domain constants.

3. Conformal, quasiconformal and Möbius invariance.

Now we gather some results concerning the invariance (or, variance) of the domain constants under various mappings. First of all, the constants M_R, I_R and K_R are conformally invariant by definition. In other words, if $f : R \to R'$ is a conformal (=biholomorphic) mapping, we have $M_R = M_{R'}, I_R = I_{R'}$, and $K_R = K_{R'}$. Moreover, constants M_R and I_R are quasi-invariant, i.e. the next theorem holds.

Theorem 3.1. *Suppose that there exists a K-quasiconformal homeomorphism ($K \geq 1$) of a hyperbolic Riemann surface R onto another surface R'. Then it follows that*

$$\frac{1}{K} M_R \leq M_{R'} \leq K M_R, \text{ and } \frac{1}{K} I_R \leq I_{R'} \leq K I_R.$$

The first part directly follows from the quasi-invariance of moduli of annuli. The second part, seeming to be less trivial than the first, is obtained by (2.2) and the next

Proposition 3.2 (Wolpert [56]). *The hyperbolic length of a closed geodesic is quasi-invariant. More presicely, let $f : R \to R'$ be a K-quasiconformal homeomophism and α a closed curve in R, then we have an inequality:*

$$\frac{1}{K} \ell_R[\alpha] \leq \ell_{R'}[f(\alpha)] \leq K \ell_R[\alpha].$$

For convenience of the reader, we shall give a proof of this proposition in Section 4. In particular, we know that the modulatedness of a hyperbolic Riemann surface is quasiconformally invariant. On the other hand, the constants N_D, M_D° and C_D for a plane domain D are not conformally invariant, but some kind of estimate can be deduced. Harmelin, Ma and Minda ([16], [30]) systematically investigated the conformal variance of the constants N_D and C_D and other important ones. We notify the reader that they use the term "quasi-invariance" in a different meaning from here.

As for the constant N_D, the following absolute estimate is obtained.

Theorem 3.3 (cf. [30]**).** *If hyperbolic domains D and D' are conformally equivalent, then we have*

$$|N_D - N_{D'}| \le 12.$$

Proof. Let $f : D \to D'$ be a conformal map and $p : \mathbb{H} \to D$ a holomorphic universal cover of D. Then $f \circ p$ is a holomorphic universal cover of D' and it holds that $S_{f\circ p} = p^*(S_f) + S_p$, where p^* denotes pull-back of quadratic differentials by p, i.e. $p^*\varphi = (\varphi \circ p)(p')^2$. Since $\|S_f\|_D := \sup_{z\in D} |S_f(z)|\rho_D(z)^{-2} \le 12$ by the theorem of Beardon-Gehring [3], we have

$$\begin{aligned} |N_D - N_{D'}| &\le \sup_{z\in\mathbb{H}} |p^* S_f(z)|\rho_{\mathbb{H}}(z)^{-2} \\ &= \sup_{z\in\mathbb{H}} |S_f(p(z))||p'(z)|^2\rho_{\mathbb{H}}(z)^{-2} \\ &= \sup_{z\in\mathbb{H}} |S_f(p(z))|\rho_D(p(z))^{-2} \le 12, \end{aligned}$$

where we used a relation $\rho_{\mathbb{H}}(z) = \rho_D(p(z))|p'(z)|$. □

The constant N_D is Möbius invariant since $S_p = S_{L\circ p}$ for any Möbius transformation L. The other constants M°_D and C_D are not even Möbius invariant, but we obtain several estimates by quite elementary calculations. In particular, it is important to know the variance of the constant M°_D under Möbius transformations when applying Theorem 7.2.

Theorem 3.4. *Let D be a hyperbolic subdomain of $\widehat{\mathbb{C}}$. For any $L \in$ Möb, we have*

$$\frac{1}{2}M^\circ_D - \log\frac{4}{3} \le M^\circ_{L(D)}. \tag{3.1}$$

Further, in case $L(D) \subset \mathbb{C}$, we have a better estimate:

$$M^\circ_D - \log 2 \le M^\circ_{L(D)}.$$

Proof. In order to prove this, we may assume that D is a round annulus. First, we note that a Möbius transformation L which is not a similarity can always be written as $L(z) = \frac{c}{z-a} + d$. Since the quantity M°_D is invariant under similarities, it is sufficient to prove the theorem in the case that $D = A_r = \{z; r < |z| < 1\}$ $(0 < r < 1)$ and $L_a(z) = \frac{1}{z-a}$ $(a \in [0,\infty))$. Further note that $L_a (a > 0)$ can be decomposed to the form

$$L_a = S \circ L_{r/a} \circ T,$$

where S is a similarity: $S(z) = -a^2 z/r - a/r$ and $T(z) = r/z$. Since $T(A_r) = A_r$, we obtain that

$$M^\circ_{L_a(A_r)} = M^\circ_{L_{r/a}(A_r)}.$$

Thus, it is sufficient to prove the following

Claim. For $A = \{z; r < |z| < 1\}$ $\quad (0 < r < 1)$ and $L(z) = \frac{1}{z-a}$, where $0 \le a \le \sqrt{r}$, it holds that

$$M^{\circ}_{L(A)} \ge \frac{1}{2} M^{\circ}_{D} + \log\frac{3}{4} = \frac{1}{2}\log\frac{1}{r} + \log\frac{3}{4}.$$

Moreover, if $L(A) \subset \mathbb{C}$, then

$$M^{\circ}_{L(A)} > M^{\circ}_{D} + \log\frac{1}{2} = \log\frac{1}{r} + \log\frac{1}{2}.$$

Now we prove the Claim. When $a \ne r$, we observe that $L(\pm 1) = \frac{1}{\pm 1 - a}$ $\quad (L(\pm r) = \frac{1}{\pm r - a})$ are the end points of a diameter of the circle $C_0 = L(\{|z| = 1\})$ (resp. $C_1 = L(\{|z| = r\})$). Thus $\frac{a}{1-a^2}$ and $\frac{a}{r^2-a^2}$ are the centers of C_0 and C_1, respectively. Further, note that under the above hypothesis, $L(A) \subset \mathbb{C}$ if and only if $0 \le a \le r$.

Case 1: $0 \le a \le r$.

Since we can treat $a = r$ as the limiting case of $a < r$, we may assume that $0 \le a < r$. In this case, the circle C_0 is contained in the inside of C_1. Since $(\frac{1}{r-a} - \frac{1}{1-a}) - (\frac{1}{-1-a} - \frac{1}{-r-a}) > 0$, we can see that

$$M^{\circ}_{L(A)} = \log\frac{\frac{a}{1-a^2} - \frac{1}{-r-a}}{\frac{a}{1-a^2} - \frac{1}{-1-a}} = \log\frac{1+ar}{a+r},$$

so we have

$$M^{\circ}_{L(A)} - \log\frac{1}{r} = \log\frac{r(1+ar)}{a+r} \ge \log\frac{r(1+r^2)}{2r} = \log\frac{1+r^2}{2} > \log\frac{1}{2},$$

which proves the second part of the Claim.

Case 2: $r < a \le \sqrt{r}$.

In this case, the circle C_1 is contained in the outside of C_1. Since

$$\left(\frac{1}{1-a} - \frac{1}{-1-a}\right) - \left(\frac{1}{-r-a} - \frac{1}{r-a}\right) = 2\frac{(a^2-r)(1+r)}{(a^2-r^2)(1-a^2)} \le 0,$$

the radius of C_1 is greater than or equal to that of C_0. Therefore, we are convinced that

$$M^{\circ}_{L(A)} = \log\frac{\frac{a}{1-a^2} - \frac{1}{-r-a}}{\frac{a}{1-a^2} - \frac{1}{-1-a}} = \log\frac{1+ar}{a+r},$$

thus

$$\begin{aligned} M^{\circ}_{L(A)} - \frac{1}{2}\log\frac{1}{r} &= \log\frac{\sqrt{r}(1+ar)}{a+r} \ge \log\frac{\sqrt{r}(1+r\sqrt{r})}{\sqrt{r}+r} \\ &= \log\left\{(\sqrt{r} - \frac{1}{2})^2 + \frac{3}{4}\right\} \ge \log\frac{3}{4}, \end{aligned}$$

which proves the first part of the Claim. Note that the equality holds if $r = \frac{1}{4}$ and $a = \frac{1}{2}$. $\quad \square$

Remark. The constant $\frac{1}{2}$ in (3.1) is best possible in the following sense. Suppose that there exist constants $c > 0$ and $d \in \mathbb{R}$ such that

$$cM_D^\circ + d \leq M_{L(D)}$$

for any hyperbolic subdomain D of $\widehat{\mathbb{C}}$ and $L \in$ Möb, then c must not be less than $\frac{1}{2}$. In fact, for any constant $c > \frac{1}{2}$, in the situation of Case 2 of the proof

$$\begin{aligned} M_{L(A)}^\circ - c\log\frac{1}{r} &\geq \log r^c \frac{1 + r\sqrt{r}}{\sqrt{r} + r} = \log r^{c-1/2} + \log\frac{1 + r\sqrt{r}}{1 + \sqrt{r}} \\ &= \left(\frac{1}{2} - c\right)\log\frac{1}{r} + O(1) \quad \text{ as } r \to +0, \end{aligned}$$

so the factor $\frac{1}{2}$ is best possible.

As to the constant C_D, we have a result as an immediate consequence of Corollary 2.14: Let D and D' be conformally equivalent hyperbolic domains in $\mathbb{C}$,then it holds that

$$C_D \leq 4C_{D'}.$$

In fact, since K_D is conformally invariant, Corollary 2.14 yields that

$$C_D \leq K_D = K_{D'} \leq 4C_{D'}.$$

This estimate follows also from the Koebe one-quarter theorem directly. On the other hand, Ma and Minda [30] showed that the above constant 4 can be replaced by a smaller one: $|1 + i\coth\frac{\pi}{3}| = 2.4335\cdots$. Since it is true that $C_D \leq 2C_{D'}$ if D and D' are simply connected, they also conjecture that $C_D \leq 2C_{D'}$ for any conformally equivalent hyperbolic subdomains D and D' of $\mathbb{C}$.

A quasiconformal variance of the other constants is obtained, at least in principle, by combining Theorem 3.1 with the comparison theorems in Section 2. For example, now we consider the contant N_D. Suppose that $f : D \to D'$ is a K-quasiconformal homeomorphism. Noting that $\coth x \leq 1 + \frac{1}{x}$ for $x > 0$, we have the following estimate by Minda's theorem:

$$\begin{aligned} N_{D'} &\leq 6\coth^2 I_D \leq 6\left(1 + \frac{1}{I_D}\right)^2 \leq 6\left(1 + \frac{K}{I_{D'}}\right)^2 \\ &\leq 6\left(1 + \frac{K}{\pi}\sqrt{2(N_D - 2)}\right)^2 \\ &\leq 12 + \frac{24K^2}{\pi^2}(N_D - 2). \end{aligned}$$

Of course, a more direct method may give a better estimate, which the author does not know unfortunately.

4. Geometry of annuli.

Here, we state standard facts about the geometry of annuli with some proofs. Let $A = A_r$ be the round annuli $\{z \in \mathbb{C}; r < |z| < 1\}$ $(0 < r < 1)$. A holomorphic universal cover $p : \mathbb{H} \to A$ is concretely given by

$$p(z) = \exp\left(\frac{i \log 1/r}{\pi} \log z\right), \quad (z \in \mathbb{H}), \tag{4.1}$$

where we take the principal branch of $\log z$, that is, $\operatorname{Im} \log z = \arg z \in (0, \pi)$. As is easily observed, the covering transformation group of $p : \mathbb{H} \to A$ is the infinite cyclic group generated by $g : z \mapsto e^{\ell} z$, where

$$\ell = \frac{2\pi^2}{\log 1/r} > 0.$$

Therefore, the image of the axis $\mathrm{ax}(g) = \{iy; y > 0\}$ under p is the core curve $\mathrm{Core}A = \{z; |z| = \sqrt{r}\}$. In particular, we can calculate the hyperbolic length of the core curve of A as

$$\ell(\mathrm{Core}A) = \int_1^{e^{\ell}} \frac{dy}{2y} = \frac{\ell}{2} = \frac{\pi^2}{\log 1/r} = \frac{\pi^2}{m(A)}, \tag{4.2}$$

thus we conclude by (2.2) that

$$I_A = \frac{\pi^2}{2m(A)}. \tag{4.3}$$

On the other hand, since the Jenkins-Strebel differential for $\mathrm{Core}A$ is a constant multiple of dz^2/z^2, we can easily deduce that

$$E(\mathrm{Core}A) = \frac{2\pi}{m(A)}.$$

Here, we prepare an elementary lemma for later use.

Lemma 4.1. *Suppose g is a hyperbolic transformation of $\mathbb{H}$ defined by $g(z) = e^{\ell} z$ with $\ell > 0$.*

Let $\theta_1, \theta_2 \in (0, \pi)$ with $\theta_1 < \theta_2$, then $\mathbb{H}(\theta_1, \theta_2)/\langle g \rangle$ is biholomorphic to an annulus with modulus $\frac{2\pi}{\ell}(\theta_2 - \theta_1)$, where $\mathbb{H}(\theta_1, \theta_2)$ denotes the set $\{z \in \mathbb{H}; \theta_1 < \arg z < \theta_2\}$.

Proof. Choose an $r \in (0, 1)$ so that $\ell = \frac{2\pi^2}{\log 1/r}$ and let $p : \mathbb{H} \to A_r$ be as in (4.1). Then, it is clear that $\mathbb{H}(\theta_1, \theta_2)/\langle g \rangle$ is biholomorphically equivalent to $p(\mathbb{H}(\theta_1, \theta_2)) = \{z; \frac{\theta_1}{\pi} \log \frac{1}{r} < \log \frac{1}{|z|} < \frac{\theta_2}{\pi} \log \frac{1}{r}\}$, which is an annulus with modulus $\frac{\theta_2 - \theta_1}{\pi} \log \frac{1}{r} = \frac{2\pi}{\ell}(\theta_2 - \theta_1)$. $\square$

As an application, we give here a proof of Proposition 3.2 (due to Wolpert [56]).

Proof of Proposition 3.2. Let $f : R \to R'$ be a K-quasiconformal homeomorphism and α a closed curve in R. We consider a holomorphic universal cover $p : \mathbb{H} \to R$ and $p' : \mathbb{H} \to R'$ with covering transformation groups Γ and Γ', respectively; and let $\tilde{f}$ be a lift of f, i.e. $\tilde{f} : \mathbb{H} \to \mathbb{H}$ is a K-quasiconformal homeomorphism satisfying that $p' \circ \tilde{f} = f \circ p$. Suppose that $\gamma \in \Gamma$ covers α, and let $\gamma' = \tilde{f} \circ \gamma \circ \tilde{f}^{-1} \in \Gamma'$. Then γ' covers $\alpha' := f(\alpha)$. Now we consider the annuli $A = \mathbb{H}/\langle\gamma\rangle$ and $A' = \mathbb{H}/\langle\gamma'\rangle$. Then $\tilde{f}$ naturally induces a K-quasiconformal homeomorphism $\hat{f} : A \to A'$. The quasi-invariance of moduli of annuli implies that

$$\frac{1}{K} m(A) \leq m(A') \leq K m(A).$$

On the other hand, $m(A) = \frac{\pi^2}{l_\gamma} = \frac{\pi^2}{\ell_R[\alpha]}$ and so on, hence we obtain that

$$\frac{1}{K} \ell_R[\alpha] \leq \ell_{R'}[\alpha'] \leq K \ell_R[\alpha].$$

□

Next, we give a concrete form of the hyperbolic metric of A. For each $z \in A$, we take a point $\zeta \in \mathbb{H}$ so that $p(\zeta) = z$. Then we have $\rho_A(z)|p'(\zeta)| = p^*\rho_A(\zeta) = \rho_{\mathbb{H}}(\zeta) = \frac{1}{2\mathrm{Im}\zeta}$. Letting $\theta = \arg\zeta \in (0, \pi)$, since $|z| = |p(\zeta)| = \exp(-\frac{\theta}{\pi}\log\frac{1}{r})$, we have $\theta = \pi\frac{\log 1/|z|}{\log 1/r}$. Therefore,

$$\begin{aligned}\frac{1}{\rho_A(z)} &= 2\mathrm{Im}\zeta \cdot |p'(\zeta)| = 2\mathrm{Im}\zeta \cdot \frac{\log 1/r}{\pi|\zeta|}|z| = \frac{2}{\pi}\log\frac{1}{r} \cdot |z|\sin\theta \\ &= \frac{2}{\pi}\log\frac{1}{r} \cdot \sin\left(\pi\frac{\log 1/|z|}{\log 1/r}\right).\end{aligned}$$

Now we obtain that

$$\rho_A(z) = \frac{\pi}{2|z|\log\frac{1}{r}\sin\left(\pi\frac{\log 1/|z|}{\log 1/r}\right)}. \tag{4.4}$$

We should remark that (4.4) remains valid if r tends to 0, i.e.

$$\rho_{A_0}(z) = \frac{1}{2|z|\log 1/|z|}.$$

Further, we note that for a general round annulus $A = \{z; r_1 < |z - a| < r_2\}$, it follows that

$$\rho_A(z) = \frac{\pi}{2|z-a|m(A)\sin\left(\pi\frac{\log r_2/|z-a|}{\log r_2/r_1}\right)}. \tag{4.5}$$

Using this explicit form of the hyperbolic metric for a round annulus, we determine the constant N_{A_r}. A direct computation shows that the Schwarzian derivative of $p : \mathbb{H} \to A_r$ has a simple form:

$$S_p(z) = \frac{1}{2}\left(\frac{(\log 1/r)^2}{\pi^2} + 1\right)\frac{1}{z^2}.$$

Therefore,

$$\begin{aligned} N_{A_r} = \|S_p\|_{\mathbb{H}} &= \sup_{z\in\mathbb{H}} \frac{1}{2}\left(\frac{(\log 1/r)^2}{\pi^2} + 1\right)\frac{1}{|z|^2}(2\mathrm{Im}\, z)^2 \\ &= 2\left(1 + \frac{(\log 1/r)^2}{\pi^2}\right) = 2 + \frac{2\pi^2}{I_{A_r}^2}, \end{aligned}$$

which is the case that the equality occurs in the left-hand side of (2.7).

5. Hyperbolic vs. extremal length.

In this section, we shall give several estimates between the hyperbolic and extremal lengths, from which we can derive some comparisons between the constants I_R and M_R. In our proof, the collar lemma plays a significant role. We say that a simple closed geodesic α in R has a *collar of width* $\omega > 0$ around it if the $\omega/2$-neighborhood $\{q \in R; \mathrm{dist}_R(q,\alpha) < \omega/2\}$ of α is homeomorphic to an annulus, where $\mathrm{dist}_R(q,\alpha)$ is the hyperbolic distance of q from α. The following form of the collar lemma fits our present aim, for which the reader may consult the book [5]. See also [15] for a short proof.

Theorem 5.1 (The collar lemma). *A simple closed geodesic α of length $\lambda > 0$ in a hyperbolic Riemann surface R has at least a collar of width ω around it, where ω is the positive number determined by the relation*

$$\sinh\omega \sinh\lambda = 1. \tag{5.1}$$

Further, this ω is best possible.

With the aid of the collar lemma, now we can show the following

Theorem 5.2 (Comparison theorem). *For any $[\alpha] \in \mathcal{S}_R$, we have the following estimate:*

$$\frac{2}{\pi}\ell[\alpha] \le E[\alpha] \le \frac{\ell[\alpha]}{\arctan\left(\frac{1}{\sinh \ell[\alpha]}\right)}. \tag{5.2}$$

Further, $\frac{2}{\pi}\ell[\alpha] = E[\alpha]$ for some $[\alpha] \in \mathcal{S}_R$ with $\ell[\alpha] > 0$ if and only if R is an annulus with $m(R) < \infty$.

By elementary, but slightly boring calculations, we can see that the function $e^x \arctan(\frac{1}{\sinh x})$ of $x > 0$ monotonically increases from $\frac{\pi}{2} = 1.57\cdots$ to 2. Therefore, we have a

Corollary 5.3 (A simpler form). *For any* $[\alpha] \in \mathcal{S}_R$, *we have a slightly weak but simpler estimate:*

$$\frac{2}{\pi}\ell[\alpha] \le E[\alpha] \le \frac{2}{\pi}\ell[\alpha]e^{\ell[\alpha]}. \tag{5.3}$$

It seems that even Corollary 5.3 is rather sharp at least when $\ell[\alpha]$ is sufficiently small. Maskit [32] has obtained a similar result:

$$\frac{2}{\pi}\ell[\alpha] \le E[\alpha] \le \frac{\ell[\alpha]}{\arcsin\left(e^{-\ell[\alpha]}\right)}$$

(in our present notations), from which he derived also that $\ell[\alpha]/E[\alpha] \to \frac{\pi}{2}$ as $\ell[\alpha] \to 0$ and that $E[\alpha] \le \ell[\alpha]e^{\ell[\alpha]}$. Further we remark that it holds that $\arcsin(e^{-x}) < \arctan(\frac{1}{\sinh x})$ for any $x > 0$, so Theorem 5.2 is an improvement of Maskit's result.

Proof of Theorem 5.2. Let α be a simple closed curve in R and $p : \mathbb{H} \to R$ be a holomorphic universal cover of R. And let $\gamma \in \mathit{\Gamma}$ be a covering transformation which covers α, i.e. there exists a lift $\tilde{\alpha} : [0,1] \to \mathbb{H}$ of α such that $\gamma(\tilde{\alpha}(0)) = \tilde{\alpha}(1)$. First, we shall show the right-hand side inequality. If γ is parabolic (i.e. $\ell[\alpha] = 0$), it holds that $E[\alpha] = 0$ because the punctured disk (so, of infinite modulus) can be conformally embedded so that an embedded loop around the puncture is freely homotopic to α. Thus, we may assume that γ is a hyperbolic transformation, in other words, γ has a form $\gamma(z) = e^{2\lambda}z$, where $\lambda > 0$ is the hyperbolic length of $[\alpha]$ and that α is a simple closed geodesic. By the collar lemma, α has a collar A of width ω satisfying (5.1). Let $\widetilde{A}$ be the connected component of $p^{-1}(A)$ which contains the axis $\mathrm{ax}(\gamma)$ of γ, i.e. the imaginary axis. Since $p(\mathrm{ax}(\gamma)) = \alpha$, we see that

$$\begin{aligned}\widetilde{A} &= \{z \in \mathbb{H}; \mathrm{dist}_{\mathbb{H}}(z, \mathrm{ax}(\gamma)) < \frac{\omega}{2}\} \\ &= \{z \in \mathbb{H}; |\arg z - \frac{\pi}{2}| < \theta\},\end{aligned}$$

where $\theta > 0$ is the angle satisfying $\tan\frac{\theta}{2} = \tanh\frac{\omega}{2}$, or equivalently,

$$\tan\theta = \sinh\omega. \tag{5.4}$$

Thus, Lemma 5.4 yields that $m(A) = \frac{2\pi\theta}{\lambda}$. Now, we conclude by Theorem 2.1 that

$$\frac{2\pi}{E[\alpha]} \ge m(A) = \frac{2\pi\theta}{\lambda} = \frac{2\pi}{\lambda}\arctan\left(\frac{1}{\sinh\lambda}\right),$$

which is equivalent to the right-hand side inequality of (5.2).

Next, we show the left-hand side inequality: $\frac{2}{\pi}\ell[\alpha] \le E[\alpha]$. However this is a known result, we shall give a different proof from Maskit's one [32] (using the length-area method) and from Matsuzaki's one [33] (using the Poincaré theta series). Let $R_0 \in \mathcal{A}_R$ with $\alpha_0 = \mathrm{Core} R_0 \in [\alpha]$ and $m(A) < \infty$. Then, since $\rho_R \le \rho_{R_0}$ on R_0, we have

$$\ell[\alpha] \le \ell(\alpha_0) = \int_{\alpha_0} \rho_R(z)|dz| \le \int_{\alpha_0} \rho_{R_0}(z)|dz| = \frac{\pi^2}{m(A)}$$

from (4.2). By virtue of the Jenkins-Strebel theorem (Theorem 2.1), we have $\ell[\alpha] \le \frac{\pi}{2}E[\alpha]$. From the above proof, we can also observe that the equality occurs only when $\ell[\alpha] = 0$ or $R = R_0$, i.e. R is an annulus itself. $\square$

Remark. We can prove that $\ell[\alpha] \le \frac{\pi}{2}E[\alpha]$ also by making use of the annular covering method. Let $h : \mathbb{H} \to A := \mathbb{H}/\langle\gamma\rangle$ be the natural projection. Note that A is an annulus with modulus $m(A) = \frac{\pi^2}{l_\gamma} = \frac{\pi^2}{\ell[\alpha]}$ (see Section 4). The universal cover $p : \mathbb{H} \to R$ induces the annular cover $q : A \to R$ with respect to α such that $p = q \circ h$. By construction of q, any other lift of α via q is *not* closed than $\hat{\alpha} = h(\tilde{\alpha})$, where $\tilde{\alpha} : [0,1] \to \mathbb{H}$ is a lift of α via p such that $\gamma(\tilde{\alpha}(0)) = \tilde{\alpha}(1)$. Let R_0 be the characteristic ring domain for α (see Theorem 2.1, and A_0 be the unique doubly connected component of $q^{-1}(R_0)$. Here we should note that $q : A_0 \to R_0$ is biholomorphic. Since A_0 is contained essentially in A, by the monotonicity of the moduli of annuli, we have

$$\frac{\pi^2}{\ell[\alpha]} = m(A) \ge m(A_0) = m(R_0) = \frac{2\pi}{E[\alpha]}.$$

As Maskit remarked in [32], if $[\alpha] \in \mathcal{S}_R$ with $\ell[\alpha] > 0$ is a boundary curve (i.e. α divides R into two parts, one of which is an annulus), then we can take a collar $A = p(\widetilde{A})$ around α in the above proof, where

$$\widetilde{A} = \{z \in \mathbb{H}; 0 < \arg z < \frac{\pi}{2} + \theta\},$$

and θ satisfies (5.4) for a suitable choice of $p : \mathbb{H} \to R$. Thus,

$$\begin{aligned}\frac{2\pi}{E[\alpha]} \ge m(A) = \frac{\pi(\theta + \pi/2)}{\lambda} &= \frac{\pi}{\lambda}\left(\frac{\pi}{2} + \arctan\left(\frac{1}{\sinh\lambda}\right)\right) \\ &\ge \frac{\pi}{\lambda}\left(\frac{\pi}{2} + \frac{\pi}{2}e^{-\lambda}\right),\end{aligned}$$

and we also have the following

Theorem 5.4. *For any boundary simple closed curve α in R with $\ell[\alpha] > 0$, we have a stronger estimate:*

$$\frac{2}{\pi}\ell[\alpha] \leq E[\alpha] \leq \frac{4}{\pi}\frac{\ell[\alpha]}{1+e^{-\ell[\alpha]}} < \frac{4}{\pi}\ell[\alpha].$$

Since any $[\alpha] \in \mathcal{S}_R$ is a boundary curve for a triply connected domain R, we obtain the next

Corollary 5.5. *For any triply connected planar Riemann surface R with $I_R > 0$, we have a better estimate:*

$$2I_R < \frac{\pi^2}{M_R} < 4I_R.$$

6. An application to quadratic differentials.

In this section, we consider complex Banach spaces of integrable and (hyperbolically) bounded holomorphic quadratic differentials on a hyperbolic Riemann surface. Presicely, let $A_2(R)$ and $B_2(R)$ be the complex Banach spaces consisting of all holomorphic quadratic differentials $\varphi = \varphi(z)dz^2$ on a hyperbolic Riemann surface R with finite norms $\|\varphi\|_1 = \iint_R |\varphi| = \iint_R |\varphi(z)|dxdy$ and $\|\varphi\|_\infty = \sup_{q\in R} |\varphi(q)|\rho_R(q)^{-2}$, respectively. A neccesary and sufficient condition for $A_2(R)$ being (continuously) contained in $B_2(R)$ is known to be that

$$\inf_{\gamma\in\Gamma:\ \text{hyperbolic}} |\mathrm{tr}\gamma| > 2,$$

where Γ is a Fuchsian group such that $R \cong \mathbb{H}/\Gamma$. In particular, we should have $A_2(R) \subset B_2(R)$ for any modulated Riemann surface. As an application of our arguments in previous sections, we shall give a simple proof for this fact with a concrete estimate for the quantity

$$\kappa_R := \sup\{\|\varphi\|_\infty; \varphi \in A_2(R) \text{ with } \|\varphi\|_1 = 1\}.$$

Here we note that $\kappa_R < \infty$ if and only if $A_2(R) \subset B_2(R)$. The following is due to Matsuzaki [33].

Theorem 6.1. *For each $[\alpha] \in \mathcal{S}_R$, we have $E[\alpha] \leq \kappa_R \ell[\alpha]^2$.*

Proof. We may assume that $E[\alpha] > 0$. Let φ_0 be the Jenkins-Strebel differential for α as in Theorem 2.1 with $\|\varphi_0\|_1 = 1$. Then, for $\alpha' \in [\alpha]$,

$$\begin{aligned} E[\alpha]^{1/2} &\leq \int_{\alpha'} |\varphi_0|^{1/2}|dz| = \int_{\alpha'} |\varphi_0\rho_R{}^{-2}|^{1/2}\rho_R|dz| \\ &\leq \|\varphi_0\|_\infty^{1/2} \int_{\alpha'} \rho_R|dz|. \end{aligned}$$

Since $\alpha' \in [\alpha]$ is arbitrary, we conclude that $E[\alpha] \leq \|\varphi_0\|_\infty \ell[\alpha]^2$, which proves the theorem. $\square$

On the other hand, by the mean value property of holomorphic functions, we can easily estimate the hyperbolic sup norm $\|\varphi\|_\infty$ by the L^1-norm $\|\varphi\|_1$ for a modulated surface as in the following way.

Proposition 6.2. *For any hyperbolic Riemann surface R, it holds that*

$$\kappa_R \leq \frac{1}{\pi}\coth^2 I_R,$$

which implies that $A_2(R) \subset B_2(R)$ for a modulated surface R.

Proof. Fix an arbitrary point $q \in R$. Let $p : \Delta = \{z \in \mathbb{C}; |z| < 1\} \to R$ be a holomorphic universal cover with $p(0) = q$. We denote by $\tilde{\varphi}$ the pull-back of $\varphi \in A_2(R)$ by p, i.e. $\tilde{\varphi}(z)dz^2 = \varphi(p(z))p'(z)^2dz^2$. Then, we can regard $\tilde{\varphi}$ as a holomorphic function satisfying the functional equations: $\tilde{\varphi}(\gamma(z))\gamma'(z)^2 = \tilde{\varphi}(z)$ for all covering transformation γ. Here, note that $|\varphi\rho_R^{-2}|(q) = |\tilde{\varphi}(0)|$ since $\rho_\Delta(0) = 1$. By the mean value property, for any $r \in (0,1)$ we have

$$\tilde{\varphi}(0) = \frac{1}{\pi r^2}\iint_{|z|<r}\tilde{\varphi}(z)dxdy.$$

Since p is injective in the disk $\{z; |z| < \tanh \iota_R(q)\} = D_\Delta(0, \iota_R(q))$, for $r = \tanh \iota_R(q)$ we have

$$\begin{aligned} |\tilde{\varphi}(0)| &\leq \frac{1}{\pi r^2}\iint_{|z|<r}|\tilde{\varphi}(z)|dxdy = \frac{1}{\pi r^2}\iint_{D_R(q,\iota_R(q))}|\varphi| \\ &\leq \frac{1}{\pi r^2}\|\varphi\|_1 = \frac{1}{\pi}\coth^2 \iota_R(q)\cdot\|\varphi\|_1. \end{aligned}$$

Thus we have the assertion that $\|\varphi\|_\infty \leq \frac{1}{\pi}\coth^2 I_R \cdot \|\varphi\|_1$. $\square$

Remarks. By the result above, we see that $\kappa_R = O({I_R}^{-2})$ as $I_R \to 0$. In fact, Matsuzaki [33] showed that $\kappa_R = O({I_R}^{-1})$ as $I_R \to 0$ by an argument using the Marden-Margulis constant. (Further, he gave a comparison theorem between κ_R and other geometric quantities.) Following Matsuzaki's method (in the torsion-free case), we can similarly show that

$$\kappa_R \leq \frac{2}{\pi}\left(\frac{\mu_0}{I_R} + 1\right),$$

where $\mu_0 = \sinh^{-1}(1) = \log(1+\sqrt{2}) = 0.8813\cdots$ is the number which appeared in [57], but this estimate is not so sharp when I_R is not small.

By combining Theorem 6.1 with Proposition 6.2, we obtain the following

Corollary 6.3. *For each* $[\alpha] \in \mathcal{S}_R$, *we have*

$$E[\alpha] \le \frac{1}{\pi} \coth^2 I_R \cdot \ell[\alpha]^2.$$

Since $I_R = \inf \frac{\ell[\alpha]}{2}$ and $\frac{2\pi}{M_R} = \inf E[\alpha]$, we see that

$$\frac{2\pi}{M_R} \le \frac{4}{\pi} {I_R}^2 \cdot \coth^2 I_R,$$

thus (2.5) is now proved.

7. A lower estimate of Hausdorff dimension.

First of all, we remind the reader definitions of the Hausdorff measures, Hausdorff contents and the Hausdorff dimension. For a positive number ε, a countable collection $(A_j)_{j=1,2,\cdots}$ of subsets of $\mathbb{R}^n$ is said to be an ε-cover of the set $E \subset \mathbb{R}^n$ if $E \subset \cup_j A_j$ and $\mathrm{diam} A_j < \varepsilon \quad (j = 1, 2, \cdots)$, where $\mathrm{diam} A_j$ denotes the Euclidean diameter of A_j. For a Borel set E in $\mathbb{R}^n$ and positive numbers α and ε, $\mathcal{H}^\alpha_\varepsilon(E)$ is defined by $\mathcal{H}^\alpha_\varepsilon(E) = \inf \sum_j (\mathrm{diam} A_j)^\alpha$, where the infimum is taken over all ε-cover (A_j) of E, and the α-dimensional Hausdorff measure $\mathcal{H}^\alpha(E)$ is defined as the limit of $\mathcal{H}^\alpha_\varepsilon(E)$ as ε tends to 0.

We also define the α-dimensional Hausdorff content $\Lambda^\alpha(E)$ by the $\inf \sum_j (\mathrm{diam} A_j/2)^\alpha = \sum_j {r_j}^\alpha$, where the infimum is taken over all countable covers (A_j) of E by closed disks with radii r_j. Since any bounded set in $\mathbb{R}^n$ with diameter d is contained by a closed ball with radius $\sqrt{3}d/2$, it is evident that $\Lambda^\alpha(E) \le (\sqrt{3}/2)^\alpha \mathcal{H}^\alpha(E)$.

It is easy to see that $\mathcal{H}^\alpha(E)$ is a non-increasing function of α and that if $\mathcal{H}^\alpha(E)$ assumes a positive finite value for some $\alpha = \alpha_0$ then $\mathcal{H}^\alpha(E) = 0$ if $\alpha > \alpha_0$ and $\mathcal{H}^\alpha(E) = +\infty$ if $\alpha < \alpha_0$. The critical point $\alpha_0 = \inf\{\alpha > 0; \mathcal{H}^\alpha(E) = 0\}$ is called the Hausdorff dimension of E and denoted by H-dim$E$. We note here that $0 \le \alpha_0 \le n$ and $\mathcal{H}^{\alpha_0}(E)$ may assume the value 0 or ∞. For more detailed exposition of the Hausdorff measures and the Hausdorff dimenstion, we refer to [7]. For the Hausdorff contents, see, for example, [42].

Järvi and Vuorinen [21] gave a definition of uniform perfectness for the closed subsets of $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, and showed that a uniformly perfect set E in $\mathbb{R}^n$ has positive Hausdorff dimension and its lower bound depends only on the modulus of $\mathbb{R}^n \setminus E$ and (possibly) n. (The modulus of an open set in $\mathbb{R}^n$ can be defined by a similar way as M°_D in two-dimensional case, and E is uniformly perfect if and only if the modulus of $\mathbb{R}^n \setminus E$ is finite, by their definition.)

We present here an explicit lower bound for the Hausdorff dimension of uniformly perfect sets, simultaneously give another characterization of uniform perfectness in terms of the Hausdorff contents. The proof is essentially due to Järvi and Vuorinen [21], while no explicit bounds are given by them.

Theorem 7.1 (Järvi-Vuorinen [21]). *Let E be a compact set in $\widehat{\mathbb{C}}$ and D its complement. Suppose that E is uniformly perfect, i.e. $M_D^\circ < \infty$. Then, for any $a \in E$ and $0 < r < \frac{1}{2}\mathrm{diam}E$, it holds that*

$$\Lambda^{\alpha_0}(E \cap B(a,r)) \geq \frac{1}{18}\left(\frac{2r}{\sqrt{3}}\right)^{\alpha_0},$$

where α_0 is the positive constant given by $\frac{\log 2}{\log(2\exp M_D^\circ + 1)}$ and $B(a,r)$ denotes the closed disk $\{z \in \mathbb{C} : |z-a| \leq r\}$.

Conversely, we suppose that there exist positive constants α and A such that

$$\Lambda^{\alpha}(E \cap B(a,r)) \geq Ar^{\alpha}$$

for any $a \in E$ and $0 < r < \frac{1}{2}\mathrm{diam}E$. Then we have $M_D^\circ \leq \frac{1}{\alpha}\log\frac{1}{A} + \log 12 < \infty$, hence E is uniformly perfect.

As an immedeate consequence of this, we have the following.

Theorem 7.2. *Let E be a uniformly perfect set in $\widehat{\mathbb{C}}$ and M° denote the round modulus of $\widehat{\mathbb{C}} \setminus E$. Then we have the following estimate.*

$$\text{H-dim}E \geq \frac{\log 2}{\log(2e^{M^\circ}+1)} \left(\geq \frac{\log 2}{M^\circ + \log 3}\right). \tag{7.1}$$

For a closed disk $B = B(a,r)$ we write $\mathrm{rad}B = r$ and $\mathrm{cent}B = a$. First, we show the next

Lemma 7.3. *Let E be a uniformly perfect compact set in $\widehat{\mathbb{C}}$ with $M_D^\circ < \beta$, where $D = \widehat{\mathbb{C}} \setminus E$. If a closed disk B is given so that $\mathrm{rad}B = r, \mathrm{cent}B \in E$ and $E \setminus B \neq \emptyset$. Then, there exist disjoint closed disks B_1, B_2 such that for each $j = 1, 2$,*

(1) $B_j \subset B$,
(2) $\mathrm{rad}B_j = cr$, *and*
(3) $\mathrm{cent}B_j \in E$,

where $c \in (0, \frac{1}{2})$ is the constant defined by $c = \frac{1}{2e^\beta+1}$.

Proof. By invariance of the constant M_D° under similarities, we may assume that $B = \{z \in \mathbb{C}; |z| \leq 1\}$. Put $B_1 = \{|z| \leq c\}$, then (1), (2) and (3) are satisfied for $j = 1$. Let $A = \{2c < |z| < 2e^\beta c\}$, then $A \cap E \neq \emptyset$ because $M_D^\circ < \beta$. Thus we can pick a point $a \in A \cap E$ and set $B_2 = B(a,c)$. By construction, we have $B_1 \cap B_2 = \emptyset$ and (1), (2) and (3) for $j = 2$. □

Proof of Theorem 7.1. Let α be an arbitrary number satisfying that $0 < \alpha < \alpha_0$ and β the number satisfying that $\alpha = \frac{\log 2}{\log(2\exp\beta+1)}$, then we have $\beta > M_D^\circ$. Fix $a \in E$ and $0 < r < \frac{1}{2}\mathrm{diam}E$ and set $B = B(a,r)$. We note that $E \setminus B \neq \emptyset$ and $\mathrm{cent}B \in E$ by assumption. Applying Lemma 7.3 inductively, we can select a sequence $(B_{i_1,\cdots,i_k})_{(i_1,\cdots,i_k)\in I^k}$, $\quad k=0,1,2,\cdots$, where $I=\{1,2\}$, of families of closed disks with the following properties:

(a) $B_{i_1,\cdots,i_k}\subset B_{i_1,\cdots,i_{k-1}}$, $\quad \mathrm{rad}B_{i_1,\cdots,i_k}=rc^k, \mathrm{cent}B_{i_1,\cdots,i_k}\in E$, and
(b) $B_{i_1,\cdots,i_{k-1},1}\cap B_{i_1,\cdots,i_{k-1},2}=\emptyset$

for any $k=0,1,\cdots$ and $(i_1,\cdots,i_k)\in I^k$. Here we interpret $B_\emptyset=B$ for $\emptyset\in I^0$.

We set $K=\cap_{k=0}^\infty\cup_{i_1,\cdots,i_k\in I}B_{i_1,\cdots,i_k}$, then we have $K\subset E$ by construction.

Here we show the next result, which seems to be classical, at least one can find a similar or more general statement in the paper [40] of Moran. For an ultimate result, we refer the reader to Hata [17].

Proposition 7.4. *Let I be the set $\{1,\cdots,p\}$ of indices. Suppose that a sequence $(B_{i_1,\cdots,i_k})_{(i_1,\cdots,i_k)\in I^k}$ $(k=0,1,2,\cdots)$ of families of closed balls in $\mathbb{R}^n$ satisfies the following conditions.*

(a) $B_{i_1,\cdots,i_k}\subset B_{i_1,\cdots,i_{k-1}}$, $\quad \mathrm{rad}B_{i_1,\cdots,i_k}=rc^k$, *and*
(b) $B_{i_1,\cdots,i_{k-1},l}\cap B_{i_1,\cdots,i_{k-1},m}=\emptyset$ *if* $l\neq m$

for any $k=0,1,\cdots$ and $(i_1,\cdots,i_k)\in I^k$, where $r=\mathrm{rad}B>0$ $(B=B_\emptyset)$ and $c\in(0,1)$ is a positive constant. Then, the Cantor set $K=\cap_{k=0}^\infty\cup_{i_1,\cdots,i_k\in I}B_{i_1,\cdots,i_k}$ has positive finite α-dimensional Hausdorff measure $\mathcal{H}^\alpha(K)$ and content $\Lambda^\alpha(K)$ with

$$\frac{1}{3^np}\left(\frac{2r}{\sqrt{3}}\right)^\alpha\le\left(\frac{2}{\sqrt{3}}\right)^\alpha\Lambda^\alpha(K)\le\mathcal{H}^\alpha(K)\le(2r)^\alpha,$$

where $\alpha=-\frac{\log p}{\log c}$. In particular, $\text{H-dim}K=-\frac{\log p}{\log c}$.

For the reader's convenience, we include the proof of this proposition, which will go along the same line as in [21]. In view of the above proposition, we see that, for any $0<\alpha<\alpha_0=\frac{\log 2}{\log(2e^{M_D^\circ}+1)}$, $\quad E$ has a compact (Cantor) subset of Hausdorff dimension α.

Proof of Proposition 7.4. The upper estimate of $\mathcal{H}^\alpha(K)$ is almost obvious. In fact, $(B_{i_1,\cdots,i_k})_{(i_1,\cdots,i_k)\in I^k}$ is an ε-cover of K if $rc^k<\varepsilon$, so we have

$$\mathcal{H}^\alpha_\varepsilon(K)\le\sum_{(i_1,\cdots,i_k)\in I^k}(\mathrm{diam}B_{i_1,\cdots,i_k})^\alpha=p^k(2rc^k)^\alpha=(2r)^\alpha.$$

Letting $\varepsilon\to0$, we have $\mathcal{H}^\alpha(K)\le(2r)^\alpha$. In order to prove the lower estimate of $\Lambda^\alpha(K)$, we make a preliminary observation. First we observe that there exists a Borel probability measure μ on $\mathbb{C}$ with the support K such that $\mu(B_{i_1,\cdots,i_k})=\frac{1}{p^k}$ for each $k=0,1,2,\cdots$ and $i_1,\cdots,i_k\in I$. In fact, let $f:I^{\mathbb{N}}\to K$ be the map defined by

$$\{f((i_k)_{k\in\mathbb{N}})\}=\cap_{k=1}^\infty B_{i_1,\cdots,i_k},$$

then $f:I^{\mathbb{N}}\to K$ is a homeomorphism if we equip I and $I^{\mathbb{N}}$ the discrete and product topologies, respectively. We know that $I^{\mathbb{N}}$ has a standard Bernoulli measure ν so that

$\nu([i_1, \cdots, i_k]) = p^{-k}$ for each cylinder set $[i_1, \cdots, i_k] = \{(j_l)_{l \in \mathbb{N}} \in I^{\mathbb{N}}; j_1 = i_1, \cdots, j_k = i_k\}$. Then μ can be obtained as the image measure $f_* \nu$ of ν by f.

Now we shall show that for any closed ball $A = B(x, \rho)$ in $\mathbb{R}^n$, $\mu(A) \le 3^n p(\rho/r)^\alpha$. If $\rho \ge r$, we have nothing to prove. So we may assume that $\rho < r$. Choose an integer $k \ge 0$ so that $rc^{k+1} < \rho \le rc^k$. Since for each $(i_1, \cdots, i_k) \in J = \{(i_1, \cdots, i_k) \in I^k; B_{i_1,\cdots,i_k} \cap A \ne \emptyset\}$ we have $B_{i_1,\cdots,i_k} \subset B(x, \rho + 2rc^k)$, we can conclude $\cup_{(i_1,\cdots,i_k)\in J} B_{i_1,\cdots,i_k} \subset B(x, \rho + 2rc^k)$. Therefore, we see that

$$\#J \cdot (rc^k)^n \omega_n \le (\rho + 2rc^k)^n \omega_n,$$

where ω_n denotes the Euclidean volume of unit ball in $\mathbb{R}^n$ and $\#J$ the cardinality of J. From this, we conclude that $\#J \le (\frac{\rho}{rc^k} + 2)^n \le 3^n$. Therefore,

$$\mu(A) \le \mu\Big(\bigcup_{(i_1,\cdots,i_k)\in J} B_{i_1,\cdots,i_k}\Big) = \#J \cdot \frac{1}{p^k} \le 3^n p^{-k} = 3^n p (c^{k+1})^\alpha < 3^n p \left(\frac{\rho}{r}\right)^\alpha.$$

By this observation, for any countable cover (A_j) of K by closed balls with radii ρ_j, we have

$$1 = \mu(K) \le \sum_j \mu(A_j) \le 3^n p \sum_j \left(\frac{\rho_j}{r}\right)^\alpha,$$

thus, $\frac{r^\alpha}{3^n p} \le \sum_j {\rho_j}^\alpha$. Since (A_j) is arbitrary, we have $\frac{r^\alpha}{3^n p} \le \Lambda^\alpha(K)$ and this implies the desired lower estimate. $\square$

Proof of Theorem 7.1 *(continued).* We now return to the proof of Proposition 7.1. By Proposition 7.4, we know that $\Lambda^\alpha(K) \ge r^\alpha/18$. Since $K \subset E \cap B(a, r)$ by construction, we have $\Lambda^\alpha(E \cap B(a, r)) \ge r^\alpha/18$ for any $\alpha < \alpha_0$. Since, for a compact set F, $\Lambda^\alpha(F)$ is continuous with respect to α from the left [42], we see that the former part is valid. Next we shall show the latter part. Suppose that $a \in E$ and $0 < r < \frac{1}{2}\mathrm{diam} E$. Then, by assumption, $\Lambda(E \cap B(a, r)) \ge Ar^\alpha$. We note first that $\Lambda^\alpha(B(a, r)) \le (2r)^\alpha$, since $\{B(a, r)\}$ is a cover of $B(a, r)$. Therefore, for any $0 < c < \frac{1}{2}A^{1/\alpha}$, it follows that

$$\Lambda^\alpha(B(a, cr)) \le (2cr)^\alpha < Ar^\alpha \le \Lambda^\alpha(E \cap B(a, r)),$$

in particular, $E \cap B(a, r) \setminus B(a, cr) \ne \emptyset$. For $\frac{1}{2}\mathrm{diam} E \le r < \mathrm{diam} E$, we have

$$E \cap B(a, r) \setminus B(a, cr/2) \supset E \cap B(a, r/2) \setminus B(a, cr/2) \ne \emptyset.$$

Thus we conclude that $E_D \ge \frac{1}{4}A^{1/\alpha}$, where E_D is the constant introduced in Section 2. By (2.9), we obtain

$$M_D^\circ \le \log \frac{3}{E_D} \le \log \frac{12}{A^{1/\alpha}} = \frac{1}{\alpha} \log \frac{1}{A} + \log 12.$$

$\square$

Example. We consider here classical Cantor sets in $\mathbb{C}$. For any $t \in (0, \frac{1}{2})$, we define affine maps f_1 and f_2 by $f_1(x) = tx$ and $f_2(x) = tx + 1 - t$. We set $K_0 = [0,1]$, and $K_1 = f_1(K_0) \cup f_2(K_0)$. Generally, $K_j (j = 0, 1, 2, \cdots)$ can be defined inductively by $K_j = f_1(K_{j-1}) \cup f_2(K_{j-1})$. Then, $K_\infty = K_\infty(t) := \cap_{j=0}^{\infty} K_j$ is a classical Cantor set. By Proposition 7.4, H-dim$K_\infty(t) = \frac{\log 2}{\log 1/t}$. On the other hand, it is easily seen that $M_D^\circ = \log \frac{1-2t+t/2}{t/2} = \log \frac{2-3t}{t}$, where $D = \widehat{\mathbb{C}} \setminus K_\infty$. Hence, $\frac{\log 2}{\log(2e^{M_D^\circ}+1)} = \frac{\log 2}{\log 1/t + \log(4-5t)}$, so the estimate (7.1) is fairly good.

Remark. Theorem 7.2 can easily be generalized to higher dimensional case with the exactly same bound as (7.1), thus independent of the dimension. Compare with the corresponding statement in [21].

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