

# GEOMETRIC FUNCTION THEORY AND SMALE'S MEAN VALUE CONJECTURE

EGE FUJIKAWA AND TOSHIYUKI SUGAWA

ABSTRACT. We improve an estimate of the constant in Smale's mean value conjecture, by using the Bieberbach theorem for coefficients of univalent functions and an estimate of the hyperbolic density on a certain simply connected domain.

## 1. INTRODUCTION AND RESULT

Let  $P(z)$  be a complex polynomial of degree  $d \geq 2$ , and let  $z_1, z_2, \dots, z_{d-1}$  be the critical points of  $P(z)$ . Smale [11] stated that, if  $z$  is not a critical point of  $P$ , then the following inequality holds:

$$(1) \quad \min_i \left| \frac{P(z) - P(z_i)}{z - z_i} \right| \leq 4|P'(z)|.$$

Furthermore, he also formulated the following conjecture, which is known as Smale's mean value conjecture. See also [10] and [12].

**Conjecture 1.** *Let  $P(z)$  be a polynomial of degree  $d \geq 2$  and let  $z_1, z_2, \dots, z_{d-1}$  be the critical points of  $P(z)$ . If  $z$  is not a critical point of  $P$ , then*

$$(2) \quad \min_i \left| \frac{P(z) - P(z_i)}{(z - z_i)P'(z)} \right| \leq \frac{d-1}{d}.$$

A weaker version of Smale's conjecture is the inequality with constant 1 instead of  $(d-1)/d$  in (2). Let  $S(P, z)$  be the left-hand side of inequality (2), and denote by  $K(d)$ ,  $d \geq 2$ , the smallest constant such that  $S(P, z) \leq K(d)$  holds for all polynomials  $P$  of degree  $d$  and for all  $z \neq z_i$ . Inequality (1) shows that  $K(d) \leq 4$  and in view of the example  $P(z) = z^d - z$ , one has the inequality  $K(d) \geq (d-1)/d$ . Smale's mean value conjecture thus says that  $K(d) \leq (d-1)/d$ . This conjecture has been proved only for degrees  $d = 2, 3, 4$  (see [9]) and  $d = 5$  (see [3]). For  $d \geq 6$ , it has been proved only under some additional conditions. See [7], [13] and [14]. In a general case, Beardon, Minda and Ng [1] proved that

$$K(d) \leq 4^{\frac{d-2}{d-1}} =: K_1(d)$$

and Conte, Fujikawa and Lacic [2] verified that

$$K(d) \leq 4 \frac{d-1}{d+1} =: K_2(d) \quad (< K_1(d)).$$

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1991 *Mathematics Subject Classification.* Primary 30C10; Secondary 30C20, 30C50.

*Key words and phrases.* polynomial, critical point, univalent function, Poincaré density.

Furthermore, Schmeisser [8] showed that

$$K(d) \leq \frac{2^d - (d+1)}{d(d-1)} =: K_3(d).$$

In this paper, we improve these estimates.

**Theorem 1.** *Let  $P$  be a polynomial of degree  $d \geq 2$  with critical points  $z_1, z_2, \dots, z_{d-1}$ . If  $z$  is not a critical point of  $P$ , then*

$$\begin{aligned} \min_i \left| \frac{P(z) - P(z_i)}{(z - z_i)P'(z)} \right| &\leq 4 \cdot \frac{1 + (d-2)4^{\frac{1}{1-d}}}{d+1} \\ &=: K_0(d). \end{aligned}$$

*Remark.* For  $d \geq 7$ , our constant  $K_0(d)$  is better than the other ones. More precisely,

- (i)  $K_0(d) < K_2(d) < K_3(d)$  for  $d \geq 8$ ;
- (ii)  $K_0(7) = 2.48425 \dots < K_3(7) < K_2(7)$ ;
- (iii)  $K_3(d) < K_0(d) < K_2(d)$  for  $d \leq 6$ .

In particular,  $K_3(6) = 1.9$ . Note also that these results are superfluous when  $d \leq 5$  since Smale's conjecture was already proved.

For all linear transformations  $\alpha$  and  $\beta$ , we have  $S(\beta \circ P \circ \alpha, \alpha^{-1}(z)) = S(P, z)$ . Thus we have only to consider for  $z = 0$  and for polynomials  $P$  satisfying  $P(0) = 0$ ,  $P'(0) = 1$  (see [1]). Namely, Smale's mean value conjecture is equivalent to the following.

**Conjecture 2.** *Let  $P(z)$  be a polynomial of degree  $d \geq 2$  with  $P(0) = 0$  and  $P'(0) = 1$ , and let  $z_1, z_2, \dots, z_{d-1}$  be the critical points of  $P(z)$ . Then*

$$\min_i \left| \frac{P(z_i)}{z_i} \right| \leq \frac{d-1}{d}.$$

Conjecture 2 is called the normalized conjecture, and this has been proved for polynomials satisfying certain conditions. For example, either if all the critical points of  $P$  are real or if all the zeros of  $P$  but the origin have the same modulus, then the normalized conjecture is true. Furthermore, Ng [6] showed that  $S(P, 0) \leq 2$  for odd polynomials  $P$ . For a general case, we have the following, which is equivalent to Theorem 1.

**Theorem 2.** *Let  $P(z)$  be a polynomial of degree  $d \geq 2$  with  $P(0) = 0$  and  $P'(0) = 1$ , and  $z_1, z_2, \dots, z_{d-1}$  the critical points of  $P(z)$ . Then*

$$\min_i \left| \frac{P(z_i)}{z_i} \right| \leq K_0(d).$$

## 2. PROOF OF THEOREM

We have only to prove Theorem 2. We denote by  $\rho_\Omega(z)|dz|$  the hyperbolic metric of a hyperbolic domain with curvature  $-4$ . The quantity  $\rho_\Omega(z)$  is called the hyperbolic density of  $\Omega$  at  $z \in \Omega$ . For instance, the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  has the hyperbolic density

$$\rho_{\mathbb{D}}(z) = \frac{1}{1 - |z|^2}.$$

**Lemma 1** ([1]). *For every domain  $\Omega$  of the form  $\mathbb{C} - (R_1 \cup \dots \cup R_n)$  where  $R_i$  are rays of the form  $\{re^{i\theta_i} \mid r \geq 1\}$ , the hyperbolic density  $\rho_\Omega(z)$  of  $\Omega$  satisfies the inequality*

$$\rho_\Omega(0) \leq 4^{-\frac{1}{n}}.$$

We will prove our theorem by using this lemma and the Bieberbach theorem for univalent functions on  $\mathbb{D}$  (see [5]). The proof is based on the argument in [2].

*Proof of Theorem 2.* We may assume that  $\min_i |z_i| = |z_1|$  with  $z_1 > 0$  and  $\min_i |P(z_i)| = 1$  by compositions of linear transformations, see [2]. Then

$$\min_i \left| \frac{P(z_i)}{z_i} \right| \leq \left| \frac{P(z_j)}{z_j} \right| = \frac{1}{|z_j|} \leq \frac{1}{z_1},$$

where  $j$  is the integer satisfying

$$|P(z_j)| = \min_i |P(z_i)| = 1.$$

Thus we will prove that

$$\frac{1}{z_1} \leq K_0(d).$$

Since  $z_1, \dots, z_{d-1}$  are critical points of  $P$  and  $P'(0) = 1$ , we have

$$P'(z) = \left(1 - \frac{z}{z_1}\right) \left(1 - \frac{z}{z_2}\right) \dots \left(1 - \frac{z}{z_{d-1}}\right).$$

Then, since  $P(0) = 0$ , this gives

$$\begin{aligned} P(z) &= z - \left(\frac{1}{2} \sum_{i=1}^{d-1} \frac{1}{z_i}\right) z^2 + \left(\frac{1}{3} \sum_{i \neq j}^{d-1} \frac{1}{z_i z_j}\right) z^3 \\ &\quad - \dots + \frac{(-1)^{d-1} z^d}{d \cdot z_1 z_2 \dots z_{d-1}}. \end{aligned}$$

Let  $R_i$  be the ray of the form  $\{re^{i\theta_i} \mid r \geq 1\}$  that passes through  $P(z_i)$ . By Lemma 1, the hyperbolic density  $\rho_\Omega(z)$  of  $\Omega = \mathbb{C} - (R_1 \cup \dots \cup R_{d-1})$  satisfies

$$\rho_\Omega(0) \leq 4^{-\frac{1}{d-1}}.$$

Since  $\Omega$  does not contain any critical value of  $P$ , one can take a (single-valued) branch  $f$  of the inverse function  $P^{-1}$  on  $\Omega$  so that  $f(0) = 0$ . In this way, we obtain a univalent function

$$f : \Omega \rightarrow \mathbb{C} - \{z_1, \dots, z_{d-1}\}$$

such that  $f(0) = 0$  and  $P(f(w)) = w$  for all  $w \in \Omega$ . Then  $f$  has the form

$$f(w) = w + a_2 w^2 + a_3 w^3 + \dots.$$

Since  $f$  omits the value  $z_1$  in  $\Omega$ , the function

$$\begin{aligned} f_1(w) &= \frac{f(w)}{1 - f(w)/z_1} \\ &= w + \left(a_2 + \frac{1}{z_1}\right) w^2 + \dots \end{aligned}$$

is analytic in  $\Omega$ . By applying the Bieberbach theorem [5, Theorem 2.2] to the univalent function  $f_1$  on  $\mathbb{D} (\subset \Omega)$ , we have

$$(3) \quad \left| a_2 + \frac{1}{z_1} \right| \leq 2.$$

Since  $P(f(w)) = w$ , we obtain

$$-P''(0) = f''(0) = 2a_2.$$

Thus

$$a_2 = -\frac{P''(0)}{2} = \frac{1}{2} \sum_{i=1}^{d-1} \frac{1}{z_i}.$$

Therefore inequality (3) yields that

$$\left| \frac{3}{z_1} + \sum_{i=2}^{d-1} \frac{1}{z_i} \right| \leq 4.$$

Since we assumed that  $z_1$  is real, we have

$$(4) \quad \frac{3}{z_1} + \sum_{i=2}^{d-1} \operatorname{Re} \frac{1}{z_i} \leq 4.$$

Let  $\phi : \mathbb{D} \rightarrow \Omega$  be a conformal homeomorphism satisfying  $\phi(0) = 0$ , which has the form

$$\phi(\zeta) = c_1\zeta + c_2\zeta^2 + \dots.$$

Since the hyperbolic density  $\rho_\Omega$  on  $\Omega$  satisfies

$$\rho_\Omega(\phi(\zeta))|\phi'(\zeta)| = \rho_\mathbb{D}(\zeta),$$

we have  $\rho_\Omega(0)|c_1| = \rho_\mathbb{D}(0) = 1$ . Thus

$$|c_1| = \frac{1}{\rho_\Omega(0)} \geq 4^{\frac{1}{d-1}}.$$

Consider the function

$$\begin{aligned} g(\zeta) &= (f \circ \phi)(\zeta) \\ &= c_1\zeta + (c_2 + c_1^2 a_2)\zeta^2 + \dots, \end{aligned}$$

which maps  $\mathbb{D}$  conformally into  $\mathbb{C} - \{z_1, \dots, z_{d-1}\}$ . Furthermore, for  $i = 1, \dots, d-1$ , set

$$\begin{aligned} g_i(\zeta) &= \frac{g(\zeta)}{1 - g(\zeta)/z_i} \\ &= c_1\zeta + \left( c_2 + c_1^2 \left( a_2 + \frac{1}{z_i} \right) \right) \zeta^2 + \dots, \end{aligned}$$

and  $h_i(\zeta) := g_i(\zeta)/c_1$ . Then  $h_i : \mathbb{D} \rightarrow \mathbb{C}$  is a univalent function satisfying  $h_i(0) = 0$  and  $h_i'(0) = 1$ . By applying the Bieberbach theorem to  $h_i(\zeta)$ , we have

$$\left| \frac{c_2}{c_1} + c_1 \left( a_2 + \frac{1}{z_i} \right) \right| \leq 2,$$

namely,

$$\left| \frac{c_2}{c_1} + a_2 + \frac{1}{z_i} \right| \leq \frac{2}{|c_1|}.$$

In particular,

$$\left| \frac{c_2}{c_1^2} + a_2 + \frac{1}{z_1} \right| \leq \frac{2}{|c_1|}.$$

By the triangle inequality, we see that

$$\left| \frac{1}{z_i} - \frac{1}{z_1} \right| \leq \frac{4}{|c_1|} \leq 4 \cdot 4^{-\frac{1}{d-1}} = 4^{\frac{d-2}{d-1}}.$$

Since we assumed that  $z_1$  is real, we have

$$(5) \quad \frac{1}{z_1} - 4^{\frac{d-2}{d-1}} \leq \operatorname{Re} \frac{1}{z_i}.$$

Therefore, inequalities (4) and (5) yield that

$$\frac{3}{z_1} + (d-2) \left( \frac{1}{z_1} - 4^{\frac{d-2}{d-1}} \right) \leq 4.$$

This implies that

$$\frac{1}{z_1} \leq 4 \cdot \frac{1 + (d-2)4^{\frac{1}{1-d}}}{d+1}$$

and we have proved our theorem.  $\square$

### 3. CONCLUDING REMARK

The present framework can be used to show the existence of an extremal polynomial for the constant  $K(d)$ . Note that the existence of such a polynomial is not trivial. We end the article by showing the following proposition. Note that Crane [4, §5] gives essentially the same conclusion and our proof is similar to his argument.

**Proposition 1.** *Let  $d$  be an integer with  $d \geq 2$ . There exists a complex polynomial  $P(z)$  of degree at most  $d$  such that  $S(P, 0) = K(d)$ .*

*Proof.* Denote by  $\mathcal{P}_0(d)$  the set of complex polynomials  $P(z)$  of degree  $d$  satisfying  $P(0) = 0$ ,  $P'(0) = 1$  and  $\min_i |P(z_i)| = 1$ , where  $z_1, z_2, \dots, z_{d-1}$  are the critical points of  $P(z)$ . Recall then that  $S(P, 0) = \min_i |P(z_i)/z_i|$ . Set

$$\mathcal{P}(d) = \mathcal{P}_0(2) \cup \dots \cup \mathcal{P}_0(d)$$

for  $d \geq 2$ . Our goal is to find a  $P \in \mathcal{P}(d)$  such that  $S(P, 0) = K(d)$ .

First note that  $K(d-1) \leq K(d)$  for  $d \geq 3$ . Indeed, for each  $P \in \mathcal{P}_0(d-1)$  define  $P_n \in \mathcal{P}_0(d)$  so that  $P_n'(z) = P'(z)(1 - z/n)$  for  $n = 1, 2, \dots$ . Then  $S(P_n, 0) \rightarrow S(P, 0)$  as  $n \rightarrow \infty$ . Therefore,  $K(d-1) \leq K(d)$ .

For each  $P \in \mathcal{P}_0(d)$ , we take a univalent function  $f$  on  $\Omega = \mathbb{C} - (R_1 \cup \dots \cup R_{d-1})$  with  $f(0) = 0$  and  $P \circ f = \operatorname{id}$  as in the proof of Theorem 2.

As we have seen in the last section, we have

$$K(d) = \sup_{P \in \mathcal{P}_0(d)} S(P, 0).$$

Therefore, there is a sequence  $P_n$  in  $\mathcal{P}_0(d)$  such that  $S(P_n, 0) \rightarrow K(d)$  as  $n \rightarrow \infty$ . Let  $f_n$  be the univalent function on  $\Omega_n$  constructed above for  $f = f_n$ . The restriction of  $f_n$  to  $\mathbb{D}$  is a member of the well-known family  $S$  of normalized univalent functions on the unit disk (cf. [5]). Since  $S$  is normal, we may assume that  $f_n$  converges to a function  $f_\infty \in S$  uniformly on every compact subset of  $\mathbb{D}$ .

By the Koebe one-quarter theorem,  $f(\mathbb{D})$  contains the disk  $\Delta = \{|z| < 1/4\}$  for each  $f \in S$ . Thus we can define the inverse function  $f^{-1}$  of  $f$  on  $\Delta$ . It is easy to see that  $f_n^{-1} = P_n$  converges to  $f_\infty^{-1}$  uniformly on every compact subset of  $\Delta$ . If we write

$$P_n(z) = a_{n,0} + a_{n,1}z + \cdots + a_{n,d}z^d$$

and

$$f_\infty^{-1}(z) = a_0 + a_1z + \cdots$$

around  $z = 0$ , the Cauchy integral formula gives

$$\begin{aligned} a_k &= \frac{1}{2\pi i} \int_{|z|=1/8} \frac{f_\infty^{-1}(z)dz}{z^{k+1}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{|z|=1/8} \frac{P_n(z)dz}{z^{k+1}} \\ &= \begin{cases} \lim_{n \rightarrow \infty} a_{n,k} & (0 \leq k \leq d) \\ 0 & (d < k). \end{cases} \end{aligned}$$

Therefore,  $f_\infty^{-1}$  is the restriction of a polynomial  $Q$  of degree  $\leq d$  to  $\Delta$  and  $P_n$  converges to  $Q$  uniformly on every compact subset of  $\mathbb{C}$ .

The degree of the limit polynomial  $Q$  is at least 2. Indeed, we take a critical point  $\zeta_n$  of  $P_n \in \mathcal{P}(d)$  so that  $|P_n(\zeta_n)| = 1$ . Since  $K(d) \geq 1 - 1/d \geq 1/2$ , we may assume that  $S(P_n, 0) \geq 1/3$  for sufficiently large  $n$ . Since  $S(P_n, 0) \leq |P_n(\zeta_n)/\zeta_n| = 1/|\zeta_n|$ , we have  $|\zeta_n| \leq 3$ . Then we can take a subsequence so that  $\zeta_n$  converges to a point  $\zeta$ , which satisfies  $Q'(\zeta) = 0$ . In particular,  $\deg Q \geq 2$ .

Next we will prove that  $S(Q, 0) = K(d)$ . Let  $\eta \neq 0$  be a critical point of  $Q$  such that  $S(Q, 0) = |Q(\eta)/\eta|$ . By the Hurwitz theorem, we can take a critical point  $\eta_n$  of  $P_n$  so that  $\eta_n \rightarrow \eta$ , and hence,

$$|P_n(\eta_n)/\eta_n| \rightarrow |Q(\eta)/\eta| = S(Q, 0).$$

Since

$$S(P_n, 0) \leq |P_n(\eta_n)/\eta_n|$$

and

$$S(P_n, 0) \rightarrow K(d),$$

we have  $S(Q, 0) \geq K(d)$ . On the other hand,

$$S(Q, 0) \leq K(\deg Q) \leq K(d),$$

and thus,  $S(Q, 0) = K(d)$ .  $\square$

In the above proof, it seems difficult to exclude the possibility that  $Q \in \mathcal{P}(d-1)$ . However, if we knew that  $K(d-1) < K(d)$ , then we could conclude that  $Q \in \mathcal{P}(d)$ .

Note that Crane [4] pointed out that the assertion  $K(d-1) < K(d)$  would lead to several conclusions concerning extremal polynomials.

**Acknowledgements.** The authors express their gratitude to Professor Nikola Lakic for encouragement. They also thank Professor Edward Crane for his valuable comments and suggestions.

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DEPARTMENT OF MATHEMATICS, SOPHIA UNIVERSITY, 7-1 KIOI-CHO, CHIYODA-KU, TOKYO, 102-8554 JAPAN

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSIMA UNIVERSITY, HIGASHI-HIROSHIMA, 739-8526 JAPAN