EQUIVALENCE PROBLEM FOR ANNULI AND BELL REPRESENTATIONS IN THE PLANE

MOONJA JEONG, JONG-WON OH, AND MASAHIKO TANIGUCHI

Abstract. In this paper we solve the problem of equivalence between annuli and Bell representations, which are canonical planar domains of connectivity 2. We give necessary and sufficient condition for equivalence as an explicit formula in parameters that shape domains.

1. Introduction

In [4] and [5], S. Bell sought for the domain with algebraic Bergman kernel and thought of the domain

$$W_{\mathbf{a},\mathbf{b},r} := \left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < r \right\}$$

as an example for complex a_k and b_k, positive r, and $(\mathbf{a}, \mathbf{b}) := (a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1})$. Then he asked if every non-degenerate n-connected planar domain with $n > 1$ can be mapped biholomorphically onto the domain $W_{\mathbf{a},\mathbf{b},r}$.

Jeong and Taniguchi showed in [7] that it is true with $r = 1$. For every $n \geq 2$ let $\mathbf{B}_n$ be the set of $(\mathbf{a}, \mathbf{b}) \in \mathbb{C}^{2n-2}$ such that the corresponding domains $W_{\mathbf{a},\mathbf{b},1}$ are non-degenerate n-connected planar domains. We call $\mathbf{B}_n$ the *coefficient body* for non-degenerate n-connected canonical planar domains. The coefficient body $\mathbf{B}_n$ of admissible coefficients a_k and b_k is completely characterized by the same authors in [9]. $W_{\mathbf{a},\mathbf{b},1}$ are *new* canonical planar domains of n-connectivity, and hence referred to as Bell representations. From now on we denote $W_{\mathbf{a},\mathbf{b},1}$ as $W_{\mathbf{a},\mathbf{b}}$ for convenience.

This gives rise to a *problem of equivalence* between our new canonical domains and classic canonical domains. In case of connectivity $n = 2$, classic canonical domain is one parameter family of annuli for $0 < \rho < 1$

$$\Omega_{\rho^2} := \{z \in \mathbb{C} : \rho^2 < |z| < 1\}.$$

It is well known that every nondegenerate doubly-connected domain in $\mathbb{C}$ is conformally equivalent to exactly one of Ω_{ρ^2}, $0 < \rho < 1$.

For $n = 2$, Bell representation with a reduced coefficient body $W_{a,0}$, $0 < a < \frac{1}{4}$ is biholomorphic via $z \mapsto z/\sqrt{a}$ to

$$A(r) = \left\{ z \in \mathbb{C} : \left| z + \frac{1}{z} \right| < r \right\}, \quad r > 2$$

Date: January 25, 2005.

1991 *Mathematics Subject Classification.* Primary 30C20, 30C40; Secondary 30C35, 32G15.

Key words and phrases. Annulus, Bell representation, Ahlfors map, critical points, Teichmüller extremal domain, theta constants, Lambert series.

The first author was supported by Grant R04-2003-000-10045-0 from the Korea Science and Engineering Foundation.

The second author was supported by BK21-Yonsei University.

with $a = r^{-2}$. Every nondegenerate doubly-connected domain in $\mathbb{C}$ is conformally equivalent to exactly one of $A(r)$, $r > 2$, a *version* of Bell representation for the connectivity 2 (see [5]). We study in this paper an equivalence

Problem 1.1. Can we tell which of Ω_{ρ^2}, $0 < \rho < 1$ is conformally equivalent to which of $A(r)$, $r > 2$?

Conformal invariants are the first to examine. For a doubly connected domain G, let Γ be the family of closed curves in G separating bounded and unbounded components of the complement. *Extremal length* of Γ is $\lambda(\Gamma) := \sup_\phi \left(\inf_{\gamma\in\Gamma} \int_\gamma \phi\, |dz|\right)^2 / \iint \phi^2 dxdy$, where the supremum is taken over all measurable $\phi \geq 0$ on the whole plane with $\iint \phi^2 dxdy \neq 0, \infty$. *Module* of G is $M(G) := \lambda(\Gamma)^{-1}$. Our annulus is of $M\left(\Omega_{\rho^2}\right) = (\pi)^{-1} \ln \rho^{-1}$ (see [1, p.13]).

Both extremal lengths and modules of two doubly connected domains coincide if and only if the two domains are conformally equivalent. However, extremal length or module of a given domain is often hardly computable, which is the case of Bell representation $G = A(r)$.

In this paper we look into a new conformal invariant for G by Jeong and Taniguchi [8]. The invariant is a pair (f, J), where $f : G \to U$ is a branched double covering of the unit disc and J is a biholomorphic involution of G satisfying $f \circ J = f$. The map f is critical at the fixed points of J. If G 's are conformally equivalent, the sets of fixed points of J 's are equivalent, and then the critical values of f are equivalent up to automorphism of the unit disc.

By means of pair (f, J) and Tegtmeyer's work [11] on annuli, we in section 4 relate Bell representation $A(r)$ to conformally equivalent annulus Ω_{ρ^2} by

$$r = \frac{2}{c(\rho)}$$

with an explicit formula for $c(\rho)$, answering Problem 1.1. We also find the set $E(\Omega_{\rho^2})$ consisting of all points in $\mathbf{B}_2$ which correspond to 2-connected canonical domains biholomorphically equivalent to Ω_{ρ^2}.

In section 5, by successive conformal transformations we relate $G = A(r)$ to a Teichmüller extremal domain(see [1, p.35]). Applying identities of the theta constants we obtain another formula for $c(\rho)$. Expanding it into Lambert series, we have the expression for $c(\rho)$ similar to the one in section 4.

However, the methods to obtain the formulas for $c(\rho)$ in section 4 and section 5 are different and the two formulas are related nontrivially.

2. Property of the Ahlfors map

Let Ω be a given non-degenerate n-connected planar domain with C^∞ smooth boundary $b\Omega$. We can assume that $b\Omega$ consists of exactly n non-intersecting smooth simple closed curves with parameterization $z_j(t)$, $0 \leq t \leq 1$, $j = 1, \ldots, n$. Let T_b be the complex unit tangent function on $b\Omega$ defined by $T_b(z_j(t)) = z_j'(t)/|z_j'(t)|$.

Fix a point a in Ω, and let f_a be the Ahlfors map of Ω with base point a. Among all holomorphic functions h which map into the unit disc with $h(a) = 0$, the Ahlfors map f_a is the unique function which maximizes $|h'(a)|$ with $f_a'(a) > 0$. Here for the definition and properties of the Ahlfors map, see [3]. In particular, f_a maps Ω properly and holomorphically onto the unit disc. Moreover, f_a can be extended to a continuous map of $\overline{\Omega}$ onto the closed unit disc so that every component γ_j of $b\Omega$, where $j = 1, \cdots, n$, is mapped homeomorphically onto the unit circle.

The Ahlfors map can be expressed as the quotient of the Szegő kernel and the Garabedian kernel via

$$f_a(z) = \frac{S(z,a)}{L(z,a)} \tag{2.1}$$

for $z \in \Omega$. The Garabedian kernel $L(z,a)$ is the kernel for the orthogonal projection from $L^2(b\Omega)$ onto the orthogonal complement of $H^2(b\Omega)$ and is represented by

$$L(z,a) = \frac{1}{2\pi}\frac{1}{z-a} + H_a(z)$$

where H_a is holomorphic on a neighborhood of $\overline{\Omega}$. The Garabedian kernel $L(z,a)$ and the Szegő kernel $S(z,a)$ are related via the identity

$$\overline{S(z,a)} = -iL(z,a)T_b(z)$$

for $a \in \Omega, z \in b\Omega$.

The Szegő kernel $S(z,a)$ has exactly $n-1$ zeroes $a_1, a_2, \cdots, a_{n-1}$ in Ω and $S(a,a) > 0$. The simple zero of f_a at a comes from the simple pole of $L(z,a)$ at a. The n-to-one map f_a must have $n-1$ zeroes besides the one at a and these zeroes coincide with the zeroes of $S(z,a)$ since $L(z,a)$ is nonvanishing.

3. Annulus and Bell representation

3.1. **Annulus.** Let $0 < \rho < 1$ and f_ρ be the Ahlfors map of $\Omega_{\rho^2} = \{z \in \mathbb{C} : \rho^2 < |z| < 1\}$ with base point ρ. An orthonormal basis for $H^2\left(b\Omega_{\rho^2}\right)$ is for $n \in \mathbb{Z}$

$$\varphi_n(z) = \frac{z^n}{\sqrt{2\pi\left(1+\rho^{4n+2}\right)}}$$

and hence the Szegő kernel for Ω_{ρ^2} with base point ρ is

$$S(z,\rho) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(z\rho)^n}{1+\rho^{4n+2}}, \tag{3.1}$$

which converges absolutely and uniformly on compact subsets. The Garabedian kernel is

$$L(z,\rho) = \frac{1}{2\pi}\frac{1}{z-\rho} + \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{\rho^{4n+2}}{1+\rho^{4n+2}}\frac{z^{2n+1}-\rho^{2n+1}}{(z\rho)^{n+1}} \tag{3.2}$$

(see [12]). The Ahlfors map with base point ρ is $f_\rho(z) = S(z,\rho)/L(z,\rho)$ as in (2.1), which gives 2-sheeted branched covering of the unit disc U by Ω_{ρ^2}.

The map f_ρ induces a nontrivial automorphism J_ρ of Ω_{ρ^2} that satisfies that

$$f_\rho(z) = f_\rho(J_\rho(z)), \qquad z \in \Omega_\rho. \tag{3.3}$$

Since $f_\rho(\rho) = f_\rho(-\rho) = 0$ (see [10]), J_ρ maps ρ to $-\rho$. Hence

$$J_\rho(z) = -\rho^2/z.$$

Also the uniqueness of the Ahlfors functions implies that

$$f_{-\rho}(z) = f_\rho(J_\rho(z)) = f_\rho(z) \tag{3.4}$$

(For the details, see [10]). In [11] Tegtmeyer proved the following lemma and for most part the proof involves manipulation of power series (3.1) and (3.2). In [10] Mair and McCullough had already proved it, but not using power series. Here we provide a proof for reader's convenience.

Lemma 3.1. *The Ahlfors map f_ρ satisfies the following for $z \in \Omega_{\rho^2}$ and $z^* := \rho^2/\bar{z}$.*

(i) $f_\rho(\bar{z}) = \overline{f_\rho(z)}$. *(ii)* $f_\rho(-\bar{z}) = -\overline{f_\rho(z)}$.

(iii) $f_\rho(z^*) = -\overline{f_\rho(z)}$ *and if* $|z| = \rho$, $f_\rho(z)$ *is purely imaginary.*

(iv) $f_\rho(\pm i\rho) = \pm ci$ *for some* $c > 0$. *(v)* $f'_\rho(i\rho) = f'_\rho(-i\rho) = 0$.

Proof. First, (i) holds since $S(\bar{z}, \rho) = \overline{S(z, \rho)}$ and $L(\bar{z}, \rho) = \overline{L(z, \rho)}$.

Next, observing that $S(-\bar{z}, \rho) = \overline{S(z, -\rho)}$ and $L(-\bar{z}, \rho) = -\overline{L(z, -\rho)}$, we have $f_\rho(-\bar{z}) = -\overline{f_{-\rho}(z)} = -\overline{f_\rho(z)}$ by (3.4), which implies (ii).

For (iii), combine (ii) and (3.3) to obtain $f_\rho(z^*) = f_\rho(-\overline{J_\rho(z)}) = -\overline{f_\rho(J_\rho(z))} = -\overline{f_\rho(z)}$. Since $z^* = z$ on $|z| = \rho$, f_ρ is purely imaginary on $|z| = \rho$.

For (iv), we know that f_ρ maps $\{z : |z| = \rho\}$ onto a line segment on the imaginary axis by (iii) and hence the end points of the line segment are branch points. By (i), the end points can be written as $[-ci, ci]$ for some $c > 0$. If $f_\rho(z) = ci$, then $f_\rho(-\bar{z}) = ci$ by (ii). Since f_ρ is a two-to-one mapping and ci is a branch point, it is the image of a single point. Hence $f_\rho(i\rho) = ci$ and similarly $f_\rho(-i\rho) = -ci$.

Finally, we have (v) since $\pm i\rho$ map to the branch points. □

The involution $J_\rho(z)$ in (3.3) fixes critical points $z = \pm\rho i$ of f_ρ in view of (v) in the above lemma.

3.2. **Bell representation for $n = 2$.** For $r > 2$, Bell representation

$$A(r) = \left\{ z \in \mathbb{C} : \left| z + \frac{1}{z} \right| < r \right\}$$

is a doubly connected domain with smooth real analytic boundary curves. Mapping

$$f_r(z) = \frac{1}{r}\left(z + \frac{1}{z} \right)$$

is a proper holomorphic map from $A(r)$ onto the unit disc which gives a 2-sheeted branched covering of U by $A(r)$ (see [5]). Moreover, f_r has critical points $z = \pm 1$ with critical values $\pm 2/r$. Hence f_r gives a 2-sheeted covering of the Riemann sphere $\widehat{\mathbb{C}}$ by itself branched over $\pm 2/r$ for each positive r.

The proper map $f_r : A(r) \to U$ is associated with a biholomorphic involution in $A(r)$

$$J(z) = \frac{1}{z},$$

which fixes critical points $z = \pm 1$ of f_r and satisfies that

$$f_r(z) = f_r(J(z)), \quad z \in A(r). \tag{3.5}$$

4. Solution to equivalence problem

The first theorem that solves the equivalence problem is

Theorem 4.1. *Annulus Ω_{ρ^2}, $0<\rho<1$, is conformally equivalent to Bell representation $A(r)$, $r>2$, if and only if $r=2/c(\rho)$, where*

$$c(\rho) := \frac{2\rho \sum_{k=0}^{\infty} (-1)^{\left[\frac{k+1}{2}\right]} \rho^{2k} / \left(1+\rho^{4k+2}\right)}{1+2\sum_{k=0}^{\infty} (-1)^{\left[\frac{k+2}{2}\right]} \rho^{4k+2} / \left(1+\rho^{4k+2}\right)}. \tag{4.1}$$

Both series in the numerator and denominator above converge uniformly on every compact subset of $(0,1)$. The denominator is $2\pi\,(i-1)\,\rho L(i\rho,\rho)$ (see the proof), which is nonvanishing for $0<\rho<1$ since the kernel L is. $c(\rho)$ is continuous with $c(0^+)=0$. One to one correspondence between $0<\rho<1$ and $r>2$ for conformal equivalence implies that $c(\rho)$ is a positive increasing function ranged in $(0,1)$. This observation immediately leads to

Corollary 4.2. *Module of Bell representation $A(r)$, $r>2$, is*

$$M\left(A(r)\right) = \frac{1}{\pi}\log\frac{1}{c^{-1}\left(\frac{2}{r}\right)}.$$

The first step for the proof of Theorem 4.1 is to match the critical values of the covering maps.

Lemma 4.3. *Fix $r>2$. Then $A(r)$ is biholomorphic to an annulus $\Omega_{\rho^2}=\{z\in\mathbb{C}:\rho^2<|z|<1\}$ for some $\rho<1$ if and only if there is a biholomorphic map $T(z)$ of the unit disc U onto itself such that*

$$T(\{\pm ci\}) = \{\pm 2/r\}$$

where f_ρ maps $\{z\in\mathbb{C}:|z|=\rho\}$ onto a line segment with the endpoints ci and $-ci$.

Proof. First assume that $A(r)$ is biholomorphic to an annulus Ω_{ρ^2}. Now choose a biholomorphic mapping $w_{r,\rho}(z)$ of $A(r)$ onto Ω_{ρ^2}. Recall that the proper map $f_r: A(r)\to U$ is associated with the canonical biholomorphic involution in $A(r)$

$$J(z) = \frac{1}{z},$$

which fixes $\{\pm 1\}$ pointwise, the image of which by f_r is $\{\pm 2/r\}$, and interchanges the sheets of the covering $f_r: A(r)\to U$. Then $w_{r,\rho}\circ J\circ w_{r,\rho}^{-1}(z)$ is a biholomorphic involution of Ω_{ρ^2}. Since any involution on annulus is of the form $z\mapsto e^{2i\theta}\rho^2/z$ for some real θ, put

$$-e^{2i\theta}J_\rho(z) = w_{r,\rho}\circ J\circ w_{r,\rho}^{-1}(z).$$

Replacing z in Ω_{ρ^2} by $e^{i\theta}z$, composing $z\mapsto e^{-i\theta}z$ with $w_{r,\rho}$, and multiplying -1 on $w_{r,\rho}$ if necessary, we may assume that $w_{r,\rho}(\pm 1)=\pm\rho i$ and

$$J_\rho(z) = w_{r,\rho}\circ J\circ w_{r,\rho}^{-1}(z). \tag{4.2}$$

For every $\alpha\in U$, the preimage $f_\rho^{-1}(\alpha)=\{w, J_\rho(w)\}$ for some $w\in\Omega_{\rho^2}$ by (3.3) and $w_{r,\rho}^{-1}$ maps $\{w, J_\rho(w)\}$ bijectively onto $\{w_{r,\rho}^{-1}(w),\ J(w_{r,\rho}^{-1}(w))\}$ by (4.2). Also f_r maps $\{w_{r,\rho}^{-1}(w),\ J(w_{r,\rho}^{-1}(w))\}$ to a single point $\beta\in U$ by (3.5). This implies that $w_{r,\rho}^{-1}$ induces

$$T := f_r\circ w_{r,\rho}^{-1}\circ f_\rho^{-1}$$

a well-defined bijection of U onto itself. T is biholomorphic with $T(\{\pm ci\}) = \{\pm 2/r\}$ as is seen from the construction.

Next suppose that there is a biholomorphic map $T(z)$ of the unit disc U onto itself such that

$$T(\{\pm ci\}) = \{\pm 2/r\}.$$

We will show that the map $T(z)$ can be lifted to a biholomorphic map of Ω_{ρ^2} onto $A(r)$. Recall that every $A(r)$ has the canonical anticonformal automorphism

$$\Pi(z) = \frac{1}{\bar{z}},$$

which fixes the unit circle S^1 pointwise, and the image $f_r(S^1)$ is the segment $L = [-2/r, 2/r]$.

Now cut U by L, then the preimage $f_r^{-1}(U - L)$ consists of two connected components, say $D_r^{\pm}$, each of which is biholomorphic to $U - L$ and bounded by two analytic Jordan curves. Similarly, cut U by $T^{-1}(L)$, then since $T^{-1}(L)$ is a circular arc connecting $\pm ci$, the preimage $f_\rho^{-1}(U - T^{-1}(L))$ also consists of two connected components, say $D_\rho^{\pm}$, each of which is biholomorphic to $U - T^{-1}(L)$ and bounded by two analytic Jordan curves.

In particular, f_r^{-1} has single-valued branches $h_r^{\pm}$ which map $U - L$ biholomorphically onto $D_r^{\pm}$, respectively. Thus, on $D_\rho^{\pm}$ set

$$h^{\pm}(z) = h_r^{\pm} \circ T \circ f_\rho.$$

Then we can see that $h^{\pm}(z)$ has the same continuous boundary values on the common boundary of $D_\rho^{\pm}$. Thus the classical theorem of Painlevé implies that $h^{\pm}(z)$ determines a biholomorphic map of Ω_{ρ^2} onto $A(r)$. □

Proof of Theorem 4.1. Notice that any biholomorphic map $T(z)$ of the unit disc U onto itself satisfying

$$T(\{\pm ci\}) = \{\pm 2/r\}$$

in Lemma 4.3 is a rotation with $|T'(0)| = 1$. Such $T(z)$ exists if and only if $c = 2/r$. Then the theorem follows from Lemma 3.1. Formula for $c = c(\rho)$ is read in $f_\rho(i\rho) = ci$ after computing $f_\rho(i\rho)$ by means of (2.1), (3.1) and (3.2). Actually,

$$f_\rho(i\rho) = \frac{S\,(i\rho, \rho)}{L\,(i\rho, \rho)} = i\,\frac{2\pi\,(i+1)\,\rho S\,(i\rho, \rho)}{2\pi(i-1)\rho L\,(i\rho, \rho)} = c\,i,$$

where the denominator of the third expression is

$$1 + 2\sum_{k=1}^{\infty} (-1)^k \left(\frac{\rho^{8k-6}}{1+\rho^{8k-6}} + \frac{\rho^{8k-2}}{1+\rho^{8k-2}} \right) = 1 + 2\sum_{k=0}^{\infty} (-1)^{\left[\frac{k+2}{2}\right]} \frac{\rho^{4k+2}}{1+\rho^{4k+2}}$$

and the numerator of the third expression is

$$\frac{2\rho}{1+\rho^2} + 2\sum_{k=1}^{\infty} (-1)^k \left(\frac{\rho^{4k-1}}{1+\rho^{8k-2}} + \frac{\rho^{4k+1}}{1+\rho^{8k+2}} \right) = 2\rho \sum_{k=0}^{\infty} (-1)^{\left[\frac{k+1}{2}\right]} \frac{\rho^{2k}}{1+\rho^{4k+2}},$$

and the formula for $c(\rho)$ follows. □

Let W be a non-degenerate n-connected planar domain and let the subset $E(W)$ of $\mathbf{B}_n$ be consisted of all points which correspond to n-connected canonical domains biholomorphically equivalent to W. We call $E(W)$ the *leaf* in $\mathbf{B}_n$ for W.

Remark 4.4. For every non-degenerate n-connected planar domain W, the set $E(W)$ is a non-empty proper subset of $\mathbf{B}_n$ and contains an element with $a_1 > 0$.

Theorem 4.1 implies the following

Corollary 4.5. *For every given $\rho < 1$ and $f_\rho(\rho i) = ci$ with $0 < c < 1$,*

$$E(\Omega_{\rho^2}) = \left\{ (a,b) \in \mathbf{B}_2 : \left| \frac{4a'}{1 - \overline{(b+2a')}(b-2a')} \right| = \frac{2c}{1+c^2} \right\}$$

where a' is any complex number such that $(a')^2 = a$.

In particular,

$$E(\Omega_{\rho^2}) \cap \{(a,0) \in \mathbb{C}^2\} = \{(a,0) \in \mathbb{C}^2 : |a| = c^2/4\}.$$

Proof. By Theorem 4.1, an annulus Ω_{ρ^2}, $0 < \rho < 1$, is conformally equivalent to Bell representation $A(r)$ where $r = 2/c(\rho)$. Therefore $E(\Omega_{\rho^2}) = E(A(2/c(\rho)))$. We denote that $c(\rho) = c$. By [8],

$$E(A(r)) = \left\{ (a,b) \in \mathbf{B}_2 : \left| \frac{4a'}{1 - \overline{(b+2a')}(b-2a')} \right| = \frac{4r}{4+r^2} \right\}$$

where a' is any complex number such that $(a')^2 = a$. Hence we get desired result with $r = 2/c$.

In particular, by [8],

$$E(A(r)) \cap \{(a,0) \in \mathbb{C}^2\} = \{(a,0) \in \mathbb{C}^2 : |a| = r^{-2}\}$$

and it implies that

$$E(\Omega_{\rho^2}) \cap \{(a,0) \in \mathbb{C}^2\} = \{(a,0) \in \mathbb{C}^2 : |a| = c^2/4\}$$

since $r = 2/c$. □

Remark 4.6. For a real θ and a real positive a,

$$W_{e^{i\theta}a,0} = \left\{ z \in \mathbb{C} : \left| z + \frac{e^{i\theta}a}{z} \right| < 1 \right\}$$

is biholomorphic to $W_{a,0}$ by the map $z \to e^{i\theta/2}z$. Hence in the family $\{W_{e^{i\theta}a,0}\}$ with $0 < a < 1/4$, there are no pair of mutually biholomorphic domains and the set $\{(e^{i\theta}a, 0) \in \mathbb{C}^2 : 0 < a < 1/4\}$ contains a point of every $E(\Omega_{\rho^2})$.

5. Solution via Teichmüller extremal domain and theta constants

The function $c(\rho)$ in Theorem 4.1 can be described in another manner by using theta constants.

By the reflection principle, we see that there is a biholomorphic map of $A(r)$ onto

$$D = \mathbb{C} - [-2/r, 2/r] - (-\infty, -r/2] - [r/2, +\infty).$$

and D is mapped onto the Teichmüller extremal domain

$$D_P = \mathbb{C} - [-1, 0] - [P, +\infty)$$

with

$$P=\frac{((r/2)-(2/r))^2}{4}=\frac{(r^2-4)^2}{16r^2}$$

by the Möbius transformation

$$T_P(z)=\frac{(r/2)-(2/r)}{4/r}\frac{z-(2/r)}{z+(r/2)}.$$

On the other hand, considering the Weierstrass $\wp$ function with the periods

$$\omega_1=2\int_{-1}^{0}\frac{dx}{\sqrt{x(x+1)(x-P)}}$$

and

$$\omega_2=2i\int_{0}^{P}\frac{dx}{\sqrt{x(x+1)(P-x)}},$$

the module $M\left(D_P\right)$ of D_P satisfies that $\omega_2/\omega_1=2i\,M\left(D_P\right)$. By setting

$$q:=\exp\left(i\pi\omega_2/\omega_1\right),$$

we have

$$\frac{P}{P+1}=\prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^8$$

(see for instance, [1] and [2]). Here, since

$$\frac{P}{P+1}=\frac{(r^2-4)^2}{(r^2+4)^2},$$

$c(\rho)=2/r$, and

$$q=\exp(-2\pi M\left(D_P\right))=\exp\left(-2\pi M\left(\Omega_{\rho^2}\right)\right)=\rho^2,$$

we have

$$\frac{1-c(\rho)^2}{1+c(\rho)^2}=\prod_{n=1}^{\infty}\left(\frac{1-\rho^{4n-2}}{1+\rho^{4n-2}}\right)^4.$$

Such infinite products can be expressed by theta constants. We recall some of such expressions. For the basic facts on theta constants, see for instance, [6].

By using the well-known product expressions of the theta constants and the Jacobi quartic identity, we have

$$c(\rho)=\frac{\theta_2^2}{\theta_3^2+\theta_0^2},$$

where we set

$$\theta_0=\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1-q^{2n-1})^2,$$

$$\theta_2=\sum_{n\in\mathbb{Z}}q^{(n-(1/2))^2}=2q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1+q^{2n})^2,$$

and

$$\theta_3=\sum_{n\in\mathbb{Z}}q^{n^2}=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1+q^{2n-1})^2.$$

with $q=\rho^2$. In particular, $c(\rho)$ can be expressed by

$$c(\rho)=\frac{4\rho\prod_{n=1}^{\infty}\left(1+\rho^{4n}\right)^4}{\prod_{n=1}^{\infty}\left(1+\rho^{4n-2}\right)^4+\prod_{n=1}^{\infty}\left(1-\rho^{4n-2}\right)^4} \tag{5.1}$$

and

$$c(\rho)=\frac{4\rho(\sum_{n=1}^{\infty}\rho^{2n(n-1)})^2}{(1+2\sum_{n=1}^{\infty}\rho^{2n^2})^2+(1+2\sum_{n=1}^{\infty}(-1)^n\rho^{2n^2})^2}. \tag{5.2}$$

Moreover, by using Lambert series, we express $c(\rho)$ in a way similar to the one in Theorem 4.1, and we have the following

Theorem 5.1. *The function $c(\rho)$ can be written by*

$$c(\rho)=\frac{2\rho\displaystyle\sum_{k=0}^{\infty}(-1)^k\rho^{2k}/\left(1-\rho^{4k+2}\right)}{1+4\displaystyle\sum_{k=0}^{\infty}(-1)^k\rho^{8k+4}/\left(1-\rho^{8k+4}\right)}.$$

Proof. Since it is known (cf. [6] p.477-478) that

$$(1+2\sum_{n=1}^{\infty}x^{n^2})^2=1+4\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{1-x^{2k+1}},$$

and hence

$$(1+2\sum_{n=1}^{\infty}(-1)^n x^{n^2})^2=1+4\sum_{k=0}^{\infty}(-1)^k\frac{-x^{2k+1}}{1+x^{2k+1}},$$

the denominator of (5.2) can be expressed as

$$2+8\sum_{k=0}^{\infty}(-1)^k\frac{\rho^{8k+4}}{1-\rho^{8k+4}}.$$

On the other hand, by another theta identity (cf. [6] (7.16)), we have

$$x(2\sum_{n=1}^{\infty}x^{2n(n-1)})^2=(1+2\sum_{n=1}^{\infty}x^{n^2})^2-(1+2\sum_{n=1}^{\infty}x^{2n^2})^2$$

$$=4\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{1-x^{2k+1}}-4\sum_{k=0}^{\infty}(-1)^k\frac{x^{4k+2}}{1-x^{4k+2}}=4\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{1-x^{4k+2}}.$$

Hence the numerator of (5.2) is expressed as

$$4\rho\sum_{k=0}^{\infty}(-1)^k\frac{\rho^{2k}}{1-\rho^{4k+2}}.$$

□

Thus we have derived the formulas for $c(\rho)$ representing the relation between $A(r)$ and an annulus Ω_{ρ^2} in several ways. The formula for $c(\rho)$ in section 4 is not derived from the formulas in this section even if it has the expression similar to the one in Theorem 5.1. By equating the formulas for $c(\rho)$ in Theorem 4.1 and Theorem 5.1 we get the following

Corollary 5.2. *For every z with $|z| < 1$, we have the following equation.*

$$\left(\sum_{k=0}^{\infty} \frac{(-1)^{[\frac{k}{2}]} z^k}{1 - z^{2k+1}} \frac{(-1)^{[\frac{k+1}{2}]} z^k}{1 + z^{2k+1}}\right) \left(1 + \sum_{k=0}^{\infty} \frac{2(-1)^{[\frac{k+2}{2}]} z^{4k+2}}{1 + z^{4k+2}}\right)$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-1)^{[\frac{k+1}{2}]} z^{2k}}{1 + z^{4k+2}}\right) \left(1 + \sum_{k=0}^{\infty} \frac{2(-1)^{[\frac{k+2}{2}]} z^{4k+2}}{1 + z^{4k+2}} \frac{2(-1)^{[\frac{k+3}{2}]} z^{4k+2}}{1 - z^{4k+2}}\right).$$

References

[1] L. Ahlfors, *Quasiconformal mappings*, Van Nostrand, Princeton, 1966.

[2] G. D. Anderson, M. K. Vamanamurthy, and M. K. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, Inc., 1997.

[3] S. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mapping*, CRC Press, Boca Raton, 1992.

[4] S. Bell, *Finitely generated function fields and complexity in potential theory in the plane*, Duke Math. J., **98**, (1999), 187-207.

[5] S. Bell, *A Riemann surface attached to domains in the plane and complexity in potential theory*, Houston J. Math., **26** (2000) 277-297.

[6] H. M. Farkas and I. Kra, *Theta Constants, Riemann Surfaces and the Modular Group*, Graduate Studies in Math. 7, AMS, 2001.

[7] M. Jeong and M. Taniguchi, *Bell representation of finitely connected planar domains*, Proc. AMS. **131**, (2003), 2325-2328.

[8] M. Jeong and M. Taniguchi, *Algebraic kernel functions and representation of planar domains*, J. Korean math. Soc., **40**, (2003), 447-460.

[9] M. Jeong and M. Taniguchi, *The coefficient body of Bell representations of finitely connected planar domains*, J. Math. Anal. Appl.**295**, (2004), 620-632.

[10] B.A.Mair and Scott McCullough, *Invariance of Extreme harmonic functions on an annulus; Applications to theta functions*, Houston J. Math. **20**, (1994), 453-473.

[11] T. Tegtmeyer, *The Ahlfors map and Szegő kernel in multiply connected domains*, Ph.D. Thesis, Purdue University, West Lafayette, 1998.

[12] T. Tegtmeyer and A. Thomas, *The Ahlfors map and Szegő kernel for an annulus*, Rocky Mountain J. Math. **29**, (1999), 709–723.

Department of Mathematics, The University of Suwon, Suwon P.O.Box 77, Kyungkido 440-600, Korea

E-mail address: `mjeong@suwon.ac.kr`

Department of Mathematics, Yonsei University, Seoul 120-749, Korea

E-mail address: `jwoh@math.snu.ac.kr`

Department of Mathematics, Graduate school of Science, Kyoto University, Kyoto 606, Japan

E-mail address: `tanig@kusm.kyoto-u.ac.jp`