# CHARACTERIZATIONS OF HYPERBOLICALLY CONVEX REGIONS

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ABSTRACT. Let $X$ be a simply connected and hyperbolic subregion of the complex plane $\mathbb{C}$. A proper subregion $\Omega$ of $X$ is called hyperbolically convex in $X$ if for any two points $A$ and $B$ in $\Omega$, the hyperbolic geodesic arc joining $A$ and $B$ in $X$ is always contained in $\Omega$. We establish a number of characterizations of hyperbolically convex regions $\Omega$ in $X$ in terms of the relative hyperbolic density $\rho_\Omega(w)$ of the hyperbolic metric of $\Omega$ to $X$, that is the ratio of the hyperbolic metric $\lambda_\Omega(w)|dw|$ of $\Omega$ to the hyperbolic metric $\lambda_X(w)|dw|$ of $X$. Introduction of hyperbolic differential operators on $X$ makes calculations much simpler and gives analogous results to some known characterizations for euclidean or spherical convex regions. The notion of hyperbolic concavity relative to $X$ for real-valued functions on $\Omega$ is also given to describe some sufficient conditions for hyperbolic convexity.

## 1. INTRODUCTION

The Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$ and the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ have the canonical metrics $\lambda_\mathbb{P}(z)|dz|, \lambda_\mathbb{C}(z)|dz|$ and $\lambda_\mathbb{D}(z)|dz|$ of constant curvature $+4, 0$ and $-4$, respectively. Here $\lambda_\mathbb{P}(z) = 1/(1+|z|^2), \lambda_\mathbb{C}(z) = 1$ and $\lambda_\mathbb{D}(z) = 1/(1-|z|^2)$.

It is well known that a region $\Omega$ in the Riemann sphere $\mathbb{P}$ with $\#(\mathbb{P} \setminus \Omega) \geq 3$ admits a (holomorphic) universal covering projection $p$ of the unit disk $\mathbb{D}$ onto $\Omega$ and the region is called *hyperbolic*. Since the metric $\lambda_\mathbb{D}(z)|dz|$ is invariant under the covering transformation group of $p$, the region $\Omega$ carries the unique metric $\lambda_\Omega(w)|dw|$ determined from $\lambda_\Omega(p(z))|p'(z)| = \lambda_\mathbb{D}(z)$ for all $z \in \mathbb{D}$ where $w = p(z)$. The metric is independent of the particular choice of $p$. We call $\lambda_\Omega(w)|dw|$ the hyperbolic metric of $\Omega$ and $\lambda_\Omega$ the hyperbolic density of the hyperbolic metric of $\Omega$. Note that the density $\lambda_\Omega$ is real analytic and hence it is smooth. Also, the hyperbolic metric has constant Gaussian curvature $-4$, that is, $-\Delta \log \lambda_\Omega = -4\lambda_\Omega^2$.

The quantity $\lambda_\Omega(w)$ can be regarded as the ratio of the hyperbolic metric $\lambda_\Omega(w)|dw|$ to the euclidean metric $\lambda_\mathbb{C}(w)|dw|$. Thus, it is sometimes called the euclidean density of the hyperbolic metric of $\Omega$.

Let $X$ be a hyperbolic region in $\mathbb{P}$ or $\mathbb{P}$ itself or $\mathbb{C}$ and $A, B \in X$. We denote by $d_X(A, B)$ the distance between $A$ and $B$ measured by the metric $\lambda_X(w)|dw|$, namely,

$$d_X(A, B) = \inf \int_\gamma \lambda_X(w)|dw|,$$

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where the infimum is taken over all paths $\gamma$ in $X$ joining $A$ and $B$. It is known that there exists an arc $\delta$ joining two points $A$ and $B$ in $X$ such that $d_X(A,B) = \int_\delta \lambda_X(w)|dw|$. We call such an arc $\delta$ a geodesic arc joining $A$ and $B$ in $X$. Note that the geodesic arc joining $A$ and $B$ in $X$ is uniquely determined by $A$ and $B$ when $X$ is a simply connected hyperbolic region.

For example,

$$d_{\mathbb{P}}(A,B) = \arctan\left|\frac{A-B}{1+\overline{A}B}\right| \quad \text{and} \quad d_{\mathbb{D}}(A,B) = \operatorname{arctanh}\left|\frac{A-B}{1-\overline{A}B}\right|.$$

A geodesic arc is the shorter arc of the great circle joining two distinct points in $\mathbb{P}$ in the case of $\mathbb{P}$ and the part of the circular arc joining two distinct points in $\mathbb{D}$ which is perpendicular to the boundary of $\mathbb{D}$ in the case of $\mathbb{D}$.

Characterizations of (euclidean) convex regions $\Omega$ in the complex plane $\mathbb{C}$ have been given as analytic or geometric properties of the density $\lambda_\Omega(w)$ of the hyperbolic metric of $\Omega$ ([2], [3], [11], [13], [14], and [15]). We state some of those characterizations in the next theorem (see [3] for a unified, geometric approach).

**Theorem A.**

*Suppose that $\Omega$ is a hyperbolic region in $\mathbb{C}$. Then the following are equivalent:*

(i) $\Omega$ *is convex.*

(ii) $\left|\partial\dfrac{1}{\lambda_\Omega}\right| \le 1.$

(iii) $\Delta\dfrac{1}{\lambda_\Omega} \le 0.$

(iv) $1/\lambda_\Omega$ *is concave on* $\Omega$.

(v) $\dfrac{1}{\lambda_\Omega}\left|\partial^2\dfrac{1}{\lambda_\Omega}\right| + \left|\partial\dfrac{1}{\lambda_\Omega}\right|^2 \le 1.$

Analogous results for spherically convex regions $\Omega$ in the Riemann sphere $\mathbb{P}$ were also obtained in terms of the spherical density

$$\mu_\Omega(w) = \frac{\lambda_\Omega(w)|dw|}{\lambda_{\mathbb{P}}(w)|dw|} = (1+|w|^2)\lambda_\Omega(w)$$

of the hyperbolic metric of $\Omega$ by Kim and Minda [4]. Here, a region in $\mathbb{P}$ is called *spherically convex* if it is convex relative to spherical geometry determined by $\lambda_{\mathbb{P}}(w)|dw|$. In the spherical case, the differential operators $\partial, \partial^2$ and $\Delta$ should be replaced by the spherical ones $\partial_{\mathbb{P}}, \partial_{\mathbb{P}}^2$ and $\Delta_{\mathbb{P}}$, respectively, and the notion of concavity should be modified to that of spherical concavity. Precise definitions of those notions will be given in Sections 2 and 3.

The following characterizations of spherically convex regions are due to Kim and Minda [4] except for (ii) which was essentially found by Ma and Minda ([6], Theorem 4) earlier.

**Theorem B.**

*Suppose that $\Omega$ is a hyperbolic region in $\mathbb{P}$. Then the following are equivalent:*

(i) $\Omega$ *is spherically convex.*

(ii) $\left|\partial_{\mathbb{P}}\dfrac{1}{\mu_\Omega}\right| \le 1.$

(iii) $\left|\partial_{\mathbb{P}}\dfrac{1}{\mu_\Omega}\right|^2 \le 1 - \dfrac{1}{\mu_\Omega^2}.$

(iv) $\Delta_{\mathbb{P}}\dfrac{1}{\mu_\Omega} \le -\dfrac{8}{\mu_\Omega}.$

(v) $1/\mu_\Omega$ *is spherically concave on* $\Omega$.

(vi) $\dfrac{1}{\mu_\Omega}\left|\partial_{\mathbb{P}}^2\dfrac{1}{\mu_\Omega}\right| + \left|\partial_{\mathbb{P}}\dfrac{1}{\mu_\Omega}\right|^2 \le 1 - \dfrac{1}{\mu_\Omega^2}.$

On the other hand, hyperbolically convex regions in the unit disk $\mathbb{D}$ which are convex relative to the hyperbolic geometry of $\mathbb{D}$ appear quite naturally, for instance, as Dirichlet fundamental regions of Fuchsian groups acting on $\mathbb{D}$. Recently, conformal homeomorphisms of the unit disk $\mathbb{D}$ onto hyperbolically convex regions in $\mathbb{D}$, which are called *hyperbolically convex functions,* were intensively studied by Ma and Minda ([7], [9]) and by Mejía and Pommerenke [10]. Meanwhile, it seems that intrinsic characterizations of hyperbolic regions are less known.

In this article, we define the notion of hyperbolic concavity for a real-valued function on a subregion $\Omega$ of a simply connected hyperbolic region $X$ in $\mathbb{C}$ and deduce some equivalent conditions for the concavity (Section 3). For a hyperbolic subregion $\Omega$ of a simply connected hyperbolic region $X$, we consider the ratio $\rho_{\Omega,X}(w)$ of the hyperbolic metric $\lambda_\Omega(w)|dw|$ of $\Omega$ to the hyperbolic metric $\lambda_X(w)|dw|$ of $X$:

$$\rho_{\Omega,X}(w) = \frac{\lambda_\Omega(w)|dw|}{\lambda_X(w)|dw|}.$$

We call $\rho_{\Omega,X}(w)$ the relative hyperbolic density of the hyperbolic metric of $\Omega$ to the hyperbolic metric of $X$ or simply, the relative hyperbolic density of $\Omega$ to $X$. We write $\rho_\Omega$ for $\rho_{\Omega,X}$ when it is clear in the context. Note that $\rho_\Omega(w) \ge 1$ holds for all $w \in \Omega$ and equality holds at some point precisely when $\Omega = X$. We will say that $\Omega(\subset X)$ is *hyperbolically convex in* $X$ if for any two points $A, B$ in $\Omega$, the geodesic arc joining $A$ and $B$ in $X$ lies entirely in $\Omega$. We will give characterizing properties of hyperbolically convex regions of $X$ in terms of the relative hyperbolic density $\rho_\Omega(w)$ in Section 4. To this end, we use the natural differential operators $\partial_X, \partial_X^2$ and $\Delta_X$, called hyperbolic differential operators (see Section 2 for precise definitions and fundamental properties of them). Moreover, by using the notion of hyperbolic concavity (Section 4), we give some sufficient conditions for subregions of $X$ to be hyperbolically convex, which are expected to be equivalent ones. Meanwhile, the notion of hyperbolic concavity may be of independent interest.

## 2. Hyperbolic calculus

In the complex plane, the partial differential operators

$$\partial = \frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right),$$
$$\bar\partial = \frac{\partial}{\partial \bar w} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

where $w = u + iv$ and the Laplacian $\Delta = 4\partial\bar{\partial}$ are of fundamental importance. However, the following differential operators are more natural for the regions $X$ which is a hyperbolic region in $\mathbb{P}$ or the Riemann sphere $\mathbb{P}$ when we are concerned with intrinsic geometry of those regions (cf. [4]):

$$\begin{aligned}
\partial_X &= \frac{1}{\lambda_X}\partial, \\
\partial_X^2 &= \frac{1}{\lambda_X^2}\left[\partial^2 - 2(\partial_X\lambda_X)\partial\right] = \frac{1}{\lambda_X^2}\left[\partial^2 - 2(\partial\log\lambda_X)\partial\right], \\
\Delta_X &= \frac{1}{\lambda_X^2}\Delta.
\end{aligned}$$

We note that $\partial_X^2$ is not equal to $\partial_X\partial_X$ unlike the euclidean case: $\partial^2 = \partial\partial$. For instance,

$$\partial_{\mathbb{D}}^2 r(z) = \partial_{\mathbb{D}}\partial_{\mathbb{D}} r(z) - \bar{z}\partial_{\mathbb{D}} r(z)$$

for the unit disk $\mathbb{D}$. When $X = \mathbb{P}$, the above-defined operators are called spherical differential operators. When $X$ is a hyperbolic region in $\mathbb{P}$, we call them hyperbolic differential operators relative to $X$. We now observe a behavior of these operators under a holomorphic covering projection $p$. (The special case when $X = \mathbb{D}$ and $p$ is an analytic automorphism of $\mathbb{D}$ was given in Section 3.2 of [4].)

**Lemma 1.** *Let $X$ and $Y$ be hyperbolic regions in $\mathbb{C}$ and suppose that $p : Y \to X$ is a holomorphic* (*unbranched*) *covering projection of $Y$ onto $X$. Let $r$ be a function of class $C^2$ defined in a neighborhood of a point $a \in X$ and let $b \in p^{-1}(a)$. Then the following formulae hold near the point $b$ :*

$$\begin{aligned}
\partial_Y(r\circ p) &= \frac{p'}{|p'|}[(\partial_X r)\circ p], \\
\partial_Y^2(r\circ p) &= \left(\frac{p'}{|p'|}\right)^2[(\partial_X^2 r)\circ p], \\
\Delta_Y(r\circ p) &= (\Delta_X r)\circ p.
\end{aligned}$$

*Proof.* A straightforward computation together with the fundamental relation $(\lambda_X \circ p)|p'| = \lambda_Y$ yields these formulae. □

In particular, the quantities $|\partial_X r|, |\partial_X^2 r|$ and $\Delta_X r$ are invariant under analytic automorphisms of $X$. This fact will be crucial in the proof of our main theorems. We also remark that the hyperbolic differential operators $\partial_X$ and $\partial_X^2$ cannot be defined at the point at infinity although the absolute value of them is well defined there.

We now recall another kind of invariant differential operators [7]. For a holomorphic map $f : \mathbb{D} \to \mathbb{D}$, we define

$$D_{\mathrm{h}1}f(z) = \frac{(1-|z|^2)f'(z)}{1-|f(z)|^2}$$

and

$$D_{\mathrm{h}2}f(z) = \frac{(1-|z|^2)^2 f''(z)}{1-|f(z)|^2} + \frac{2\,(1-|z|^2)^2\overline{f(z)}f'(z)^2}{(1-|f(z)|^2)^2} - \frac{2\bar{z}\,(1-|z|^2)f'(z)}{1-|f(z)|^2}.$$

It is sometimes convenient to use the notation

$$Q_f(z) = \frac{D_{\mathrm{h}2}f(z)}{D_{\mathrm{h}1}f(z)} = (1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z} + \frac{2(1-|z|^2)\overline{f(z)}f'(z)}{1-|f(z)|^2}.$$

In terms of these operators, Ma and Minda (Theorems 3 and 5 in [7], Corollary 3.2 and Theorem 5.1 in [9]) established the following characterizations of hyperbolically convex regions in $\mathbb{D}$.

**Theorem C.** *Let $f$ be a holomorphic universal covering projection of the unit disk $\mathbb{D}$ onto a subregion $\Omega$ of $\mathbb{D}$. Then, the following are equivalent:*

(i) *$\Omega$ is hyperbolically convex in $\mathbb{D}$.*
(ii) $|D_{\mathrm{h}2}f(z)/(2D_{\mathrm{h}1}f(z))| = |Q_f(z)/2| < 1$ *for* $z \in \mathbb{D}$.
(iii) $|D_{\mathrm{h}2}f(z)/(2D_{\mathrm{h}1}f(z))| = |Q_f(z)/2| \le 1 - |D_{\mathrm{h}1}f(z)|^2$ *for* $z \in \mathbb{D}$.
(iv) $(1-|z|^2)^2|S_f(z)| + \dfrac{3}{4}|Q_f(z)|^2 \le 3$ *for* $z \in \mathbb{D}$.

In the above theorem, equalities hold when $\Omega$ is a hyperbolic half plane of $\mathbb{D}$ in (iii) and (iv), and $S_f$ denotes the Schwarzian derivative of $f$ :

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

It is also easy to verify the following chain rules (cf. Section 3.5 in [4]).

**Lemma 2.** *Let $\Omega$ be a subregion of the unit disk $\mathbb{D}$ and $r : \Omega \to \mathbb{R}$ be a function of class $C^2$. For a holomorphic map $f : \mathbb{D} \to \Omega$, the following formulae hold:*

$$\begin{aligned} \partial_{\mathbb{D}}(r\circ f) &= (\partial_{\mathbb{D}} r)\circ f \cdot D_{\mathrm{h}1}f, \\ \partial^2_{\mathbb{D}}(r\circ f) &= (\partial^2_{\mathbb{D}} r)\circ f \cdot (D_{\mathrm{h}1}f)^2 + (\partial_{\mathbb{D}} r)\circ f \cdot D_{\mathrm{h}2}f. \end{aligned}$$

We also have the following analog of formulae (4) and (5) in [4].

**Lemma 3.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a locally injective holomorphic map. Then*

$$\begin{aligned} \partial_{\mathbb{D}}|D_{\mathrm{h}1}f| &= \frac{1}{2}|D_{\mathrm{h}1}f|Q_f, \\ \partial^2_{\mathbb{D}}|D_{\mathrm{h}1}f| &= \frac{1}{2}|D_{\mathrm{h}1}f|\left(\lambda_{\mathbb{D}}^{-2}S_f + Q_f^2\right). \end{aligned}$$

*Proof.* It is easy to check the first formula. In order to show the second, we observe that

$$\begin{aligned} \partial^2_{\mathbb{D}}|D_{\mathrm{h}1}f| &= \partial_{\mathbb{D}}(\partial_{\mathbb{D}}|D_{\mathrm{h}1}f|) - \bar{z}\partial_{\mathbb{D}}|D_{\mathrm{h}1}f| \\ &= \partial_{\mathbb{D}}\left(\frac{1}{2}|D_{\mathrm{h}1}f|Q_f\right) - \frac{\bar{z}}{2}|D_{\mathrm{h}1}f|Q_f \\ &= \frac{1}{2}Q_f\,\partial_{\mathbb{D}}|D_{\mathrm{h}1}f| + \frac{1}{2}|D_{\mathrm{h}1}f|\partial_{\mathbb{D}}Q_f - \frac{\bar{z}}{2}|D_{\mathrm{h}1}f|Q_f \\ &= \frac{1}{2}|D_{\mathrm{h}1}f|\left(\partial_{\mathbb{D}}Q_f + \frac{1}{2}Q_f^2 - \bar{z}Q_f\right). \end{aligned}$$

A straightforward computation yields the relation,

$$(\partial_{\mathbb{D}} Q_f)(z) = (1-|z|^2)^2 S_f(z) + \frac{1}{2} Q_f(z)^2 + \bar{z} Q_f(z),$$

and hence, we obtain the desired identity. □

## 3. Hyperbolically concave functions

Concavity (or convexity) for real-valued functions is a fundamental notion in real analysis. Beckenbach [1] developed a theory of generalized convexity for real-valued functions on (one-dimensional) intervals while some people used the notion of geodesically convex functions on a Riemannian manifold which are convex along each geodesic arc with constant speed. We somewhat combine these notions to give a definition of hyperbolically concave functions in this article.

Recall that a real-valued function $r$ defined on a plane region $\Omega \subset \mathbb{C}$ is said to be *concave* in $\Omega$ if the inequality

$$r((1-t)w_0 + tw_1) \geq (1-t)r(w_0) + tr(w_1)$$

holds for every $t \in [0,1]$ whenever the line segment $[w_0, w_1]$ joining two points $w_0$ and $w_1$ in $\Omega$ is contained in $\Omega$ (cf. [3]). Note that we do not require $\Omega$ to be convex.

Similarly, we can define hyperbolic concavity. Let $\Omega$ be a subregion of a simply connected hyperbolic region $X$ in $\mathbb{C}$. A real-valued function $r$ on $\Omega$ is said to be *hyperbolically concave* relative to $X$ if the inequality

$$r(w_t) \geq \frac{\sinh[2(1-t)d]r(w_0) + \sinh[2td]r(w_1)}{\sinh[2d]} \tag{3.1}$$

holds for each $t \in [0,1]$, where $d = d_X(w_0, w_1)$ and $w_t$ is the unique point in $X$ such that $d_X(w_0, w_t) = td$ and that $d_X(w_t, w_1) = (1-t)d$ for $w_0,\ w_1 \in \Omega$, whenever the geodesic arc $\gamma$ joining $w_0$ and $w_1$ in $X$ lies entirely in $\Omega$. We remark that the point $w_t$ lies in $\gamma$ necessarily. For brevity, we will write $w_t = P_X(w_0, w_1, t)$.

First we show a continuity of this function for completeness though we do not use it in this article. In the proof of the next lemma, set $D_X(w_0, d) = \{w \in X : d_X(w_0, w) \leq d\}$ for a point $w_0 \in \Omega$.

**Lemma 4.** *A hyperbolically concave function $r : \Omega \to \mathbb{R}$ relative to $X$ is continuous.*

*Proof.* Let $w_0$ be an arbitrary point in $\Omega$. First we assume that $r \geq -M$ in the neighborhood $V = D_X(w_0, d_0) \subset \Omega$, where $M$ and $d_0$ are positive constants. Note that for $t \in [0,1]$, $D_X(w_0, td_0) = \{P_X(w_0, w, s) : 0 \leq s \leq t, w \in \partial V\}$. We take an arbitrary point $w_1$ on $\partial V$ and set $w_t = P_X(w_0, w_1, t)$. By (3.1), for each $t$ where $0 \leq t \leq 1$, we get the inequality

$$r(w_t) - r(w_0) \geq \left( \frac{\sinh[2(1-t)d_0]}{\sinh[2d_0]} - 1 \right) r(w_0) - \frac{\sinh[2td_0]}{\sinh[2d_0]} M \tag{3.2}$$

In order to get an upper bound of $[r(w_t) - r(w_0)]$, we choose a point $w_{-1} \in \partial V$ so that $w_0$ is the hyperbolic midpoint of $w_1$ and $w_{-1}$ in $X$. Since $w_0 = P_X(w_{-1}, w_t, 1/(1+t))$, we obtain the inequality

$$r(w_0) \geq \frac{\sinh[2td_0]r(w_{-1}) + \sinh[2d_0]r(w_t)}{\sinh[2(1+t)d_0]}$$

by (3.1), and hence,

$$r(w_t) - r(w_0) \leq \left(\frac{\sinh[2(1+t)d_0]}{\sinh[2d_0]} - 1\right) r(w_0) + \frac{\sinh[2td_0]}{\sinh[2d_0]}M. \tag{3.3}$$

Both of the right-hand sides in (3.2) and (3.3) tend to 0 as $t \to 0$. Hence we have shown that $r$ is continuous at $w_0$ if it is locally bounded below.

On the other hand, the local lower boundedness of $r$ can be easily shown. Indeed, consider a compact hyperbolic triangle $T$ in $\Omega$ relative to $X$. Then, by (3.1), the function $r$ is bounded below on each side of $T$. Applying (3.1) again, we can deduce that $r$ is bounded below on $T$. Since $\Omega$ is covered by such triangles, the local lower boundedness follows. □

Now, we give characterizations of hyperbolic concavity when $r$ is of class $C^2$. First, we need a kind of minimum principle for solutions to a boundary value problem for an ordinary differential equation. See [8] for a proof of the following result or apply the Strong Minimum Principle to the function $v - u$ (see [12], p. 260).

**Lemma 5.** *Let $u$ and $v$ be real-valued functions of class $C^2$ on the interval $[a, b]$ and suppose that $v'' \leq 4v$ and $u'' = 4u$ there. If $u(a) = v(b)$ and $u(b) = v(b)$, then either $v = u$ on $[a, b]$ or $v > u$ on $(a, b)$.*

We are now ready to state our important result for hyperbolic concavity.

**Theorem 1.** *Let $\Omega$ be a subregion of a simply connected hyperbolic region $X$ in $\mathbb{C}$ and $r$ be a real-valued function of class $C^2$ on $\Omega$. Then the following are equivalent:*

(i) *$r$ is hyperbolically concave on $\Omega$ relative to $X$.*
(ii) *Whenever the geodesic arc joining $w_0$ and $w_1$ in $X$ is contained in $\Omega$, the midpoint $m$ of it satisfies the inequality,*

$$r(m) \geq \frac{r(w_0) + r(w_1)}{2\cosh d_X(w_0, w_1)}. \tag{3.4}$$

(iii) *Whenever the geodesic arc $w(s)$ in $X$ parametrized by its hyperbolic arclength is contained in $\Omega$, the function $v(s) = r(w(s))$ satisfies the differential inequality $v''(s) - 4v(s) \leq 0$.*
(iv) *The inequality*

$$|\partial_X^2 r(w)| + \frac{1}{4}\Delta_X r(w) \leq 2r(w) \tag{3.5}$$

*holds on $\Omega$.*

*Proof.*
(i)⇒(ii): Just put $t = 1/2$ in the inequality (3.1).

(ii)⇒(iii): Let $w(s)$ be a geodesic arc in $\Omega$ parametrized by its hyperbolic arclength and set $s = s_0$. For $w_0 = w(s_0 - \delta)$ and $w_1 = w(s_0 + \delta)$, we obtain the inequality

$$v(s_0) \geq \frac{v(s_0-\delta) + v(s_0+\delta)}{2\cosh[2\delta]}$$

by (3.4) or, equivalently,

$$\frac{v(s_0-\delta)+v(s_0+\delta)-2v(s_0)}{\delta^2} \leq 2\,\frac{\cosh[2\delta]-1}{\delta^2}\,v(s_0).$$

Letting $\delta \to 0$, we obtain the inequality $v''(s_0) \leq 4v(s_0)$.

(iii)⇒(i): Let $w_0, w_1 \in \Omega$ and suppose that the geodesic arc joining $w_0$ and $w_1$ in $X$ is contained in $\Omega$. Set $w(s) = w_{s/d} = P(w_0, w_1, s/d)$ for $s \in [0, d]$, where $d = d_X(w_0, w_1)$ and $u(s) = (\sinh[2(d-s)]r(w_0) + \sinh[2s]r(w_1))/\sinh[2d]$. Then $u$ satisfies the differential equation $u'' - 4u = 0$ and the boundary conditions $u(0) = r(w_0), u(d) = r(w_1)$. Applying Lemma 5 to the function $v(s) = r(w(s))$ yields the inequality $r(w_{s/d}) \geq u(s)$ for $s \in [0, d]$, which is same as (3.1).

(iii)⇔(iv): Since the left-hand side of (3.5) is conformally invariant by Lemma 1, we may assume that $X = \mathbb{D}$.

Let $\gamma : s \mapsto w(s)$ be a smooth arc parametrized by hyperbolic arclength in $\mathbb{D}$ and suppose that $\gamma$ lies in $\Omega$. (For a while, we do not assume $\gamma$ to be a geodesic arc.) Letting $\theta(s) = \arg w'(s)$, we may write $w'(s) = |w'(s)|e^{i\theta(s)} = (1 - |w(s)|^2)e^{i\theta(s)}$.

Since $r$ is real-valued, the derivative of $v(s) = r(w(s))$ can be obtained by

$$\begin{aligned} v'(s) &= \partial r(w(s))w'(s) + \bar\partial r(w(s))\overline{w'(s)} \\ &= 2\,\mathrm{Re}\,\{\partial r(w(s))w'(s)\} \\ &= 2\,\mathrm{Re}\,\left\{(1-|w(s)|^2)\partial r(w(s))e^{i\theta(s)}\right\} \\ &= 2\,\mathrm{Re}\,\{e^{i\theta(s)}\partial_{\mathbb{D}} r(w(s))\}. \end{aligned}$$

Let $\kappa_{\mathbb{D}}(w(s), \gamma)$ and $\kappa_{\mathbb{C}}(w(s), \gamma)$ be the hyperbolic curvature and the euclidean curvature of $\gamma$ at $w(s)$, respectively (cf. [5]). That is,

$$\kappa_{\mathbb{C}}(w(s),\gamma) = \frac{\mathrm{Im}\left\{\frac{w''(s)}{w'(s)}\right\}}{|w'(s)|} = \frac{\mathrm{Im}\left\{\frac{w''(s)}{w'(s)}\right\}}{1-|w(s)|^2}$$

and

$$\kappa_{\mathbb{D}}(w(s),\gamma) = (1-|w(s)|^2)\kappa_{\mathbb{C}}(w(s),\gamma) + 2\,\mathrm{Im}\,\{\overline{w(s)}e^{i\theta(s)}\}.$$

Then

$$\mathrm{Im}\left\{\frac{w''(s)}{w'(s)}\right\} = \kappa_{\mathbb{D}}(w(s),\gamma) - 2\,\mathrm{Im}\,\{\overline{w(s)}e^{i\theta(s)}\}.$$

By the relation $w'(s) = (1-|w(s)|^2)e^{i\theta(s)}$, we get

$$\mathrm{Re}\left\{\frac{w''(s)}{w'(s)}\right\} = -2\,\mathrm{Re}\,\{\overline{w(s)}e^{i\theta(s)}\},$$

and hence,

$$w''(s) = -2\,\overline{w(s)}(1-|w(s)|^2)e^{2i\theta(s)} + i(1-|w(s)|^2)e^{i\theta(s)}\kappa_{\mathbb{D}}(w(s),\gamma).$$

By using the previous equality, we have

$$\begin{aligned} v''(s) &= 2\,\mathrm{Re}\,\left\{\partial^2 r(w(s))w'(s)^2\right\} + 2\,\mathrm{Re}\,\{\partial r(w(s))w''(s)\} + 2\,\partial\bar\partial r(w(s))|w'(s)|^2 \\ &= 2\,\mathrm{Re}\,\left\{\left[(1-|w(s)|^2)^2\partial^2 r(w(s)) - 2\,\overline{w(s)}(1-|w(s)|^2)\partial r(w(s))\right]e^{2i\theta(s)}\right\} \\ &\quad +2\,(1-|w(s)|^2)^2\partial\bar\partial r(w(s)) - 2\,\kappa_{\mathbb{D}}(w(s),\gamma)\mathrm{Im}\,\left\{(1-|w(s)|^2)\partial r(w(s))e^{i\theta(s)}\right\}. \end{aligned}$$

Hence,

$$\begin{aligned} v''(s) &= 2\,\mathrm{Re}\,\left\{e^{2i\theta(s)}\cdot\partial_{\mathbb{D}}^2 r(w(s))\right\} + \frac{1}{2}\Delta_{\mathbb{D}} r(w(s)) \\ &\quad -2\,\kappa_{\mathbb{D}}(w(s),\gamma)\cdot\mathrm{Im}\,\{e^{i\theta(s)}\partial_{\mathbb{D}} r(w(s))\}. \end{aligned}$$

From now on, we assume $\gamma$ to be a geodesic arc. So, $\kappa_{\mathbb{D}}(w(s),\gamma)=0$, and therefore,

$$v''(s) = 2\,\mathrm{Re}\,\{e^{2i\theta(s)}\cdot\partial_{\mathbb{D}}^2 r(w(s))\} + \frac{1}{2}\Delta_{\mathbb{D}} r(w(s)).$$

Since the argument $\theta(s)$ can be chosen arbitrarily for a given point $w=w(s)$ by taking a suitable geodesic arc passing through $w$, the inequality (3.5) holds at the point $w$. The converse is obvious by the last formula. $\square$

**Remark.** We can similarly define *spherical concavity* for a real-valued function defined on a hyperbolic region in $\mathbb{P}$ by replacing $d_X(w_0,w_1)$ and sinh by $d_{\mathbb{P}}(w_0,w_1)$ and sin, respectively in the inequality (3.1). When $r$ is of class $C^2$, our definition is equivalent to that of Kim and Minda [4]: A real-valued function $r$ of class $C^2$ is called spherically concave in $\Omega$ if the function $v$ given by $v(s)=r(w(s))$ satisfies the inequality $v''(s)+4v(s)\le 0$ for a spherical geodesic arc $w(s)$ in $\Omega$ which is parametrized by its spherical arclength.

The composition of a hyperbolically concave function with a certain function can also be hyperbolically concave. We show this in the next lemma.

**Lemma 6.** *Suppose that a function $h:(0,M)\to\mathbb{R}$ satisfies the following three conditions on $(0,M)$:* (a) *$h$ is non-decreasing,* (b) *$h(x)/x$ is non-increasing, and* (c) *$h$ is concave. Let $\Omega$ be a subregion of a hyperbolic region $X$ in $\mathbb{C}$. If a hyperbolically concave function $r:\Omega\to\mathbb{R}$ relative to $X$ takes its values in $(0,M)$, then the composed function $h\circ r$ is also hyperbolically concave on $\Omega$ relative to $X$.*

*Proof.* We need to show the inequality (3.1) for $h\circ r$. First, we put

$$c = \frac{\sinh[2(1-t)d]+\sinh[2td]}{\sinh[2d]} \quad\text{and}\quad s = \frac{\sinh[2td]}{\sinh[2(1-t)d]+\sinh[2td]}.$$

Note that $0<c\le 1$ because $\sinh x$ is super-additive: $\sinh(x+y)>\sinh x+\sinh y$ for $x,y>0$. Now the hyperbolic concavity of $r$ gives

$$r(w_t)\ge c[(1-s)r(w_0)+sr(w_1)].$$

By the condition (b), $h(cx) \geq ch(x)$ for $x \in (0, M)$. This together with (a) and (c) implies

$$\begin{aligned} h(r(w_t)) &\geq h(c[(1-s)r(w_0) + sr(w_1)]) \\ &\geq ch([(1-s)r(w_0) + sr(w_1)]) \\ &\geq c[(1-s)h(r(w_0)) + sh(r(w_1))], \end{aligned}$$

which is the desired inequality. □

## 4. Main theorems

In this section, we give characterizations of hyperbolically convex regions $\Omega$ in a simply connected hyperbolic region $X$ in $\mathbb{C}$ in terms of the relative hyperbolic density $\rho_\Omega$ of $\Omega$ to $X$.

**Theorem 2.** *Let $\Omega$ be a subregion of a simply connected hyperbolic region $X$ in $\mathbb{C}$ and let $\rho_\Omega$ be the relative hyperbolic density of $\Omega$ to $X$. Then the following are equivalent:*

(i) *$\Omega$ is hyperbolically convex in $X$.*
(ii) $\left|\partial_X \dfrac{1}{\rho_\Omega}\right| \leq 1.$
(iii) $\left|\partial_X \dfrac{1}{\rho_\Omega}\right| \leq 1 - \dfrac{1}{\rho_\Omega^2}.$
(iv) $\Delta_X \dfrac{1}{\rho_\Omega} \leq \dfrac{4}{\rho_\Omega}.$
(v) *$1/\rho_\Omega$ is superharmonic in $\Omega$.*
(vi) $\Delta_X \dfrac{1}{\rho_\Omega} \leq -\dfrac{4}{\rho_\Omega}\left(1 - \dfrac{1}{\rho_\Omega^2}\right).$
(vii) $\dfrac{1}{\rho_\Omega}\left|\partial_X^2 \dfrac{1}{\rho_\Omega}\right| \leq \dfrac{3}{2}\left(1 - \left|\partial_X \dfrac{1}{\rho_\Omega}\right|^2\right).$

Equalities hold in (iii), (vi) and (vii) when $\Omega$ is a hyperbolic half plane of $X$. Note that we do not assume $\Omega$ to be simply connected in the theorem. In euclidean convex regions, a condition analogous to (iii) cannot appear as a characterizing property of euclidean convex regions since equality holds for a half plane in (ii) of Theorem A.

We now give sufficient conditions for hyperbolic convexity in terms of hyperbolic concavity of some functions involving the relative hyperbolic density.

**Theorem 3.** *Let $\Omega$ be a subregion of a simply connected hyperbolic region $X$ in $\mathbb{C}$ and let $\rho_\Omega$ be the relative hyperbolic density of $\Omega$ to $X$. Then, concerning the following conditions, the implication relations* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) *hold:*

(i) $\dfrac{1}{\rho_\Omega}\left|\partial_X^2 \dfrac{1}{\rho_\Omega}\right| + \left|\partial_X \dfrac{1}{\rho_\Omega}\right|^2 \leq 1 + \dfrac{1}{\rho_\Omega^2}.$
(ii) *$1/\rho_\Omega$ is hyperbolically concave on $\Omega$ relative to $X$.*
(iii) *$\tanh(1/\rho_\Omega)$ is hyperbolically concave on $\Omega$ relative to $X$.*
(iv) *$\Omega$ is hyperbolically convex in $X$.*

In view of the equivalence of the condition (iv) in Theorem A or B to the corresponding convexity of the subregion, it is expected that the conditions in Theorem 3 are all equivalent. The authors, however, did not succeed in showing it.

Next, we establish characterizations of hyperbolically convex regions in $X$ in terms of two-point distortion. Theses are hyperbolic analogs of the results in [15] for the euclidean case and in [4] for the spherical case.

**Theorem 4.** *Let $\Omega$ be a subregion of a hyperbolic region $X$ in $\mathbb{C}$ and denote by $\rho_\Omega$ the relative hyperbolic density of $\Omega$ to $X$. Then the following are equivalent:*

(i) *$\Omega$ is hyperbolically convex in $X$.*

(ii) $\left|\dfrac{1}{\rho_\Omega(A)} - \dfrac{1}{\rho_\Omega(B)}\right| \le 2\, d_X(A,B)$ *for all $A, B \in \Omega$.*

(iii) $\left|\operatorname{arctanh}\left(\dfrac{1}{\rho_\Omega(A)}\right) - \operatorname{arctanh}\left(\dfrac{1}{\rho_\Omega(B)}\right)\right| \le 2\, d_X(A,B)$ *for all $A, B \in \Omega$.*

To prove the above theorems, we first establish a formula related to the hyperbolic Laplacian of $\rho_\Omega$. This is an analogous result to [4, Lemma 1 (8)] for the spherical case.

**Lemma 7.** *Let $\Omega$ be a subregion of a hyperbolic region $X$ in $\mathbb{C}$. Then the relative hyperbolic density $\rho_\Omega$ of $\Omega$ to $X$ satisfies the relation*

$$\frac{1}{4\,\rho_\Omega}\,\Delta_X\left(\frac{1}{\rho_\Omega}\right) = \left|\partial_X\left(\frac{1}{\rho_\Omega}\right)\right|^2 - \left(1 - \frac{1}{\rho_\Omega^2}\right). \tag{4.1}$$

*Proof.* By the curvature equation for the hyperbolic metric, we have

$$\begin{aligned}
\Delta_X \log \rho_\Omega &= \frac{1}{\lambda_X^2}\Delta \log \rho_\Omega \\
&= \frac{1}{\lambda_X^2}(\Delta \log \lambda_\Omega - \Delta \log \lambda_X) \\
&= \frac{1}{\lambda_X^2}(4\lambda_\Omega^2 - 4\lambda_X^2) \\
&= 4\,(\rho_\Omega^2 - 1).
\end{aligned}$$

On the other hand, by definition of $\Delta_X$, we have

$$\begin{aligned}
\Delta_X \log \rho_\Omega &= -\frac{1}{\lambda_X^2}\,\Delta \log \frac{1}{\rho_\Omega} = -\frac{4}{\lambda_X^2}\,\partial\bar\partial \log \frac{1}{\rho_\Omega} \\
&= -\frac{4}{\lambda_X^2}\,\partial\left[\rho_\Omega\,\bar\partial\left(\frac{1}{\rho_\Omega}\right)\right] \\
&= -\frac{4\rho_\Omega}{\lambda_X^2}\,\partial\bar\partial\left(\frac{1}{\rho_\Omega}\right) - \frac{4}{\lambda_X^2}\,\partial\rho_\Omega\,\bar\partial\left(\frac{1}{\rho_\Omega}\right) \\
&= -\rho_\Omega\,\Delta_X\left(\frac{1}{\rho_\Omega}\right) + \frac{4\rho_\Omega^2}{\lambda_X^2}\,\partial\left(\frac{1}{\rho_\Omega}\right)\cdot\bar\partial\left(\frac{1}{\rho_\Omega}\right) \\
&= -\rho_\Omega\,\Delta_X\left(\frac{1}{\rho_\Omega}\right) + 4\rho_\Omega^2\left|\partial_X\left(\frac{1}{\rho_\Omega}\right)\right|^2.
\end{aligned}$$

In view of these two representations of $\Delta_X \log \rho_\Omega$, we obtain

$$\rho_\Omega \, \Delta_X \left(\frac{1}{\rho_\Omega}\right) = 4\rho_\Omega^2 \left|\partial_X \left(\frac{1}{\rho_\Omega}\right)\right|^2 - 4(\rho_\Omega^2 - 1).$$

□

Let $\Omega$ be a subregion of a hyperbolic region $X$ in $\mathbb{C}$ and $g : X \to Y$ be a conformal homeomorphism. Then, we have the obvious relation $\rho_{\Omega,X} = \rho_{\Omega',Y} \circ g$ on $\Omega$ where $\Omega' = g(\Omega)$. Therefore, the conditions which appear in Theorem 2 are invariant under conformal mappings and hence, it is enough to prove the theorem for $X = \mathbb{D}$.

By combining the formulae in the following lemma with conditions in Theorem C, we can obtain some of our conditions in Theorem 2.

**Lemma 8.** *Let $f$ be a holomorphic universal covering projection of the unit disk $\mathbb{D}$ onto a subregion $\Omega$ of $\mathbb{D}$. Then the relative hyperbolic density $\rho_\Omega$ of $\Omega$ to $\mathbb{D}$ satisfies the following relations:*

$$\left(\frac{1}{\rho_\Omega}\right) \circ f = |D_{\mathrm{h}1} f|, \tag{4.2}$$

$$\left(\partial_{\mathbb{D}} \frac{1}{\rho_\Omega}\right) \circ f = \frac{|f'|}{f'} \cdot \frac{D_{\mathrm{h}2} f}{2\, D_{\mathrm{h}1} f} = \frac{|f'|}{f'} \cdot \frac{Q_f}{2}, \tag{4.3}$$

$$\left(\partial_{\mathbb{D}}^2 \frac{1}{\rho_\Omega}\right) \circ f = \frac{|f'|}{f'} \cdot \frac{\lambda_{\mathbb{D}}^{-2} S_f}{2\, D_{\mathrm{h}1} f}. \tag{4.4}$$

*Proof.* By using $\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z) = 1/(1-|z|^2)$, we obtain (4.2) immediately. The second relation is deduced by Lemmas 2 and 3. Now, we show the third one. By Lemma 2, we have

$$\begin{aligned} \left(\partial_{\mathbb{D}}^2 \frac{1}{\rho_\Omega}\right) \circ f &= (D_{\mathrm{h}1} f)^{-2} \left\{ \partial_{\mathbb{D}}^2 \left(\frac{1}{\rho_\Omega} \circ f\right) - \partial_{\mathbb{D}} \left(\frac{1}{\rho_\Omega} \circ f\right) \frac{D_{\mathrm{h}1} f}{D_{\mathrm{h}2} f} \right\} \\ &= (D_{\mathrm{h}1} f)^{-2} \left\{ \partial_{\mathbb{D}}^2 |D_{\mathrm{h}1} f| - \partial_{\mathbb{D}} |D_{\mathrm{h}1} f| Q_f \right\} \end{aligned}$$

We then apply Lemma 3 to get (4.4). □

We are now ready to prove our main theorems by making use of the previous lemmas.

*Proof of Theorem 2.* In view of Lemma 8, the equivalence of (i), (ii), (iii) and (vii) in Theorem 2 is just a restatement of Theorem C. Furthermore, by (4.1) in Lemma 7, we have the relations

$$\begin{aligned} \rho_\Omega \, \Delta_X \left(\frac{1}{\rho_\Omega}\right) &= 4\rho_\Omega^2 \left\{ \left|\partial_X \left(\frac{1}{\rho_\Omega}\right)\right|^2 - 1 \right\} + 4 \\ &= 4\rho_\Omega^2 \left\{ \left|\partial_X \left(\frac{1}{\rho_\Omega}\right)\right|^2 - \left(1 - \frac{1}{\rho_\Omega^2}\right)^2 \right\} - 4\left(1 - \frac{1}{\rho_\Omega^2}\right), \end{aligned}$$

which show the equivalence of (ii) and (iv) and the equivalence of (iii) and (vi). Hence, (iv) and (vi) are equivalent. We recall that $1/\rho_\Omega$ is superharmonic if and only if $\Delta(1/\rho_\Omega) \le 0$, equivalently, $\Delta_X(1/\rho_\Omega) \le 0$. Clearly, (vi) implies (v) and (v) implies (iv). Therefore, we have shown that (i) through (vii) are all equivalent. $\square$

*Proof of Theorem 3.* Set $r = 1/\rho_\Omega$ for simplicity. Then, (4.1) in Lemma 7 yields the relation

$$r\left(|\partial_X^2 r| + \frac{1}{4}\Delta_X r - 2r\right) = r|\partial_X^2 r| + |\partial_X r|^2 - 1 - r^2.$$

By Theorem 1, we see that the conditions (i) and (ii) are equivalent.

In order to show (ii)⇒(iii), we use Lemma 6. For $h(x) = \tanh x$, it is sufficient to check that the function $h(x)$ satisfies conditions (a), (b) and (c) in the lemma. We omit the proof for this since it is quite elementary.

Finally, we show that (iii) implies (iv). Since these conditions are conformally invariant, we may assume that $X = \mathbb{D}$. Suppose that the condition (iii) is satisfied but $\Omega$ is not hyperbolically convex in $\mathbb{D}$. Then, there are distinct points $a$ and $b$ in $\Omega$ such that the geodesic arc joining $a$ and $b$ in $\mathbb{D}$ is not contained in $\Omega$. Without loss of generality, we may further assume that $a = 0$. Note that any geodesic arc in $\mathbb{D}$ passing through 0 is a line segment. Since $\Omega$ is connected, 0 and $b$ can be connected by a smooth simple arc $\gamma : s \mapsto w(s), 0 \le s \le 1$, with $w(0) = 0$ and $w(1) = b$ in $\Omega$. Let $\sigma_s$ be the closed line segment joining 0 and $w(s)$. Since 0 is an interior point of $\Omega$, the segment $\sigma_s$ is contained in $\Omega$ for sufficiently small $s > 0$. Let $s_0$ be the smallest number such that $\sigma_{s_0}$ is not contained in $\Omega$. It is clear that $0 < s_0 \le 1$. Choose a point $c$ in $\sigma_{s_0} \cap \partial\Omega$ and put $d_0 = d_{\mathbb{D}}(0, w(s_0))$ and $t_0 = d_{\mathbb{D}}(0, c)/d_0$. For any number $0 < s < s_0$, let $c(s)$ be the point lying in $\sigma_s$ such that $t_0 = d_{\mathbb{D}}(0, c(s))/d(s)$, where $d(s) = d_{\mathbb{D}}(0, w(s))$. It is evident that $c(s) \to c$ as $s \to s_0-$. Since the hyperbolic metric is complete, it follows that $\lambda_\Omega(c(s)) \to +\infty$, and hence, $\rho_\Omega(c(s)) \to +\infty$ as $s \to s_0-$. On the other hand, by (3.1) for the function $r = \tanh(1/\rho_\Omega)$, we have

$$\tanh\frac{1}{\rho_\Omega(c(s))} \ge \frac{\sinh[2(1-t_0)d(s)]\tanh(1/\rho_\Omega(0)) + \sinh[2t_0 d(s)]\tanh(1/\rho_\Omega(w(s)))}{\sinh[2d(s)]}.$$

Clearly, the right-hand side of the above inequality tends to a positive number as $s \to s_0-$, which contradicts the divergence of $\rho_\Omega(c(s))$. Thus, $\Omega$ must be hyperbolically convex in $\mathbb{D}$. $\square$

*Proof of Theorem 4.* Suppose that $\Omega$ is a hyperbolically convex region in $\mathbb{D}$. Fix $A, B \in \Omega$. Let $\gamma : w = w(s)$, $0 \le s \le L$, be the hyperbolic geodesic arc from $A$ to $B$ parametrized by hyperbolic arclength. Then $\gamma \subset \Omega$ and $L = d_{\mathbb{D}}(A, B)$. If we set $v(s) = 1/\rho_\Omega(w(s))$, then

$$|v'(s)| = 2\,\mathrm{Re}\left\{\partial\frac{1}{\rho_\Omega}(w(s))w'(s)\right\} \le 2\left|\partial_{\mathbb{D}}\frac{1}{\rho_\Omega}(w(s))\right|.$$

By using the condition (iii) of Theorem 2, we obtain

$$|v'(s)| \le 2\left(1 - \frac{1}{\rho_\Omega^2(w(s))}\right) = 2(1 - v^2(s))$$

or, equivalently,

$$-2 \le \frac{v'(s)}{1-v^2(s)} \le 2.$$

By integrating these inequalities over $[0, L]$, we obtain

$$|\operatorname{arctanh}(v(L)) - \operatorname{arctanh}(v(0))| \le 2L,$$

namely,

$$\left| \operatorname{arctanh}\left(\frac{1}{\rho_\Omega(B)}\right) - \operatorname{arctanh}\left(\frac{1}{\rho_\Omega(A)}\right) \right| \le 2d_{\mathbb{D}}(A, B). \tag{4.5}$$

Next, we prove that if the preceding inequality (4.5) holds for all $A, B \in \Omega$, then $\Omega$ is hyperbolically convex in $\mathbb{D}$. We show that the inequality (4.5) implies

$$\left| \partial_{\mathbb{D}} \frac{1}{\rho_\Omega} \right| \le 1 - \frac{1}{\rho_\Omega^2}$$

and so $\Omega$ is hyperbolically convex by Theorem 2. Fix $w_0 \in \Omega$. Let $\gamma : w = w(s)$ be a hyperbolic geodesic arc in $\Omega$ parametrized by hyperbolic arclength on some interval containing 0 with $w_0 = w(0)$. Let $v(s) = 1/\rho_\Omega(w(s))$, then we have

$$|\operatorname{arctanh}(v(s)) - \operatorname{arctanh}(v(0))| \le 2\, d_{\mathbb{D}}(w(s), w(0)) = 2s$$

for all $s$ sufficiently small. We now divide the both sides by $s$ and let $s$ tend to 0 to obtain

$$\frac{|v'(0)|}{1 - v^2(0)} \le 2.$$

Since

$$v'(0) = 2\,\mathrm{Re}\left\{ \left( \partial_{\mathbb{D}} \frac{1}{\rho_\Omega} \right)(w_0)\, e^{i\theta} \right\},$$

where $\theta = \arg w'(0)$, and we can find a geodesic arc containing $w_0$ in any direction there, we conclude that

$$\left| \left( \partial_{\mathbb{D}} \frac{1}{\rho_\Omega} \right)(w_0) \right| \le 1 - \frac{1}{\rho_\Omega^2(w_0)}$$

holds by choosing a suitable $\theta$. Thus, we have shown that (i) and (iii) are equivalent. In a similar way (even more easily), we can also show that (i) and (ii) are equivalent. □

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