GEOMETRIC PROPERTIES OF NONLINEAR INTEGRAL TRANSFORMS OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. By using norm estimates of the pre-Schwarzian derivatives for certain analytic functions defined by a nonlinear integral transform, we shall give several interesting geometric properties of the integral transform.

1. Introduction

Let $\mathcal{H}$ denote the class of all analytic functions in the open unit disk $\mathbb{D} = \{|z| < 1\}$ and $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ normalized by $f(0) = 0 = f'(0) - 1$. Also let $\mathcal{S}$ denote the class of all *univalent* functions in $\mathcal{A}$. For $0 \le \alpha < 1$, let $\mathcal{S}^*$ and $\mathcal{K}$ denote the familiar classes of functions in $\mathcal{A}$ that are *starlike* (with respect to origin) and *convex*, respectively. As is well known (cf. [4]), these two classes are analytically characterized, respectively, by

$$f \in \mathcal{K} \;\Leftrightarrow\; \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D},$$

and

$$f \in \mathcal{S}^* \;\Leftrightarrow\; \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Note that $f \in \mathcal{S}^* \Leftrightarrow J[f] \in \mathcal{K}$, where $J[f]$ denotes the Alexander transform [1] of $f \in \mathcal{A}$ defined by

$$J[f](z) = \int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta = \int_0^1 f(tz)\frac{dt}{t}.$$

In 1960, Biernacki claimed that $f \in \mathcal{S}$ implies $J[f] \in \mathcal{S}$, but this turned out to be wrong (see [4, Theorem 8.11]). This means that the Alexander integral operator J does not preserve the class $\mathcal{S}$.

A function $f \in \mathcal{A}$ is said to be *close-to-convex* if there exists a (not necessarily normalized) convex function g such that

$$\mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

We shall denote by $\mathcal{C}$ the class of close-to-convex functions in $\mathbb{D}$. It is well known that a close-to-convex function is univalent (cf. [4]).

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In [10], Y. J. Kim and Merkes considered the nonlinear integral transform J_α defined by

$$J_\alpha[f](z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\alpha d\zeta$$

for complex numbers α and for functions f in the class

$$\mathcal{ZF} = \{f \in \mathcal{A} : f(z) \neq 0 \text{ for all } 0 < |z| < 1\}$$

and showed that

$$J_\alpha(\mathcal{S}) = \{J_\alpha[f] : f \in \mathcal{S}\} \subset \mathcal{S}$$

when $|\alpha| \leq 1/4$. Up to now, the best constant is not known for this reult. Also, Merkes [12] proved that, for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1/2$, the inequality

$$J_\alpha(\mathcal{S}^*) \subset \mathcal{S}$$

holds, where $1/2$ is sharp. Note also that the authors has recentely proved in [7] that $J_\alpha(\mathcal{S}^*) \subset \mathcal{S}$ precisely when either $|\alpha| \leq 1/2$ or $\alpha \in [1/2, 3/2]$. More generally, for a given constant $\beta > 0$, it may be interesting to find a subclass $\mathcal{F}$ of $\mathcal{A}$ such that $J_\alpha(\mathcal{F}) \subset \mathcal{S}$ for all $\alpha \in \mathbb{C}$ with $|\alpha| \leq \beta$. The main purpose of this note is to give such classes $\mathcal{F}$ in a concrete way.

Let $f : \mathbb{D} \to \mathbb{C}$ be analytic and locally univalent. The pre-Schwarzian derivative T_f of f is defined by

$$T_f(z) = \frac{f''(z)}{f'(z)}.$$

Also, with respect to the Hornich operation [5], the quantity

$$\|f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)|T_f(z)|$$

can be regarded as a norm of the space of uniformly locally univalent analytic functions f in $\mathbb{D}$ (see [7] for details). Here, an analytic function f on $\mathbb{D}$ is said to be *uniformly locally univalent* if f is univalent on each hyperbolic disk in $\mathbb{D}$ with a fixed radius. Note, in fact, that f is uniformly locally univalent if and only if $\|f\| < \infty$ (see [17]). In connection with the above norm, the following result is important to note.

Theorem A. *Let f be analytic and locally univalent in $\mathbb{D}$. Then*

(i) *if $\|f\| \leq 1$ then f is univalent, and*
(ii) *if $\|f\| < 2$ then f is bounded.*

The constants are sharp.

The part (i) is due to Becker [2] and sharpness of the constant 1 is due to Becker and Pommerenke [3]. The part (ii) is obvious (see [9, Corollary 2.4]). Note also that, recently, Kari and Per Hag [6] gave a necessary and sufficient condition for $f \in \mathcal{S}$ to have a John disk as the image in terms of the pre-Schwarzian derivative of f. Also, the norm estimates for typical subclasses of univalent functions are investigated by many authors ([18], [9], [8], and so on).

In the present paper, first we estimate the norm of $J_\alpha[f]$ for a function f in a subclass of $\mathcal{A}$ and then make use of Theorem A to obtain boundedness and univalence of the nonlinear integral transform $J_\alpha[f]$ of f. We give also conditions for $J_\alpha[f]$ to be in typical subclasses of univalent functions such as $\mathcal{S}^*, \mathcal{K}$ and $\mathcal{C}$.

2. Main Results

For a constant $0 < \lambda \le 1$, consider the class $\mathcal{U}(\lambda)$ defined by

$$\mathcal{U}(\lambda) = \{f \in \mathcal{A} : \, |f'(z)(z/f(z))^2 - 1| < \lambda, \; z \in \mathbb{D}\}.$$

The class $\mathcal{U}(\lambda)$ looks natural through the transformation $F(\zeta) = 1/f(1/\zeta)$, where $|\zeta| > 1$. In fact, $F'(1/z) = f'(z)(z/f(z))^2$ and therefore, $f \in \mathcal{U}(\lambda)$ if and only if $|F'(\zeta) - 1| < \lambda$ in $|\zeta| > 1$. Note that $f \in \mathcal{U}(\lambda)$ has no zeros in $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, namely, $\mathcal{U}(\lambda) \subset \mathcal{ZF}$, because $z^2 f'(z)/f(z)^2$ is analytic in $\mathbb{D}$. It is known [16] that $\mathcal{U}(\lambda) \subset \mathcal{S}$ for $0 < \lambda \le 1$ and that every $f \in \mathcal{U}(\lambda)$ admits a K-quasiconformal extension to the Riemann sphere when $K = (1+\lambda)/(1-\lambda) < \infty$ (see [11]). In particular, the Bieberbach theorem yields that $|a_2| = |f''(0)/2| \le 2$ for $f \in \mathcal{U}(1)$. Set

$$\mathcal{U}_\sigma(\lambda) = \{f \in \mathcal{U}(\lambda) : \, |f''(0)| \le 2\sigma\}$$

for $\sigma \ge 0$. Recently, the class $\mathcal{U}(\lambda)$ and its related classes have been studied extensively by M. Obradović and Ponnusamy [14]. Furthermore, it is shown in [15] that $\mathcal{U}_0(\lambda) \subset \mathcal{S}^*$ for $0 < \lambda \le 1/\sqrt{2}$, and that, for $1/\sqrt{2} < \lambda \le 1$, every function in $\mathcal{U}_0(\lambda)$ is starlike in $|z| < 1/\sqrt{2\lambda}$.

Theorem 2.1. *Let λ, μ and σ be non-negative numbers with $\mu = \sigma + \lambda \le 1$. For a function $f \in \mathcal{U}_\sigma(\lambda)$, one obtains the estimate*

$$\|J_\alpha[f]\| \le \frac{2|\alpha|\mu}{1 + \sqrt{1-\mu^2}} \tag{2.2}$$

for every $\alpha \in \mathbb{C}$, where equality holds precisely when $f(z) = z/(1-az)$ with $|a| = \mu$. In particular, $J_\alpha[f] \in \mathcal{S}$ whenever $|\alpha| \le (1 + \sqrt{1-\mu^2})/2\mu$.

Proof. Taking a logarithmic differentiation, we obtain $J_\alpha[f] = \alpha J[f]$ and thus

$$\|J_\alpha[f]\| = |\alpha| \|J[f]\|.$$

Hence it suffices to show the inequality (2.2) in the case $\alpha = 1$. Let $f(z) = z + a_2 z^2 + \cdots$ be in $\mathcal{U}_\sigma(\lambda)$ and set $F = J[f]$. Since $f'(z)(z/f(z))^2 = 1 + (a_3 + 3a_2^2)z^2 + \cdots$, we can write

$$f'(z)\left(\frac{z}{f(z)}\right)^2 = 1 + \lambda z^2 \omega(z),$$

where ω is an analytic function in $\mathbb{D}$ with $|\omega(z)| \le 1$. If we set $g(z) = 1/f(z) - 1/z$, then we see that g is analytic in $\mathbb{D}$ and $g(0) = -a_2$. Using the identity

$$g'(z) = -\frac{f'(z)}{f^2(z)} + \frac{1}{z^2} = -\lambda\omega(z),$$

we get the representation

$$\frac{z}{f(z)} = 1 - a_2 z - \lambda z^2 \int_0^1 \omega(tz)\, dt, \tag{2.3}$$

of f. (Conversely, for an arbitrary analytic function $\omega : \mathbb{D} \to \mathbb{C}$ with $|\omega(z)| \le 1$, the function f given by (2.3) belongs to the class $\mathcal{U}(\lambda)$ as long as the right-hand side of (2.3) does not vanish in $\mathbb{D}$. The requirement that $f \in \mathcal{ZF}$ is guaranteed by $|a_2| \le \sigma = 1 - \lambda$.)

Since $|a_2| + \lambda \le 1$, by (2.3), we get

$$\left| \frac{z}{f(z)} - 1 \right| \le |a_2 z| + \lambda |z|^2 < \mu.$$

This implies that $F'(z) = f(z)/z$ is subordinate to the function $p(z) = 1/(1 + \mu z)$. By the Schwarz-Pick lemma, we easily obtain

$$\|F\| \le \sup_{z\in\mathbb{D}} (1 - |z|^2) \left| \frac{p'(z)}{p(z)} \right|,$$

see [9, Theorem 4.1]. Since

$$\frac{p'(z)}{p(z)} = -\frac{\mu}{1 + \mu z},$$

a computation shows that

$$\sup_{z\in\mathbb{D}} (1 - |z|^2) \left| \frac{p'(z)}{p(z)} \right| = \mu \sup_{0<t<1} \frac{1 - t^2}{1 - \mu t} = \frac{2\mu}{1 + \sqrt{1 - \mu^2}},$$

where the supremum is attained by $z = t = \mu/(1 + \sqrt{1 - \mu^2})$. (This calculation has been done in [8, Lemma 4.2] in a general situation.) Thus inequality (2.2) follows. The case of equality can be easily analyzed in the above.

Because $\|J_\alpha[f]\| \le 2|\alpha|\mu/(1 + \sqrt{1 - \mu^2}) \le 1$, Becker's univalence criterion (Theorem A) yields the second assertion. $\square$

Letting $a_2 = 0$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.4. *Let $0 < \lambda \le 1$, and $\alpha \in \mathbb{C}$ with $|\alpha| \le (1 + \sqrt{1 - \lambda^2})/2\lambda$. Then, $J_\alpha(\mathcal{U}_0(\lambda)) \subset \mathcal{S}$ holds.*

We may rewrite the last corollary in the following equivalent form.

Corollary 2.5. *For $\beta \ge 0$, set $\mathcal{F}_\beta = \mathcal{U}_0(4\beta/(1 + 4\beta^2))$. Then $J_\alpha(\mathcal{F}_\beta) \subset \mathcal{S}$ holds for all $\alpha \in \mathbb{C}$ with $|\alpha| \le \beta$.*

When α is real, we can deduce a stronger conclusion.

Theorem 2.6. *Let f be a function in $\mathcal{U}_{1-\lambda}(\lambda)$ for some $\lambda \in (0, 1]$. Then $J_\alpha[f]$ is a close-to-convex function for each $\alpha \in [-1, 1]$.*

Proof. By (2.3), we have

$$\operatorname{Re} \frac{z}{f(z)} > 1 - (|a_2| + \lambda) \ge 0, \quad z \in \mathbb{D}.$$

Therefore, both $J_{-1}[f]$ and $J[f] = J_1[f]$ are close-to-convex functions. Convexity of the class $\mathcal{C}$ with respect to the Hornich operation (cf. [13]) implies that $J_\alpha[f] \in \mathcal{C}$ for $\alpha \in [-1, 1]$. $\square$

Next, we consider a function $f \in \mathcal{A}$ satisfying the condition $|f''(z)/2| \le \mu,\ z \in \mathbb{D}$, for a positive constant μ. As we see below, if $\mu \le 1/2$, then f is starlike, and thus, univalent. Otherwise, however, f may not be locally univalent as the example $f(z) = z + \mu z^2$ shows.

Theorem 2.7. *Let f be a function in $\mathcal{A}$ such that $|f''(z)| \le 2\mu,\ z \in \mathbb{D}$, holds for some constant $0 < \mu \le 1$. Then $f \in \mathcal{ZF}$ and the sharp inequality*

$$\|J_\alpha[f]\| \le \frac{2|\alpha|\mu}{1+\sqrt{1-\mu^2}} \tag{2.8}$$

holds for each $\alpha \in \mathbb{C}$. If, in addition, $\mu < 1$, then equality holds above precisely when $f(z) = z + az^2$ for a constant a with $|a| = \mu$. Moreover,

(i) *$J_\alpha[f] \in \mathcal{S}$ if $|\alpha| \le (1+\sqrt{1-\mu^2})/2\mu$*
(ii) *$J_\alpha[f] \in \mathcal{K}$ if $|\alpha| \le (1-\mu)/\mu$.*

Note that $(1+\sqrt{1-\mu^2})/2\mu > (1-\mu)/\mu$ holds for all $\mu > 0$.

Proof. We may write $f''(z) = 2\mu\omega(z)$, where $|\omega| \le 1$. By integration, we have

$$f'(z) = 1 + 2\mu z \int_0^1 \omega(tz)\,dt \quad \text{and} \quad f(z) = z + 2\mu z^2 \int_0^1 (1-t)\omega(tz)\,dt.$$

Since $|\int_0^1 (1-t)\omega(tz)\,dt| \le 1/2$, we conclude that $|f(z)/z - 1| \le \mu|z| < 1$. In particular, $f \in \mathcal{ZF}$. Furthermore,

$$\frac{zf'(z)}{f(z)} - 1 = \frac{2\mu z \int_0^1 t\omega(tz)\,dt}{1 + 2\mu z \int_0^1 (1-t)\omega(tz)\,dt},$$

and hence,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\mu|z|}{1-\mu|z|}.$$

In particular, it turns out that f is starlike when $\mu \le 1/2$. Since

$$1 + \frac{z(J_\alpha[f])''(z)}{(J_\alpha[f])'(z)} = 1 + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right),$$

we obtain the convexity of $J_\alpha[f]$ under the assumption $|\alpha|\mu/(1-\mu) \le 1$. In addition, we have the estimate

$$\|J[f]\| \le \sup_{0<t<1} \mu\frac{1-t^2}{1-\mu t} = 2\frac{1-\sqrt{1-\mu^2}}{\mu} = \frac{2\mu}{1+\sqrt{1-\mu^2}}$$

in the same way as in the proof of Theorem 2.1, where the supremum is taken by $t_0 = \mu/(1+\sqrt{1-\mu^2})$. When $\mu < 1$, this point is contained in $\mathbb{D}$. Therefore, we can examine the equality case through the above proof. The univalence of $J_\alpha[f]$ under the hypothesis $|\alpha| \le (1+\sqrt{1-\mu^2})/2\mu$ follows from Theorem A (i) because $\|J_\alpha[f]\| = |\alpha|\,\|J[f]\| \le 1$. $\square$

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