THE ORDER OF PERIODIC ELEMENTS OF TEICHMÜLLER MODULAR GROUPS

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Abstract. We consider a quasiconformal automorphism of a Riemann surface, which fixes the homotopy class of a simple closed geodesic. Under certain conditions on the injectivity radius of the surface and bounds on the dilatation of the map, the automorphism induces a periodic element of the Teichmüller modular group. We may also estimate the order of the period.

1. Introduction

Let R be an arbitrary Riemann surface with possibly infinitely generated fundamental group. An element χ of the Teichmüller modular group $\mathrm{Mod}(R)$ is induced by a quasiconformal automorphism f of R. We would like to determine when the order of χ is finite. When f is a conformal automorphism of R, then the element χ of $\mathrm{Mod}(R)$ induced by f fixes the base point of the Teichmüller space $T(R)$. In [3], we proved that, for a Riemann surface R with non-abelian fundamental group, a conformal automorphism f of R has finite order if and only if f fixes either a simple closed geodesic, a puncture or a point on R. In each case, we obtained a concrete estimate for the order of f in terms of the injectivity radius on R. One of our results is the following. For the definition of the upper bound condition, see the next section.

Theorem 1.1 ([3], [4]). *Let R be a hyperbolic Riemann surface with non-abelian fundamental group. Suppose that R satisfies the upper bound condition for a constant $M > 0$ and a connected component R_M^* of R_M. Let f be a conformal automorphism of R such that $f(c) = c$ for a simple closed geodesic c on R with $c \subset R_M^*$ and $l(c) = l > 0$. Then the order n of f satisfies*

$$n < (e^M - 1)\cosh(l/2).$$

The purpose of this paper is to extend Theorem 1.1 to a quasiconformal automorphism f. One of the difficulties that arise is that the element $\chi \in \mathrm{Mod}(R)$ induced by f need not have a fixed point on $T(R)$. However, we will show that if the maximal dilatation of f is smaller than some constant, then χ is periodic.

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2. Statement of Theorem

Let $\mathbb{H}$ be the upper half-plane equipped with the hyperbolic metric $|dz|/\mathrm{Im}\, z$. Throughout this paper, we assume that a Riemann surface R is *hyperbolic.* Namely, it is represented as $\mathbb{H}/\Gamma$ for some torsion-free Fuchsian group Γ acting on $\mathbb{H}$. Furthermore, we also assume that R has a non-abelian fundamental group. The hyperbolic distance on $\mathbb{H}$ is denoted by d, and the hyperbolic length of a curve c on R by $l(c)$. For the axis L of a hyperbolic element of the Fuchsian group Γ, we denote by $\pi_\Gamma(L)$ the projection of L to $\mathbb{H}/\Gamma$. When there is no fear of confusion, we denote this simply by $\pi(L)$. Also, for a quasiconformal automorphism $\tilde{f}$ of $\mathbb{H}$, we denote by $\tilde{f}(L)_*$ the geodesic having the same end points as those of $\tilde{f}(L)$.

We recall the definition of Teichmüller spaces and Teichmüller modular groups. Fix a Riemann surface R. We say that two quasiconformal maps f_1 and f_2 on R are *equivalent* if $f_2 \circ f_1^{-1}$ is homotopic to a conformal map of $f_1(R)$ onto $f_2(R)$. The *reduced Teichmüller space* $T(R)$ with the base Riemann surface R is the set of all equivalence classes $[f]$ of quasiconformal maps f on R. The Teichmüller distance d_T on $T(R)$ is defined by $d_T([f_1],[f_2]) = \log K(g)$, where g is an extremal quasiconformal map in the sense that its maximal dilatation $K(g)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. This is a complete metric on $T(R)$. The *reduced Teichmüller modular group* $\mathrm{Mod}(R)$ of R is a group of the homotopy classes $[h]$ of quasiconformal automorphisms h of R. Each element $[h]$ of $\mathrm{Mod}(R)$ induces an automorphism of $T(R)$ by $[f] \mapsto [f \circ h^{-1}]$, which is an isometry with respect to d_T.

We now make a couple of definitions given in terms of the hyperbolic geometry of Riemann surfaces.

Definition. For a constant $M > 0$, we define R_M to be the set of points $p \in R$ for which there exists a non-trivial simple closed curve c_p passing through p with $l(c_p) < M$. The set R_ϵ is called the *ϵ-thin part* of R if $\epsilon > 0$ is smaller than the Margulis constant. Furthermore, a connected component of the ϵ-thin part corresponding to a puncture is called the *cusp neighborhood*.

Remark. The *injectivity radius* at a point $p \in R$ is the supremum of radii of embedded hyperbolic discs centered at p. Note that R_M coincides with the set of those points having the injectivity radius less than $M/2$.

Definition. We say that R satisfies the *lower bound condition* if there exists a constant $\epsilon > 0$ such that *ϵ-thin part* of R consists of only cusp neighborhoods or neighborhoods of geodesics which are homotopic to boundary components. We also say that R satisfies the *upper bound condition* if there exist a constant

$M > 0$ and a connected component R_M^* of R_M such that the homomorphism of $\pi_1(R_M^*)$ to $\pi_1(R)$ induced by the inclusion map of R_M^* into R is surjective.

Remark. The lower and upper bound conditions are quasiconformally invariant notions (see [5, Lemma 8]).

We shall obtain a range of maximal dilatations of quasiconformal automorphisms f inducing periodic elements $\chi \in \mathrm{Mod}(R)$. Moreover, we get a concrete estimate for the order of χ.

Theorem 2.1. *Let R be a Riemann surface satisfying the lower bound condition for a constant $\epsilon > 0$ as well as the upper bound condition for a constant $M > 0$ and a connected component R_M^* of R_M. For a given constant $l > 0$, there exists a constant $K_0 = K_0(\epsilon, M, l) > 1$ depending only on ϵ, M and l that satisfies the following: Let f be a quasiconformal automorphism of R such that $f(c)$ is homotopic to c for a simple closed geodesic c on R with $c \subset R_M^*$ and $l(c) = l$. Suppose $K(f) < K_0$. Then there exists a positive integer $n \leq N_0$ such that f^n is homotopic to the identity. Here*

$$N_0 = N_0(M, l) = -\frac{l}{\log(\tanh(D + 13.5))},$$

$$D = D(M, l) = \begin{cases} 2\,\mathrm{arccosh}\left(\dfrac{\sinh(M/2)}{\sinh(l/2)}\right) + M & if\ \ l \leq M, \\ M & if\ \ l \geq M. \end{cases}$$

In particular, when $K(f) = 1$, we have the following:

Theorem 2.2. *Let R be a Riemann surface satisfying the upper bound condition for a constant $M > 0$ and a connected component R_M^* of R_M as well as the lower bound condition. Let f be a conformal automorphism of R such that $f(c) = c$ for a simple closed geodesic c on R with $c \subset R_M^*$ and $l(c) = l > 0$. Then the order n of f satisfies*

$$n \leq -\frac{l}{\log(\tanh(D/2))},$$

where $D = D(M, l)$ is the same constant as in Theorem 2.1.

Note that for $M \geq \mathrm{arcsinh}(2/\sqrt{3}) = 0.98\cdots$ and every $l > 0$, we have

$$-\frac{l}{\log(\tanh(M/2))} < (e^M - 1)\cosh(l/2).$$

Here the constant $\mathrm{arcsinh}(2/\sqrt{3})$ is the smallest possible value of M for which R satisfies the upper bound condition (see [6]). Hence when $l \geq M$, the upper bound of the order of f obtained in Theorem 2.2 is smaller than that in Theorem 1.1. However, when $l < M$, the estimate in Theorem 1.1 is still better than that in Theorem 2.2 for all sufficiently small l. In fact, $(e^M - 1)\cosh(l/2)$ converges to $e^M - 1$ as $l \to 0$, while $-l/(\log(\tanh(D/2)))$ diverges to $+\infty$.

In connection with Theorems 2.1 and 2.2, we would like to mention the result about the discreteness of the orbit of a certain subgroup of the Teichmüller modular group.

Proposition 2.3 ([5]). *Let R be a Riemann surface satisfying the lower and upper bound conditions. For a simple closed geodesic c on R, let G be a subgroup of* $\mathrm{Mod}(R)$ *such that $g(c)$ is homotopic to c for every $[g] \in G$. Then for every point $p \in T(R)$, the orbit $G(p)$ of p is a discrete subset in $T(R)$. Furthermore, for any point $p \in T(R)$, there exist only finitely many elements $[g]$ in G that fix p.*

3. Proof of Theorems

For a proof of these theorems, we first prove some properties on the hyperbolic geometry of Riemann surfaces.

Proposition 3.1. *Let $R = \mathbb{H}/\Gamma$ be a Riemann surface satisfying the upper bound condition for a constant $M > 0$ and a connected component R_M^* of R_M. Suppose that L is the axis of a hyperbolic element of Γ such that the projection $\pi(L)$ is a simple closed geodesic c on R with $c \subset R_M^*$ and $l(c) = l > 0$. Then there exists an axis L' of a hyperbolic element of Γ such that $L \cap L' = \emptyset$, $d(L, L') \le D$ and $\pi(L') = \pi(L)$. Here $D = D(M, l)$ is the same constant as in Theorem* 2.1.

Proof. First we assume that $l > M$. Since $c \subset R_M^*$, there exists a non-trivial simple closed curve α passing through $p \in c$ with $l(\alpha) < M$. It follows from the assumption $l > M$ that α is not homotopic to c, which implies that there exists an axis L' $(\neq L)$ such that $\pi(L') = c$ and $d(L, L') < M$.

Next we assume that $l \le M$. We further assume that there exists an annular neighborhood $A(c)$ of c with width $\omega(c)$, where

$$\omega(c) = \operatorname{arccosh}\left(\frac{\sinh(M/2)}{\sinh(l/2)}\right).$$

Then, for any $q \in \partial A(c)$, the boundary of $A(c)$, the shortest simple closed curve γ passing through q and homotopic to c has length M.

Indeed, we may assume that $L = \{iy \mid y > 0\}$, and $\tilde{q} = e^{i\theta}$ and $\tilde{q}' = e^{l+i\theta}$ are lifts of q to $\mathbb{H}$. Then, by the equality (7.20.3) in [2], we have

$$\frac{1}{\sin\theta} = \frac{1}{\cos(\pi/2 - \theta)} = \cosh d(\tilde{q}, L) = \cosh\omega(c) = \frac{\sinh(M/2)}{\sinh(l/2)}.$$

Thus, by Theorem 7.2.1 in [2], we see that

$$\sinh \frac{1}{2} d(\tilde{q}, \tilde{q}') = \frac{|\tilde{q} - \tilde{q}'|}{2\,(\mathrm{Im}\tilde{q}\,\mathrm{Im}\tilde{q}')^{1/2}} = \frac{e^l - 1}{2\,e^{l/2} \sin\theta} = \frac{\sinh(l/2)}{\sin\theta} = \sinh\frac{M}{2},$$

which implies that $l(\gamma) = d(\tilde{q}, \tilde{q}') = M$.

We can take a point $q_0 \in \partial A(c)$ such that $q_0 \in R_M^*$. Indeed, otherwise, $\partial A(c) \cap R_M^* = \emptyset$. Since $c \subset R_M^*$, this means that R_M^* is an annular neighborhood of c, contradicting the upper bound condition.

By the definition of R_M, there exists a non-trivial simple closed curve β passing through q_0 with $l(\beta) < M$. By the consideration above, we see that the curve β is not homotopic to c. Hence there exists an axis L' ($\neq L$) such that $\pi(L') = c$ and $d(L, L') < 2\,\omega(c) + M$.

Finally, we assume that $l \le M$ and that the width of the maximal annular neighborhood $A(c)$ of c is less than $\omega(c)$. Then there exists an axis L' ($\neq L$) such that $\pi(L') = c$ and $d(L, L') < 2\,\omega(c)$. □

We now estimate the number of axes satisfying Proposition 3.1.

Definition. For an element γ of a Fuchsian group, we say that two axes L_1 and L_2 are γ-equivalent if $\gamma^n(L_1) = L_2$ for some $n \in \mathbb{Z}$.

Proposition 3.2. *Let $R = \mathbb{H}/\Gamma$ be a Riemann surface and $D_0 > 0$ a constant. Furthermore, let L be the axis of a hyperbolic element $\gamma \in \Gamma$ such that the projection $\pi(L)$ is a simple closed geodesic c on R with $l(c) = l > 0$. Let S be the set of axes L' of hyperbolic elements of Γ satisfying the following:* (i) $L \cap L' = \emptyset$, (ii) $d(L, L') \le D_0$, (iii) $\pi(L') = c$ *and* (iv) *there exists an arc α connecting L and L' whose projection to R has no intersection with c except at the end points. Then the number of γ-equivalence classes of axes in S is dominated by*

$$-\frac{l}{\log(\tanh(D_0/2))}.$$

Proof. We may assume that $L = \{iy \mid y > 0\}$. We take θ_0 $(0 < \theta_0 < \pi/2)$ so that $\cosh D_0 = (\cos\theta_0)^{-1}$ and set $\theta = \pi/2 - \theta_0$. Furthermore, we set

$$T_+ = \{re^{i\theta} \mid 1 \le r < e^l\} \qquad \text{and} \qquad T_- = \{re^{i(\pi-\theta)} \mid 1 \le r < e^l\}.$$

Then $d(L, T_+) = D_0$ and $d(L, T_-) = D_0$. To estimate the number of γ-equivalence classes of elements in S, we have only to consider the maximal number n of disjoint axes L' that are tangent to T_+ or T_-.

Let C be the Euclidean circle on $\mathbb{C}$ that is tangent to the segment T_+ and has center $a > 0$ with radius r. Then $r = a \sin\theta$, and the circle C passes through two points,

$$x_1 = (1 - \sin\theta)a \qquad \text{and} \qquad x_2 = (1 + \sin\theta)a.$$

The ratio of these points is given by

$$s = \frac{x_2}{x_1} = \frac{1+\sin\theta}{1-\sin\theta} = \frac{1+\cos\theta_0}{1-\cos\theta_0} = \frac{\cosh D_0 + 1}{\cosh D_0 - 1} = \frac{1}{(\tanh(D_0/2))^2}.$$

Hence it is easy to see that

$$n \quad \leq \quad 2\cdot\frac{l}{\log s} = -\frac{l}{\log(\tanh(D_0/2))}.$$

□

The following proposition gives a relationship between the hyperbolic distance of two axes and that of their images under a quasiconformal map.

Proposition 3.3 ([1])**.** *Let f be a K-quasiconformal automorphism of $\mathbb{H}$. Then there exists a constant $C = C(K) > 0$ depending only on K such that, for any two geodesics L_1 and L_2 in $\mathbb{H}$, the inequality*

$$K^{-1}\cdot d(L_1, L_2) - C \leq d(f(L_1)_*, f(L_2)_*) \leq K\cdot d(L_1, L_2) + C$$

holds. The constant $C(K)$ satisfies $C(K) \to 0$ as $K \to 1$, and may be taken to be

$$(1/2)\mathrm{arccosh}\left(2^{-(K-1)^2} e^{6(K+1)^2\sqrt{K-1}}\right).$$

The following proposition gives a sufficient condition for the maximal dilatations of quasiconformal maps to be bounded away from one.

Proposition 3.4 ([4])**.** *Let $R = \mathbb{H}/\Gamma$ be a Riemann surface. Suppose that R satisfies the lower bound condition for a constant $\epsilon > 0$ as well as the upper bound condition for a constant $M > 0$ and a connected component R_M^* of R_M. Let $B > 0$ and $l > 0$ be constants. Then there exists a constant $A_0 = A_0(\epsilon, M, B, l) > 1$ depending only on ϵ, M, B, l and satisfying the following conditions: Given a quasiconformal automorphism f of R, suppose that there exist three disjoint axes L_i $(i = 1, 2, 3)$ of hyperbolic elements of Γ such that*

1. *their projections $\pi(L_i)$ on R are simple closed geodesics c_i $(i = 1, 2, 3)$ with $c_i \subset R_M^*$ and $l(c_i) \leq l$,*
2. *$d(L_1, L_2) \leq B$,*
3. *$\tilde{f}(L_1)_* = L_1$, $\tilde{f}(L_2)_* = L_2$, $\tilde{f}(L_3)_* \neq L_3$ for a lift $\tilde{f}$ of f to $\mathbb{H}$.*

Then $K(f) \geq A_0$.

We now prove our theorems.

Proof of Theorem 2.1. We set $B := D = D(M, l)$ in Proposition 3.4 and let $A_0 = A_0(\epsilon, M, l) > 1$ be a constant depending only on ϵ, M and l obtained in Proposition 3.4. Setting $A = \min\{A_0, 2\}$, we prove the statement for $K_0 =$

$A^{1/(N_0+1)}$. Namely, we show that, if $K(f) < K_0$, then there exists an integer $n \le N_0$ such that f^n is homotopic to the identity.

Let Γ be a Fuchsian model of R. Furthermore let L_1 be an axis such that $\pi(L_1) = c$ and γ_1 the primitive hyperbolic element of Γ with axis L_1. By applying Proposition 3.1 to L_1, we see that there exists an axis L_2 of a hyperbolic element γ_2 of Γ such that $L_1 \cap L_2 = \emptyset$, $d(L_1, L_2) \le D$ and $\pi(L_1) = \pi(L_2)$.

Let $\tilde{f}$ be a lift of f to $\mathbb{H}$ satisfying $\tilde{f}(L_1)_* = L_1$. Since $K(f) < K_0 = A^{1/(N_0+1)}$, we have $K(f^k) < A$ for $k \le N_0 + 1$. Then, by Proposition 3.3,

$$
\begin{aligned}
(1) \qquad d(L_1, \tilde{f}^k(L_2)_*) &= d(\tilde{f}^k(L_1)_*, \tilde{f}^k(L_2)_*) \le A \cdot d(L_1, L_2) + C(A) \\
&\le 2D + C(2) = 2D + (1/2)\text{arccosh}(e^{54}/2) \\
&\le 2D + 27
\end{aligned}
$$

for all $k \le N_0 + 1$.

We consider the set S_0 of all axes L' of hyperbolic elements of Γ satisfying the following conditions: (i) $L_1 \cap L' = \emptyset$, (ii) $d(L_1, L') \le 2D+27$, (iii) $\pi(L') = c$ and (iv) there exists an arc α connecting L_1 and L' such that the projection of α to R has no intersection with c except at the end points. We see that the set $S' = \{\tilde{f}^k(L_2)_*\}_{k=1}^{N_0+1}$ is contained in S_0. Indeed, by the proof of Proposition 3.1, the axis L_2 satisfies the property (iv), and since $\tilde{f}^k$ is a homeomorphism, the axes $\tilde{f}^k(L_2)_*$ satisfy the same property. The other properties (i), (ii), (iii) are also satisfied.

By Proposition 3.2, the number of γ_1-equivalence classes of elements in S_0 is dominated by N_0. Hence there exist at least two elements in S', say $\tilde{f}^{m_1}(L_2)_*$ and $\tilde{f}^{m_2}(L_2)_*$ $(1 \le m_1 < m_2 \le N_0 + 1)$, that are γ_1-equivalent to each other. Thus there exists $j \in \mathbb{Z}$ such that $\gamma_1^j \circ \tilde{f}^n(L_2)_* = L_2$, where $n = m_2 - m_1$ $(\le N_0)$. With this n, we will prove that f^n is homotopic to the identity. We set $F = \gamma_1^j \circ \tilde{f}^n$, which is a lift of f^n to $\mathbb{H}$.

Suppose to the contrary that f^n is not homotopic to the identity. We set $\chi(\gamma) = F \circ \gamma \circ F^{-1}$ for $\gamma \in \Gamma$. Then there exists $\gamma_3 \in \Gamma$ such that $\chi(\gamma_3) \ne \gamma_3$. Setting $\gamma_i' = \gamma_3 \circ \gamma_i \circ \gamma_3^{-1}$ for $i = 1, 2$, we claim that either $\chi(\gamma_1') \ne \gamma_1'$ or $\chi(\gamma_2') \ne \gamma_2'$ is satisfied. Suppose that both $\chi(\gamma_1') = \gamma_1'$ and $\chi(\gamma_2') = \gamma_2'$ are satisfied. Since $\chi(\gamma_i) = \gamma_i$, we have $\beta \circ \gamma_i \circ \beta^{-1} = \gamma_i$ $(i = 1, 2)$, where $\beta = \gamma_3^{-1} \circ \chi(\gamma_3)$. Thus, β fixes all fixed points of γ_1 and γ_2. Since γ_1 and γ_2 are non-commutative, the Möbius transformation β fixes four points and must be the identity. This contradicts that $\chi(\gamma_3) \ne \gamma_3$.

Hence either $F(\gamma_3(L_1))_* \ne \gamma_3(L_1)$ or $F(\gamma_3(L_2))_* \ne \gamma_3(L_2)$ is satisfied, and we may assume without loss of generality that $F(\gamma_3(L_1))_* \ne \gamma_3(L_1)$. Since $\pi(\gamma_3(L_1)) = \pi(L_1) = c$, we can apply Proposition 3.4 to the lift F of f^n and to the three axes L_1, L_2 and $\gamma_3(L_1)$. Then we have $K(f^n) \ge A_0$, a contradiction, since we assumed $K(f^n) < A \leq A_0$. Hence if $K(f) < A^{1/(N_0+1)}$, then f^n is homotopic to the identity. $\square$

Proof of Theorem 2.2. In the proof of Theorem 2.1, we can replace the inequality (1) with

$$d(L_1, \tilde{f}^k(L_2)_*) \;=\; d(\tilde{f}^k(L_1)_*, \tilde{f}^k(L_2)_*) = d(L_1, L_2) = D.$$

Hence we have only to replace the constant $2D+27$ with D in Theorem 2.1. $\square$

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