# UNIFIED APPROACH TO CONFORMALLY INVARIANT METRICS ON RIEMANN SURFACES

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Abstract. In this article, we will survey the relationship between various conformally invariant (pseudo-)metrics on Riemann surfaces and classical extremal problems. As a simple application, we will give universal estimates for (pseudo-)metrics which satisfy a contraction property with respect to a certain class of holomorphic maps.

## 1. Introduction

In the unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$, the Poincaré metric $|dz|/(1-|z|^2)$ plays quite an important role in modern function theory. Various generalizations of this metric to Riemann surfaces have been given, e.g., hyperbolic (Kobayashi) metric, Carathéodory metric, Hahn metric, Begman metric and so on. In this article, we will provide a unifying treatment of such invariant metrics and explain a general principle dominating those metrics by some universal metrics.

A family of conformal pseudo-metrics $\rho_R$ defined for all Riemann surfaces $R$ is said to be *holomorphically contractive* if $f^*\rho_{R'} \le \rho_R$ holds for any holomorphic map $f : R \to R'$ and is said to be normalized if $\rho_{\mathbb{D}}(0) = 1$ holds for the unit disk and its canonical coordinate $z$. The following result is well known (cf. [5]).

**Theorem 1.1.** *Any normalized, holomorphically contractive, conformal pseudo-metric $\rho$ satisfies $c_R \le \rho_R \le k_R$ for all Riemann surfaces $R$, where $c_R$ and $k_R$ denote the Carathéodory and the Kobayashi pseudo-metrics, respectively.*

Many invariant metrics need not be holomorphically contractive but satisfy a weaker contraction property. A family of conformally invariant, conformal pseudo-metrics $\rho_R$ is said to be *monotone* if $\rho_{R_0} \ge \rho_R$ holds for each Riemann surface $R$ and for each subdomain $R_0$ of $R$.

**Theorem 1.2.** *Any normalized, conformally invariant, monotone, conformal metric $\rho$ satisfies $a_R \le \rho_R \le h_R$ for all Riemann surfaces $R$, where $a_R$ and $h_R$ denote the Ahlfors-Beurling and the Hahn pseudo-metrics, respectively.*

Precise definitions and fundamental properties for the above metrics will be given in Sections 2 and 3. Section 2 will be devoted to the formal definition of (conformal) pseudo-metircs and explanations of principles generating invariant pseudo-metrics from various classical function spaces. We will state also a more general result about pseudo-metrics with some contraction property. As corollaries, we will show the above two theorems in

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Section 3, where we will provide characterizations of many classical (pseudo-)metrics in our context. A similar approach can be found in the book [10] by Sario and Oikawa.

Some of the formulations here can be generalized to the case of several complex variables although many of them would only be Finsler metrics instead of Riemannian metrics, but we will not pursue it here.

## 2. Principles

First, we recall the definition of differential forms on Riemann surfaces. Let $R$ be a Riemann surface with an atlas $\mathcal{A}$ consisting of local coordinates $\alpha : U_\alpha \to V_\alpha$ from an open set $U_\alpha$ of $R$ onto an open set $V_\alpha$ of $\mathbb{C}$ such that $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = R$ and that $\beta\circ\alpha^{-1} : \alpha(U_\alpha\cap U_\beta)\to\beta(U_\alpha\cap U_\beta)$ is holomorphic for each $\alpha$ and $\beta$ in $\mathcal{A}$.

Let $m$ and $n$ be half integers, i.e., $2m,\ 2n\in\mathbb{Z}$, with $m+n$ being an integer. A family $(\omega_\alpha : V_\alpha\to\mathbb{C})_{\alpha\in\mathcal{A}}$ of functions is said to be an $(m,n)$-form on $R$ and will be denoted conventionally by $\omega$ or $\omega_\alpha(z)dz^m d\bar{z}^n$ if for two local coordinates $z=\alpha(p)$ and $w=\beta(p)$ in $\mathcal{A}$ these functions satisfy the transition relations

$$\omega_\beta(w)(w')^m(\overline{w'})^n = \omega_\alpha(z)$$

for $z\in\alpha(U_\alpha\cap U_\beta)$, where $w=\beta\circ\alpha^{-1}(z)$ and $w'=dw/dz$ and the branch of ${w'}^{1/2}\overline{w'}^{1/2}$ should be chosen to equal $|w'|$ in the case that $m$ and $n$ are not integers. A differential form $\omega$ is said to be, for example, continuous or holomorphic if any coefficient $\omega_\alpha$ is continuous or holomorphic. An $(m,0)$-form is called an $m$-form or $m$-differential. If no confusion can occur, we often write $\omega=\omega(z)dz^m d\bar{z}^n$ for simplicity. When $R$ is a plane domain, we will take the canonical coordinate $z$ as a local coordinate unless a specification will be stated. A $(0,0)$-form $\omega$ is nothing but a function on $R$, so we can write $\omega=\omega(p)$ for $p\in R$ without refering to any local coordinate. If all coefficients $\omega_\alpha$ of an $(m,m)$-form are positive (or non-negative) for each local coordinate $\alpha$, we will say that $\omega$ is positive (or non-negative). A positive (or non-negative) $(\frac{1}{2},\frac{1}{2})$-form on $R$ will be called a *conformal metric* (or *conformal pseudo-metric*) on $R$. In this article, a (pseudo-)metric will always be conformal. We note finally that a non-negative $(1,1)$-form $\omega=\omega(z)|dz|^2$ can be regarded as an area element on the surface under the identification $dzd\bar{z}=|dz|^2=dx\wedge dy$, where $z=x+iy$.

Let $f : R\to R'$ be a holomorphic map. Then we can define the pullback of an $(m,n)$-form $\omega$ on $R'$ by $f$, which will be denoted by $f^*\omega$, by

$$(f^*\omega)_\alpha(z) = \omega_\beta(w)(w')^m(\overline{w}')^n,$$

where $\alpha$ and $\beta$ are local coordinates of $R$ and $R'$, respectively, such that $f(U_\alpha)\subset U_\beta$ and $w=\beta\circ f\circ\alpha^{-1}(z)$.

Let $\mathcal{R}$ denote the category of all (connected) Riemann surfaces with holomorphic maps between them as morphisms. A family $\rho=(\rho_R)$ of pseudo-metrics $\rho_R$ on $R$ for all Riemann surfaces $R\in\mathcal{R}$ will be simply called a *pseudo-metric.* A pseudo-metric $\rho$ is said to be *conformally invariant* if $f^*\rho_{R'}=\rho_R$ holds for all conformal isomorphism $f : R\to R'$. A conformally invariant pseudo-metric $\rho$ is said to be *normalized* if $\rho_{\mathbb{D}}(0)=1$ for the unit disk $\mathbb{D}$. By conformal invariance, this means also that $\rho_{\mathbb{D}}=|dz|/(1-|z|^2)$ for the canonical coordinate $z$ of $\mathbb{D}$.

Let $\mathcal{C}$ mean a class of holomorphic maps between Riemann surfaces, namely, $\mathcal{C}(R, R')$ is a set of holomorphic maps from $R$ to $R'$ for any pair of Riemann surfaces. A pseudo-metric $\rho$ is said to be *$\mathcal{C}$-contractive* if $f^*\rho_{R'} \le \rho_R$ holds for any $f \in \mathcal{C}(R, R')$.

In the sequel, we will use mainly the next three classes:

$$\begin{aligned}
\mathcal{O}(R, R') &= \{f : R \to R' \text{ holomorphic}\}, \\
\mathcal{S}(R, R') &= \{f \in \mathcal{O}(R, R'); \text{injective or constant}\}, \text{ and} \\
\mathcal{I}(R, R') &= \{f : R \to R' \text{ biholomorphic}\}.
\end{aligned}$$

Note that $\mathcal{I}(R, R')$ is empty if $R$ and $R'$ are not holomorphically equivalent.

By definition, $\rho$ is $\mathcal{I}$-contractive if and only if $\rho$ is conformally invariant. Similarly, $\rho$ is $\mathcal{O}$-contractive if and only if $\rho$ is holomorphically contractive, and $\rho$ is $\mathcal{S}$-contractive if and only if $\rho$ is conformally invariant and monotone.

We now give an idea to systematically construct conformally invariant pseudo-metrics from the natural classes of functions on Riemann surfaces in the spirit of Ahlfors-Beurling [1].

Let $m$ be a positive integer and $\mathcal{C}$ be a class of holomorphic maps between Riemann surfaces. A class $\mathcal{F} = (\mathcal{F}(R))_{R\in\mathcal{R}}$ of subsets of holomorphic $m$-forms on $R$ will be called *$\mathcal{C}$-admissible* if the following conditions are fulfilled:

(i) if $f \in \mathcal{C}(R, R')$ then $f^*(\mathcal{F}(R')) \subset \mathcal{F}(R)$, and

(ii) the set $\mathcal{F}(R)$ is compact in the sense that any sequence in $\mathcal{F}(R)$ has a subsequence converging to some element of $\mathcal{F}(R)$ uniformly on each compact subset of $R$.

Given a $\mathcal{C}$-admissible class $\mathcal{F}$ of holomorphic $m$-forms, we can define the pseudo-metric $\rho^{\mathcal{F}}$ by

$$(\rho^{\mathcal{F}}_{R,\alpha}(z))^m = \sup\{|\varphi_\alpha(z)|; \varphi \in \mathcal{F}(R)\},$$

where $z = \alpha(p)$ is a local coordinate of a Riemann surface $R$. In fact, for each point $p \in R$, by property (ii), we see that there exists an element $\varphi \in \mathcal{F}(R)$ such that $\rho^{\mathcal{F}}_R = |\varphi|^{1/m}$ at $p \in R$, in particular, $\rho^{\mathcal{F}}_R(p) < +\infty$. We then have the following result, a prototype of which can be found in [1, Theorem 1].

**Proposition 2.1.** *If $\mathcal{F}$ is $\mathcal{C}$-admissible, then the pseudo-metric $\rho^{\mathcal{F}}$ is continuous and $\mathcal{C}$-contractive. Moreover, $\log \rho^{\mathcal{F}}_R$ is subharmonic on $R$ unless $\rho^{\mathcal{F}}_R = 0$.*

***Remark.*** We say that $\log \rho_R$ is subharmonic if $\log \rho_{R,\alpha}$ is subharmonic on $V_\alpha$ for each local coordinate $\alpha$. Note that subharmonicity of $\log \rho_R$ is conformally invariant because $\log |dw/dz|$ is harmonic for a biholomorphic transition function $w = w(z)$.

*Proof.* By definition, $\log \rho^{\mathcal{F}}_{R,\alpha} = \frac{1}{m} \sup_{\varphi\in\mathcal{F}(R)} \log |\varphi_\alpha|$ and each $\log |\varphi_\alpha|$ is subharmonic in $V_\alpha$. So the latter part of the proposition is now clear. In particular, $\rho^{\mathcal{F}}_R$ is lower semi-continuous. In order to show the upper semi-continuity of $\rho^{\mathcal{F}}_R$, it is sufficient to see that $\rho_R(p) \ge \alpha$ for a sequence $p_n$ converging to $p$ in $R$ such that $\rho_R(p_n) \ge \alpha$ for all $n$. Let $\varphi_n \in \mathcal{F}(R)$ be an extremal differential for which $\rho_R(p_n) = |\varphi_n(p_n)|$ holds. Then, by compactness of $\mathcal{F}(R)$, we may assume that the sequence $\varphi_n$ converges to an element $\varphi$ of $\mathcal{F}(R)$ uniformly on each compact subset of $R$. Therefore, we have $\rho^{\mathcal{F}}_R(p) \ge |\varphi(p)| \ge \alpha$. Finally, $\mathcal{C}$-contractivity is straightforward to see. □

We now explain a construction of pseudo-metrics which is dual with the above in some sense.

The complex tangent space $T_pR$ of a Riemann surface $R$ at a point $p \in R$ is defined as the set of derivations from the space of holomorphic germs $\mathrm{Hol}_p(R)$ at $p$ into $\mathbb{C}$, where a map $v : \mathrm{Hol}_p(R) \to \mathbb{C}$ is called a derivation if $v$ is linear and satisfies $v(fg) = g(p)v(f) + f(p)v(g)$ for all $f, g \in \mathrm{Hol}_p(R)$. Let $f : \mathbb{D} \to R$ be a holomorphic map with $f(0) = p$. We can then assign to $f$ a tangent vector $v_f$ defined by

$$v_f(g) = \frac{d}{dz}(g \circ f)(z)\Big|_{z=0} \quad (g \in \mathrm{Hol}_p(R)).$$

Note that $v_f$ is nothing but the image of the canonical tangent vector $(d/dz)|_{z=0}$ of $T_0\mathbb{D}$ under the tangent map $f_*$ of $f$.

A class $\mathcal{C}$ of holomorphic maps between Riemann surfaces will be called *ample* if the following four conditions are satisfied:

(a) for each point $p$ of an arbitrary Riemann surface $R$ there exists a non-constant map $f \in \mathcal{C}(\mathbb{D}, R)$ such that $f(0) = p$,
(b) each $f \in \mathcal{C}(R, R')$ satisfies $f^*(\mathcal{C}(R', \mathbb{D})) \subset \mathcal{C}(R, \mathbb{D})$,
(c) for every Riemann surface $R$, $\mathcal{C}(R, \mathbb{D})$ is compact in the sense that any sequence in $\mathcal{C}(R, \mathbb{D})$ converging uniformly on compacta has its limit in $\mathcal{C}(R, \mathbb{D})$, and
(d) $\mathrm{id}_{\mathbb{D}} \in \mathcal{C}(\mathbb{D}, \mathbb{D})$.

We set $\mathcal{C}_p(R) = \{f \in \mathcal{C}(\mathbb{D}, R); f(0) = p\}$. We now define a pseudo-metric $\sigma^{\mathcal{C}}$ by

$$\sigma^{\mathcal{C}}_{R,\alpha}(z)^{-1} = \sup\{|v_f(\alpha)|; f \in \mathcal{C}_p(R)\},$$

where $\alpha$ is a local coordinate of $R$ with $z = \alpha(p)$. Note that the ampleness guarantees $\sigma_R < +\infty$. The pseudo-metric $\sigma^{\mathcal{C}}$ is actually normalized and $\mathcal{C}$-contractive as is easily seen by property (d) the Schwarz lemma.

***Remark.*** Formally, we may think that a tangent vector is a $-1$-form, so the above definition agrees with the former one for $\rho^{\mathcal{F}}$ in a formal sense.

We can also associate a pseudo-metric with a class $\mathcal{C}$. Set

$$\mathcal{C}^\sharp(R) = \{df; f \in \mathcal{C}(R, \mathbb{D})\}$$

for an arbitrary Riemann surface $R$. Then $\mathcal{C}^\sharp$ is $\mathcal{C}$-admissible by properties (b) and (c). So, we now set $\tau^{\mathcal{C}} = \rho^{\mathcal{C}^\sharp}$. Then the pseudo-metric $\tau^{\mathcal{C}}$ is normalized and $\mathcal{C}$-contractive. Remember that $\tau^{\mathcal{C}}_R(p) = \sup_{f \in \mathcal{C}(R,\mathbb{D})} |df|(p)$ by definition.

The pseudo-metrics $\sigma^{\mathcal{C}}$ and $\tau^{\mathcal{C}}$ are the minimum and the maximum among normalized $\mathcal{C}$-contractive pseudo-metrics $\rho$, respectively.

**Theorem 2.2.** *Let $\mathcal{C}$ be an ample class of holomorphic maps between Riemann surfaces. A normalized, $\mathcal{C}$-contractive pseudo-metric $\rho$ satisfies $\tau^{\mathcal{C}}_R \le \rho_R \le \sigma^{\mathcal{C}}_R$ for each Riemann surface $R$.*

*Proof.* Fix a point $p_0$ in $R$ and a local coordinate $\alpha$ around $p_0$. Set $z_0 = \alpha(p_0)$. Let $f \in \mathcal{C}^\sharp(R) = \mathcal{C}(R, \mathbb{D})$ and put $f_\alpha = f \circ \alpha^{-1}$. By assumption, we have

$$|f'_\alpha(z_0)| \le \frac{|f'_\alpha(z_0)|}{1 - |f_\alpha(z_0)|^2} = (f^*\rho_{\mathbb{D}})_\alpha \le \rho_{R,\alpha}(z_0).$$

This implies $\tau^{\mathcal{C}}_{R,\alpha}(z_0) \le \rho_{R,\alpha}(z_0)$.

We next take an $f \in \mathcal{C}_{p_0}(R)$ and set $f_\alpha = \alpha \circ f$ on $f^{-1}(U_\alpha)$. Then

$$\rho_{R,\alpha}(z_0)|f'_\alpha(0)| = f^*\rho_R(0) \le \rho_{\mathbb{D}}(0) = 1.$$

Noting $f'_\alpha(0) = v_f(\alpha)$, we obtain $\rho_{R,\alpha}(z_0) \le \sigma^{\mathcal{C}}_{R,\alpha}(z_0)$. □

## 3. Examples

In this section, we will demonstrate various classical invariant metrics can be re-defined in our framework. In this section, a metric, say $\rho$, is sometimes called *metrics* instead of *pseudo-metrics* when $\rho_R$ is actually a metric unless $\rho_R$ identically vanishes.

**Example 1** (Kobayashi pseudo-metric $k_R$)**.** The pseudo-metric $k := \sigma^{\mathcal{O}}$ is called the Kobayashi pseudo-metric (cf. [5]). By definition, we can easily see that this metric is holomorphically contractive. Furthermore, we have the next result.

**Lemma 3.1.** *If $R$ is hyperbolic, then $k_R$ coincides with the hyperbolic metric, that is, $\pi^* k_R = |dz|/(1-|z|^2)$ for a holomorphic universal covering $\pi : \mathbb{D} \to R$ of $R$. Otherwise, namely, $R$ is conformally equivalent to either the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, the punctured complex plane $\mathbb{C}^* = \mathbb{C} - \{0\}$ or a complex torus (an elliptic curve), we have $k_R = 0$.*

*Proof.* Suppose that $R$ is hyperbolic and a point $p \in R$ is given. Let $\alpha$ be a local coordinate with $\alpha(p) = z_0$. Then we can take a holomorphic universal covering $\pi$ from the unit disk with $\pi(0) = p$. It suffices to show that $k_{R,\alpha}(z_0) = 1/|v_\pi(\alpha)|$. Let $f \in \mathcal{O}_p(R)$. We can then take a lift $\tilde{f} : \mathbb{D} \to \mathbb{D}$ of $f$ such that $\tilde{f}(0) = 0$ via the universal covering $\pi$. The Schwarz lemma now implies $|v_\pi(\alpha)/v_f(\alpha)| = |\tilde{f}'(0)| \le 1$, which means that $k_{R,\alpha}(z_0)^{-1} = |v_\pi(\alpha)|$.

The latter part is obvious because we can take a holomorphic immersion $f$ of $\mathbb{C}$ into $R$ with $f(0)$ being equal to a given point in $R$. □

**Example 2** (Carathéodory pseudo-metric)**.** The pseudo-metric $c = \tau^{\mathcal{O}}$ is called the Carathéodory (-Reiffen) pseudo-metric. By definition, this pseudo-metric is normalized and holomorphically contractive. We directly see that $c_R = 0$ if and only if $R \in O_{AB}$, namely, $R$ carries no non-constant bounded analytic functions.

The quantity $c_R(p)$ is sometimes called the analytic capacity. An extremal function $f : R \to \mathbb{D}$ satisfying $|df|(p) = c_R(p)$ is usually called the *Ahlfors function* at $p$ and known to be unique up to unimodular constants (see [4]). We remark that the condition $c_R(p) = 0$ at *some* point $p$ need not imply that $c_R(p) = 0$ at *every* point $p$ in the case that $R$ is non-planar. A counterexample was constructed by Virtanen [13] (see also [10, X. 2K]). The same can be said to the span metric $s_R$ defined below.

Applying $\mathcal{C} = \mathcal{O}$ to Theorem 2.2, we now have Theorem 1.1.

**Example 3** (Hahn metric)**.** The pseudo-metric $h := \sigma^{\mathcal{S}}$ is called the Hahn metric (see [6] for details). This metric is normalized, conformally invariant and monotone. It follows that $h_R = 0$ if and only if $R$ is conformally equivalent to either $\widehat{\mathbb{C}}$ or $\mathbb{C}$.

This metric is greatly useful because of conformal invariance and comparability with the quasi-hyperbolic metric in the case of plane domains.

**Lemma 3.2** ([6]). *For a proper subdomain $D$ of $\mathbb{C}$, we have the estimate*

$$\frac{1}{4\delta_D(z)} \leq h_D(z) \leq \frac{1}{\delta_D(z)},$$

*where $\delta_D(z) = \inf\{|z-a|; a \in \partial D\}$.*

**Example 4** (Ahlfors-Beurling pseudo-metric). The pseudo-metric $a := \tau^{\mathcal{S}}$ has been used in the paper [1] by Ahlfors and Beurling. This is normalized, conformally invariant and monotone. Note that $a_R = 0$ if and only if $R$ is either of positive genus or planar and the boundary is of class $N_{SB} = N_{SD}$. For other characterizations of the property $a_R = 0$ for plane domains $R$, see [10, VII. 5C].

Applying $\mathcal{C} = \mathcal{S}$ to Theorem 2.2, we have Theorem 1.2.

**Example 5** (Bergman metric). Let $\Omega(R)$ be the space of square integrable holomorphic 1-forms $\omega$ on a Riemann surface $R$ with norm

$$\|\omega\|_{\Omega(R)} = \left(\frac{i}{2}\iint_R \omega \wedge \bar{\omega}\right)^{1/2} = \left(\iint_R |\omega(z)|^2 dxdy\right)^{1/2}.$$

We define the class $\mathcal{W}$ by $\mathcal{W}(R) = \{\omega \in \Omega(R); \|\omega\|_{\Omega(R)} \leq \sqrt{\pi}\}$. Then the pseudo-metric $b := \rho^{\mathcal{W}}$ is called the Bergman metric. This metric is also normalized, conformally invariant and monotone. Note that $b_R = 0$ if and only if $R$ is planar and $R \in O_G$ (see [10]). Virtanen [13] showed that if $b_R(p) = 0$ at some point $p \in R$ then $b_R$ identically vanishes (see also [10, II. 3C]).

The space $\Omega(R)$ is actually a Hilbert space with the inner product

$$(\omega, \delta)_R = \frac{i}{2}\iint_R \omega \wedge \bar{\delta} = \iint_R \omega(z)\overline{\delta(z)}dxdy.$$

For a local coordinate $\beta$ of $R$, the linear functional $\omega \mapsto \omega_\beta(w)$ is bounded on $\Omega(R)$ for each $w \in V_\beta$, and hence there exists an element $\kappa^{\beta,w} \in \Omega(R)$ such that $\omega_\beta(w) = (\omega, \kappa^{\beta,w})_R$ for all $\omega \in \Omega(R)$ by the Riesz representation theorem. For another local coordinate $z = \alpha(p)$, we write $K_{R,\alpha,\beta}(z,w) = \kappa_\alpha^{\beta,w}(z)$ for $(z,w) \in V_\alpha \times V_\beta$. For brevity, we sometimes write $K_R(p,q) = K_R(z,w)dzd\bar{w}$ and call it the Bergman kernel of $R$. Note that $\overline{K_R(p,q)} = K_R(q,p)$ and $K_R(w,w) = (K_R(\cdot,w), K_R(\cdot,w))_R = \|K_R(\cdot,w)\|_{\Omega(R)}$. The Schwarz inequality implies that $|\omega(w)| = |(\omega, K_R(\cdot,w))_R| \leq \sqrt{K_R(w,w)}\|\omega\|_R$ with equality being valid if $\omega$ is a constant multiple of $K_R(\cdot,w)$. It then follows that $b_R(w) = \sqrt{\pi K_R(w,w)}$.

The reader should be careful of the fact that the Bergman (pseudo-)metric sometimes refers to the Hermitian form $\sum_{jk} \partial_{z_j}\partial_{\bar{z}_k} \log K_R(z,z)dz_j d\bar{z}_k$ in the theory of several complex variables.

The following proposition was proved by Suita [12]. (For finite Riemann surfaces, Hejhal proved it earlier.)

**Proposition 3.3.** *The inequality $c_R \leq b_R$ holds for every Riemann surface $R$.*

**Example 6** (span pseudo-metric)**.** Define the space $\Omega_{\mathrm{e}}(R)$ as the closed subspace of $\Omega(R)$ consisting of exact differentials with the same norm as $\Omega(R)$. Set $\mathcal{W}_{\mathrm{e}} = \Omega_{\mathrm{e}}(R) \cap \mathcal{W}$. Then the pseudo-metric $s := \rho^{\mathcal{W}_{\mathrm{e}}}$ is called the span (pseudo-)metric. This is also normalized, conformally invariant and monotone. By definition, we know $s_R = 0$ if and only if $R \in O_{AD}$.

Since $\Omega_{\mathrm{e}}$ is a closed subspace of the Hilbert space $\Omega(R)$, it has a reproducing kernel, written by $\tilde{K}_R(p,q) = \tilde{K}_R(z,w)dzd\bar{w}$, which is called the exact (or reduced) Bergman kernel. As before, we can prove that $s_R(z) = \sqrt{\pi \tilde{K}_R(z,z)}$. It is clear that $s_R \le b_R$. The next result is due to Ahlfors and Beurling [1] in the case that $R$ is planar. For the general case, see [9].

**Proposition 3.4.** *We have $s_R \le c_R$ for every Riemann surface $R$.*

Since $c_R \le k_R$, we also have the estimate $s_R \le k_R$, which was used effectively in the paper [2] by Beardon and Gehring.

**Example 7** (quadratic differentials)**.** Let $A(R)$ be the set of integrable holomorphic quadratic differentials (2-forms) $\varphi = \varphi(z)dz^2$ on $R$ with norm

$$\|\varphi\|_{A(R)} = \iint_R |\varphi| = \iint_R |\varphi(z)|dxdy.$$

Define the class $\mathcal{Q}$ by $\mathcal{Q}(R) = \{\varphi \in A(R); \|\varphi\|_{A(R)} \le \pi\}$. Then the pseudo-metric $q := \rho^{\mathcal{Q}}$ is normalized, conformally invariant and monotone. (In this case, $m = 2$.) In particular, by Theorem 1.2, we see $q_R \le h_R$. It is essentially well known that $q_R = 0$ if and only if $R$ is conformally equivalent to either $\widehat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$ or $\mathbb{C} - \{0,1\}$. For more details of this pseudo-metric, see [11].

For every element $\omega \in \Omega(R)$, we see $\omega^{\otimes 2} = \omega(z)^2 dz^2$ belongs to $A(R)$ and satisfies $\|\omega^{\otimes 2}\|_{A(R)} = \|\omega\|^2_{\Omega(R)}$. Therefore, we have the following:

**Proposition 3.5.** *The inequality $b_R \le q_R$ holds for every Riemann surface $R$.*

The next example may be out of our formulation, however the way to understand logarithmic capacity as in the following is sometimes important.

**Example 8** (logarithmic capacity)**.** Let $\mathcal{B}^p(R)$ be the set of multivalued holomorphic functions $f$ on $R$ such that $|f|$ is single-valued, bounded and satisfies $f(p) = 0$. Then, the pseudo-metric $r$ defined by

$$r_R(p) = \sup\{|df|(p); f \in \mathcal{B}^p(R)\}$$

is normalized, conformally invariant and monotone. We should note here that $|df|$ is also single-valued for $f \in \mathcal{B}^p(R)$. We will understand below that $r_R = 0$ if and only if $R \in O_G$.

The quantity $r_R(q)$ is called the (logarithmic) capacity. In fact, let $\gamma$ be the Robin constant of $R$ at a point $q$ with respect to a local coordinate $\alpha$, in other words, Green's function $G_R(p,q)$ of $R$ with pole at $q$ has the local behaviour

$$G_R(p,q) = -\log|\alpha(p) - \alpha(q)| + \gamma + u(p)$$

near the point $q$, where $u$ is a harmonic function near $q$ with $u(q) = 0$. It suffices to see $r_{R,\alpha}(w) = e^{-\gamma}$, where $w = \alpha(q)$. Set $v(p) = G_R(p,q)$ for $p \in R$. First observe that

$\omega := 2(\partial v/\partial z)dz$ is a holomorphic 1-form on $R-\{q\}$ and $\omega_\alpha(z)+(z-w)^{-1}$ is holomorphic near $w=\alpha(q)$. We fix a point $p_0$ in $R$ other than $q$. Then the holomorphic function

$$f(p) = \exp\left(-\int_{p_0}^{p} \omega - v(p_0)\right)$$

is multivalued in $R$ but has single-valued modulus and satisfies $|f| = e^{-v} < 1$. Thus $f \in \mathcal{B}^q(R)$ and we can easily see $df_\alpha(w) = e^{-\gamma}$.

Let $g$ be an arbitrary multivalued holomorphic function on $R$ with $|g|$ being single-valued and $|g|<1$. Then $\psi = -\log|g|$ is positive and superharmonic in $R$. Since $\psi - v$ is harmonic near the point $q$ or takes the value $+\infty$ at $q$, we conclude that $\psi - v > 0$ on $R$ by the minimum principle. This implies $|dg|(q) \le |df|(q)$ for such a $g$. Now we have $r_{R,\alpha}(w) = |df|_{R,\alpha}(w) = e^{-\gamma}$.

By this characterization, we have $c_R \le r_R$ for all Riemann surfaces $R$. We also have the next result due to Burbea [3].

**Proposition 3.6.** *For every Riemann surface $R$, the inequality $r_R \le k_R$ holds.*

Suita [12] conjectured that $r_R \le b_R$ for all Riemann surfaces $R$. Although this conjecture is not solved yet, recently Ohsawa [7] proved $r_R \le K b_R$ for all Riemann surfaces $R$ with an absolute constant $K \le \sqrt{750}$. (In [8], the estimate was improved to $K \le 16\sqrt{2}$.)

Summarizing the above results, we have the chain of inequalities

$$a_R \le s_R \le c_R \le \left\{ \begin{array}{c} r_R \le k_R \\ b_R \le q_R \end{array} \right\} \le h_R$$

for every Riemann surface $R$.

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