Conformal mapping and universal teichmuller space

Wang Yanyan
Nagoya university

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Introduction

- Bieberbach conjecture
- Krushkal's paper
  (i) Main theorem
  (ii) Main idea

- 1916. Bieberbach: $|a_n| \le n$ for $\forall f \in S$

  "=" only for $K_\theta := z + \sum_{2}^{\infty} n e^{-i(n-1)\theta} z^n \quad \theta \in [0, 2\pi)$

- 1923. Loewner: $\forall f \in S$. $|a_3| \le 3$
- 1984. de. Branges: proved
- Zalcman Conjecture: $\forall f \in S$. $|a_n^2 - a_{2n-1}| \le (n-1)^2$

  "=" only for K_θ

(Hayman) $\Downarrow$

Bieberbach Conjecture.

- Hayman theorem: $\forall f \in S$

$$\lim_{n \to \infty} \frac{|a_n|}{n} = \alpha \le 1, \text{ "=" only for } K_\theta$$

$$\alpha = \lim_{r \to 1} (1-r)^2 \max_{|z|=r} |f(z)|$$

Krashkal's main result.

- $J_n(f) := a_n^p - a_{p(n-1)+1} + P(a_4, a_5, \cdots, a_{p(n-1)-3})$

  for $\forall f \in S$. $p \geq 2$.

  - P is a polynomial of indicated coefficients of f which is homogeneous of degree $p(n-1)$ with respect to the stretching $f(z) \longmapsto f_t(z) := \frac{1}{t} f(tz)$ $t \in [0,1]$

- Thm: $\forall J_n(f)$ with $n \geq 3$. $f \in S$.

$$|J_n(f)| \leq \max\{|J_n(K_\theta)|, |J_n(K_{2,\theta})|\}$$

  If $P \equiv 0$ "=" only for K_θ.

  - $K_{2,\theta} := \sqrt{K_\theta(z^2)} = \frac{z}{1-e^{i\theta}z^2} = \sum_0^\infty e^{in\theta} z^{2n+1}$

- $P \equiv 0$ $p=2$. Krashkal's theorem $\Rightarrow$ Zalcman conjecture

  $\Rightarrow$ Bieberbach conjecture.

$$S := \{ f \mid f : \mathbb{D} \xrightarrow{\text{cont}} \mathbb{C},\ f(0)=0,\ f'(0)=1 \}$$

$$S^{\circ} := \{ f \in S \mid f \text{ has q.c. extension to } \mathbb{C} \}$$

$$\Sigma := \{ F \mid F = \frac{1}{f(\frac{1}{z})},\ f \in S \}$$

$$\Sigma^{\circ} := \{ F \in \Sigma \mid F \text{ has q.c. extension to } \mathbb{C} \}$$

$$F_1 \in \Sigma^0 \to S_{F_1}(z) \xrightarrow{\phi_P(\mu) = S_{F_1^\mu}} \Gamma \to F_1(\Gamma) \subset T \times \Delta(0,2)$$

$$\downarrow$$

$$\left(F_1 = \frac{1}{f(\frac{1}{z})} \right) \qquad J_n(f) =: \tilde{J}_n(S_{F_1}, a_2)$$

$$\downarrow$$

$$\frac{\tilde{J}_n(S_{F_1}, a_2)}{\max_S |J_n(f)|} \qquad \Leftrightarrow : \tilde{J}_n^0(S_{F_1}, a_2)$$

$$\downarrow$$

$$\tilde{J}_n^0(S_{F_1^\mu}, F_1^\mu(z_m)) \qquad =: J_{n,m}(S_{F_1^\mu}) \qquad \begin{cases} m = 1, 2, \dots \\ n \text{ fixed} \end{cases}$$

$$\downarrow$$

$$J_n(S_{F_1}) := \sup |J_{n,m}(S_{F_1})|^{2/p(n-1)}$$

- $g_m(\zeta) := J_{n,m}(\zeta)^{2/p(n-1)}$. $(J_{n,m}(\zeta)|_{\partial D})$

$$\downarrow$$

$$\lambda g_m(\zeta) := g_m^* \lambda_\Delta(\zeta) = \frac{|g_m'(\zeta)|}{1 - |g_m(\zeta)|^2}, \quad \zeta \in \mathbb{D})$$

$$\downarrow$$

$$\lambda_\Gamma(\zeta) := \sup \lambda g_m(\zeta)$$

- $\log J(\varphi) \le g_\Gamma(0, \varphi)$

$D = h(\mathbb{D})$ distinguished disk. $h'(\zeta) \neq 0$ on $\mathbb{D} \setminus \{0\}$

- a holomorphic disk in $\mathcal{P}$ is called distinguished if it touch the zero-set

$$Z_J := \bigcup_m \{ S_{F_0} \in \mathcal{P} : J_{n,m}(S_{F_0}) = 0 \}$$

only at the origin.

Preliminary

- Quasiconformal mapping
- Complex dilatation
- Schwarzian derivative

Quasiconformal mapping

- A sense preserving homeomorphism with a finite maximal dilatation is quasiconformal. If the maximal dilatations is bounded by a number K, the mapping is said to be K-quasiconformal.

quadrilateral :

$Q(z_1, z_2, z_3, z_4)$ Jordan domain

$\{z_1, z_2, z_3, z_4\} \subset \partial Q$ following each other determine a positive orientation of ∂Q. with respect to Q

R : canonical rectangle of $Q(z_1, z_2, z_3, z_4)$

$\{$ i.e. $\exists f : Q \xrightarrow{\text{conf}} R$. s.t. z_1, z_2, z_3, z_4 correspond to the vertices of R $\}$

If $R=\{x+iy \mid 0<x<a.\ \ 0<y<b\}$

(z_1, z_2) corresponds to $0 \le x \le a$

module of $Q(z_1, z_2, z_3, z_4)$:

$$M(Q(z_1, z_2, z_3, z_4)) = a/b$$

maximal dilatation:

$$K := \sup_{Q} \frac{M(f(Q)(f(z_1), f(z_2), f(z_3), f(z_4)))}{M(Q(z_1, z_2, z_3, z_4))}$$

$f: A \to A'$: sense-preserving homeomorphism

$\bar{Q} \subset A$.

- K invariant under conformal
- $K \geq 1$. $K=1$ iff f conf.

Complex dilatation :

- $f: A \to A'$ K.q.c. Suppose $J_f(z) > 0$ then $\partial f(z) \neq 0$

define: $\mu(z) = \dfrac{\bar{\partial} f(z)}{\partial f(z)}$: complex dilatation of f.

$$|\mu(z)| \le \frac{K-1}{K+1} < 1.$$

- $\mu(z) = 0$ iff f: conformal.

f, g : q.c. of A with μ_f, μ_g.

$$\Rightarrow \mu_{f\circ g^{-1}}(\zeta) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \left(\frac{\partial g(z)}{|\partial g(z)|}\right)^2 \quad \zeta = g(z)$$

$\Rightarrow$ $\mu_f = \mu_g$ a.e in A iff $f\circ g^{-1}$ is conformal

(existence theorem): μ : measurable in A with $\|\mu\|_\infty < 1$, then $\exists$ q.c f of A s.t. $\mu_f = \mu$ a.e. in A.

Schwarzian derivative

- Definition
- Existence and uniqueness
- Norm of the schwarzian derivative
- Convergence of schwarzian derivatives

Definition: $f: A \to \hat{\mathbb{C}}$, mero, locally injective.

$f'(z) \neq 0$

$$S_f(z) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \left(\frac{f'''}{f'}\right) - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \quad z \in A$$

If $\infty \in A$ let $\varphi(z) = f\left(\frac{1}{z}\right)$, $S_f(\infty) = \lim_{z \to 0} z^4 S_\varphi(z)$

- S_f holo in A
- $S_g = 0$ if $g \in$ Möb
- $S_{f \circ g} = S_g$ if $f \in$ Möb
- $S_{f \circ g} = (S_f \circ g)(g')^2 + S_g$

Thm : A : simply connected

φ : holo in A

$\Rightarrow \exists f$: mero in A s.t. $S_f = \varphi$

The solution is unique up to an arbitrary Möbius transformation.

norm: A: simply connected, conformally equivalent to A

η_A: Poincare density of A

$$\eta_A := \frac{f'(z)}{1-|f(z)|^2} \cdot \quad f: A \xrightarrow{\text{conf}} \mathbb{D} .$$

hyperbolic sup-norm:

$$\|S_f\|_A := \sup_{z \in A} \eta(z)^{-2} |S_f(z)|$$

- $|S_f|\eta^{-2}$ is a function on Riemann Surface

Convergence :

f_n, f : mero, locally injective in A

If $f_n \longrightarrow f$ locally uniformly in A

$\Rightarrow S_{f_n} \longrightarrow S_f$ locally uniformly in A

$\nRightarrow \lim_{n\to\infty} \|S_{f_n} - S_f\|_A = 0$

Models of the universal Teichmuller space T

1. T is the set of the equivalence classes of For B.
2. T is the set of all normalized quasisymmetric fucntions.
3. T is the normalized conformal mappings.
4. T is the collection of all normalized quasidiscs.

• $F_1 := \{ f \mid f : \mathbb{H} \xrightarrow{q.c} \mathbb{H},\ \text{fix } 0, 1, \infty \}$

$f_1 \sim f_2$ if $f_1|_{\mathbb{R}} = f_2|_{\mathbb{R}}$. $f_1, f_2 \in F_1$.

$B := \{ \mu(z) \mid \mu(z) \text{ measurable}, \ \|\mu(z)\| < 1, \ z \in \mathbb{H} \}$

$\mu_{f_1} \sim \mu_{f_2} :\Leftrightarrow f_1 \sim f_2$.

(a) $T := \{ [f] \mid f \in F_1 \} = \{ [\mu] / \mu \in B \}$

• $X := \{ h \mid h : \mathbb{R} \to \mathbb{R} \text{ quasisymmetric}, \ \text{fix } 0, 1 \}$

$$\begin{array}{ccc} T & \xrightarrow[\text{onto}]{\text{bijective}} & X \\ \Downarrow & & \\ [f] & \longmapsto & f|_{\mathbb{R}}. \end{array}$$

(b) $T := \{ h \mid h \in X \}$

- $B^* := \left\{ \mu^* \;\middle|\; \mu^*(z) = \begin{cases} \mu(z) & z \in H \\ 0 & z \notin H \end{cases} \right\}$

- $F^* := \left\{ f_\mu \;\middle|\; \mu \in B^*,\ f_\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ fix } 0, 1, \infty,\ f_\mu|H' \text{ conf} \right\}$

- $f_\mu, f_\nu \in F^*.\quad f_\mu \sim f_\nu :\Leftrightarrow f_\mu|H' = f_\nu|H'$

- $\mathcal{T} := \{ [f_\mu] \mid f_\mu \in F^* \} = \{ [f^\mu] \mid f^\mu \in F \}$

- $\Delta := \{ A \mid A \text{ is a normalized quasidisc} \}$

$$f \in F^* \qquad \begin{array}{ccc} [f] & \xrightarrow{\text{bijec}} & f(H') \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{T} & \longrightarrow & \Delta \end{array}$$

- $\mathcal{T} := \{ f(H') \mid f(H') \in \Delta \}$

Normalized quasidiscs

- We call a quasidisc normalized if its boundary passes through the points 0,1,infinity,and is so oriented that the direction from o to 1 to infinity is negtive with respect to the domain.

quasicircle

A quasicircle in the extended plane is the image of a circle under a quasiconformal mapping of the plane.A domain bounded by a quasicircle is called a quasidisc.

Metric of T

- T has a natural metric, we obtain this metric by measuring the distance between quasiconform mappings in terms of their maximal dilatations.
- Some properties

(i) teichmuller distance and complex dilatation

(ii) geodesics , contractibility, incompatibility metrit

metric of $\mathbb{P}$:

$p, q \in \mathbb{P}$

$$\tau(p, q) = \frac{1}{2}\inf\{\log K_{g\circ f^{-1}} \mid f \in p, g \in q\}$$

$$= \frac{1}{2}\inf\{\log K_{g_0\circ f_0^{-1}} \mid f_0 \in p, g_0 \in q\}$$

$$= \frac{1}{2}\min\{\log K_h \mid h = g_0\circ f_0^{-1}|\mathbb{R}\}$$

- τ makes $\mathbb{P}$ into a complete metric space.

Teichmuller distance and complex dilatation

$$\tau(p,q)=\frac{1}{2}\min\left\{\log\frac{1+\left\|\frac{\mu-\nu}{1-\bar{\mu}\nu}\right\|_\infty}{1-\left\|\frac{\mu-\nu}{1-\bar{\mu}\nu}\right\|_\infty}\;\middle|\;\mu\in p,\ \nu\in q\right\}$$

- $\beta(p,q)=\min\left\{\left\|\frac{\mu-\nu}{1-\bar{\mu}\nu}\right\|_\infty\;\middle|\;\mu\in p,\ \nu\in q\right\}$
- β makes T into a metric space
- $\beta=\tanh\tau \Rightarrow (T,\tau),\ (T,\beta)$ topologically equivalent
- (T,β) arcwise connected, complete

Geodesics for the teichmuller metric

- The length of an arc
- An arc is a geodesic if the length of every subarc is equal to the distance between the endpoints.
- Geodesic of T can be described explicitly with the help of extremal complex dilatation
- Theorem :

$$\gamma : [0,1] \longrightarrow (P, \tau)$$

$$l(\gamma) := \sup_{0=t_0<t_1<\cdots<t_n=1} \left\{ \sum \tau(\gamma(t_{j-1}), \gamma(t_j)) \right\}$$

<u>Thm</u> : μ is extremal. for $p \in T$. then.

$$\mu_t = \frac{(1+|\mu|)^t - (1-|\mu|)^t}{(1+|\mu|)^t + (1-|\mu|)^t} \frac{\mu}{|\mu|} \qquad t \in [0,1]$$

is extremal for $p_t = [\mu_t]$

the arc $t \to p_t$ is a geodesic from 0 to p

and $\tau(p_t, 0) = t\,\tau(p, 0)$

$\mu \in P \in \mathcal{P}$ is extremal. if

$$\|\mu\|_\infty = \min\{\|\nu\|_\infty \mid \nu \in P\}$$

Contractibility of T

- T is contractible.

- $\exists \omega : T \times [0,1] \xrightarrow{\text{conti}} T$
- $\exists \ \omega : T$
- $\exists \ \pi : T \times [0,1] \xrightarrow{\text{conti}} T$

  s.t $\pi(p, 0) = p$. $\quad \pi(p, 1) = \text{constant}$

Distance between quasisymmetric functions

- $$K_h^* := \sup \frac{M(\mathbb{H}(h(x_1), h(x_2), h(x_3), h(x_4)))}{M(\mathbb{H}(x_1, x_2, x_3, x_4))}$$

  $x_1, x_2, x_3, x_4 \in \overline{\mathbb{R}}$ determine the positive orientation with respect to $\mathbb{H}$

- $\rho(h_1, h_2) := \frac{1}{2} \log K^*_{h_2 \circ h_1^{-1}}$, $h_1, h_2 \in X$

- The group isomorphism

$$\begin{array}{ccc} [f] & \longmapsto & f|_{\mathbb{R}} \\ \uparrow & & \Cap \\ (T, \tau) & \longrightarrow & (X, \rho) \end{array}$$

homeomorphism.

Incompatibility of the group structure with the metric

- The topological structure and the group structure of X are not compatible.
- T is not a topological group.

$$\exists\, [f] \in \mathcal{T},\ [g_n] \in \mathcal{T} \text{ s.t. } [g_n] \to [g] \text{ but } [f \circ g_n] \not\to [f \circ g]$$

- $f \in F_1 \Rightarrow f^{-1} \in F_1$.

  $f, g \in F_1 \Rightarrow f \circ g \in F_1$.

  So. F_1 can be regarded as a group.

- P inherits this group structure.

  the rule $[f] \circ [g] = [f \circ g]$ defines

  the group operation in P.

- the point of P determined by the identity mapping is called the origin of P and denoted by O.

- counterexample:

$$f(x) = \begin{cases} x & x \ge 0 \\ x/2 & -2 \le x < 0 \\ x+1 & x < -2 \end{cases}$$ 2-quasisymmetric of X.

$$l_n(x) = \begin{cases} x & x \ge 0 \\ (1+\frac{1}{n})x & x < 0 \end{cases}$$ $n = 1, 2, \cdots$. $l_n(x) \in X$.

$(1+\frac{1}{n})$-quasisymmetric. so $(1+\frac{1}{n})^2$ q.c.

$l(x) = x \quad (x \in \mathbb{R})$

$\rho(l_n, l) \le \log(1 + 1/n)$

let $g_n = l_n \circ f^{-1}$ $\quad \because \rho(g_n, f^{-1}) = \rho(l_n, l)$

$\Rightarrow \lim\limits_{n\to\infty} \rho(g_n, f^{-1}) = 0$

but $f \circ g_n = f \circ l_n \circ f^{-1} \not\longrightarrow f \circ (\lim g_n) = l$ in the ρ-metric.

Mapping into the space of schwarzian derivatives

- Comparison of distance
- Imbedding of T

$\cdot\ T(1) := \{ S_{f_\mu|\mathbb{H}} \mid [\mu] \to S_{f_\mu|\mathbb{H}},\ \mu \in B \}$

$= \{ S_f \mid f \text{ is conf in } \mathbb{H} \text{ has q.c. to } \mathbb{C} \}$

$\therefore\ [\mu] \longmapsto S_{f_\mu|\mathbb{H}}$ homeo

$\Uparrow$

$(T, \rho) \longrightarrow (T(1), q)$

(Bers imbedding of Teichmuller space).

Comparison of distance

$f_\mu : \mathbb{C} \xrightarrow{q.c.} \mathbb{C}$ $\quad f_\mu|_{\mathbb{H}}$: conf

- $Q := \{\varphi \mid \varphi$ holo in $\mathbb{H}$ with $\|\varphi\| = \sup_{z\in\mathbb{H}} 4y^2|\varphi(z)|$ finite$\}$

- $g(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|$ $\quad \varphi_1, \varphi_2 \in Q$.

$S_\mu := S_{f_\mu|\mathbb{H}}$

- $\varepsilon(K_0)\,\beta([\mu],[\nu]) \le g(S_\mu, S_\nu) \le \sigma_0(A_\nu)\,\beta([\mu],[\nu])$

$\sigma_0(A = 6 + \delta(A_\nu)$ $\qquad (\delta(A) = \|S_f\|_A$).

- β - and g - metrics are topologically equivalent.

$$g(z) = i\,\frac{1+z}{1-z} : \mathbb{D} \longrightarrow \mathbb{H}$$

(inverse of the Cayley transform)

$$z \longmapsto \frac{z-i}{z+i}$$

thank you very much!