Loewner 方程式の応用について
Applications of Loewner Equation

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Semigroup of analytic germs at the origin

Let $\mathscr{O}_0$ be the set of analytic germs $f(z)$ of one complex variable z with $f(0) = 0$ and let

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Our main aim is to interpret the Loewner equation in the context of these structures.

Key references

"The power matrix, coadjoint action and quadratic differentials", by Eric Schippers, J. Anal. Math. **98** (2006) 249–277.

"A Green's inequality for the power matrix", by M. Schiffer and O. Tammi, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **501** (1971) 15 pages.

Grunsky coefficients

Let $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ be an element of $\mathscr{O}_0^\times$. The Grunsky coefficients $A_{m,n}$ of f are defined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=0}^{\infty} A_{m,n} z^m \zeta^n$$

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We also consider the expansion

$$\log \frac{1}{1 - f(z)\overline{f(\zeta)}} = \sum_{m,n=0}^{\infty} B_{m,n} z^m \bar{\zeta}^n.$$

Grunsky-Nehari inequality

Theorem (Nehari 1953)

If $f : \mathbb{D} \to \mathbb{D}$ is univalent and $f(0) = 0$, then

$$\mathrm{Re} \sum_{l,m=1}^{N} (A_{l,m} t_l t_m + B_{l,m} t_l \bar{t}_m) \leq \sum_{l=1}^{N} \frac{|t_l|^2}{l}$$

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Schiffer and Tammi extended it by using the power matrix and Loewner equation.

Power matrix

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I. Schur (1945), Jabotinsky (1953), and so on.

Connection with Loewner equation

Let

$$f_t(z) = \sum_{k=1}^{\infty} c_k(t) z^k$$

be a solution to the (radial) Loewner equation

$$\dot{f}_t = -z \frac{1 + u(t)z}{1 - u(t)z} f_t', \quad t \geq 0,$$

for a continuous $u(t)$ with $|u(t)| = 1$. Then,

$$\sum_{k=1}^{\infty} \dot{c}_k(t) z^k = -(1 + 2 \sum_{k=1}^{\infty} u^k z^k) \sum_{k=1}^{\infty} k c_k(t) z^k.$$

Therefore,

$$\dot{c}_k(t) = -kc_k(t) - 2\sum_{m=1}^{k-1} mu(t)^{k-m} c_m(t)$$

for $k \geq 1$. (In particular, $c_1(t) = e^{-t}$.)

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$$\dot{c}_{n,k}(t) = -kc_{n,k}(t) - 2\sum_{m=1}^{k-1} mu(t)^{k-m}c_{n,m}(t)$$

for $k \geq n$, where $c_{n,k}(t) = [f_t]_k^n$.

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for $k \geq n$, where $c_{n,k}(t) = [f_t]_k^n$. From these equations, Schiffer and Tammi (1971) obtained an extension of Grunsky-Nehari inequalities.

Law of composition

The power matrix has the following remarkable property:

$$[f \circ g] = [f][g]$$

for $f, g \in \mathscr{O}_0$. Also, $[\mathrm{id}] = \mathrm{id}$. Therefore, this gives a matrix representation of $\mathscr{O}_0$ and $\mathscr{O}_0^\times$.

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Also note that

$$n[f]_n^{k+1} = (k+1) \sum_{m=1}^{n-k} m[f]_m^1 [f]_{n-m}^k.$$

Lie group G and Lie algebra $\mathfrak{g}$

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$$\langle h \rangle_n^m = \begin{cases} m[h]_{n-m+1}^1 & (m \le n) \\ 0 & (m > n). \end{cases}$$

Then $\mathfrak{g} = \{\langle h \rangle : h \in \mathscr{O}_0\}$ can be identified with the Lie algebra of G. The Lie bracket is given by

$$[\langle h \rangle, \langle j \rangle] = \langle h \rangle \langle j \rangle - \langle j \rangle \langle h \rangle.$$

An observation

We set $\mathbf{e}_n = \langle z^{n+1} \rangle$ for $n \geq 0$. Then

$$[\mathbf{e}_k, \mathbf{e}_l] = (k - l)\mathbf{e}_{k+l}.$$

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Thus $\mathfrak{g}$ can be regarded as the "positive part" of the Virasoro algebra with zero central charge.

Loewner equations

Loewner ODE:

$$\frac{d}{dt}w_t = -w_t p_t(w_t).$$

Loewner PDE:

$$\frac{d}{dt}f_t = z f_t' p_t.$$

Matrix forms of Loewner equations

Matrix form (Schippers 2006)

$$\frac{d}{dt}[w_t] = -\langle zp_t\rangle [w_t].$$

$$\frac{d}{dt}[f_t] = [f_t]\langle zp_t\rangle$$