The ‘maximal range’ problem for complex polynomials

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Abstract

Let Ω be some domain in the complex plane, with $0 \in \Omega$, and $\mathbb{D}$ the unit disk. For $n \in \mathbb{N}$ we set

$$\mathcal{P}_n(\Omega) := \{P \in \mathcal{P}_n \,:\, P(0) = 0,\, P(\mathbb{D}) \subset \Omega\},$$

where $\mathcal{P}_n$ denotes the set of complex polynomials of degree $\leq n$. The *maximal range* of degree n with respect to Ω is then defined as

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D}).$$

We give a complete description of the extremal polynomials in $\mathcal{P}_n(\Omega)$, i.e. such P with $P(\partial\mathbb{D}) \cap (\partial\Omega_n \setminus \partial\Omega) \neq \emptyset$. This identification leads in many cases to a unified approach to new and old estimates for polynomials with certain range restrictions (most elemenary example: polynomials with positive real part in $\mathbb{D}$). Furthermore, if Ω is simply connected, the extremal polynomials are very sensible candidates for a polynomial approximation to the conformal mappings of $\mathbb{D}$ onto Ω. We discuss special cases and an intriguing ‘arc’-conjecture, which is closely related to a special subordination condition for polynomials.