Löwner Equations
and
Dispersionless Integrable Hierarchies

Takashi Takebe

Faculty of Mathematics/ International Laboratory
of Representation Theory and Mathematical Physics,
National Research University — Higher School of Economics,
Moscow, Russia

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§0. Introduction

- **Integrable** hierarchies = 'solvable' systems with infinitely many variables (e.g., $t = (t_1, t_2, t_3, \ldots)$).
- Dispersionless **integrable** hierarchies = quasi-classical limits of certain **integrable** hierarchies.
- One-variable reduction: solutions depend on $\infty$-many variables only through one function, e.g., $\lambda(t)$.

**Today's topic**

one-variable reduction of the dispersionless KP (resp. Toda, BKP, DKP) hierarchy

$\Leftrightarrow$

the chordal (resp. radial, quadrant, annulus) Löwner equation.
Plan of the talk:

1. Brief introduction to integrable systems.
2. KP hierarchy and Toda lattice hierarchy.
3. Dispersionless hierarchies.
5. dKP hierarchy and chordal Löwner equation.
6. Other examples.

Disclaimer: In this talk everything is quite “algebraic”:

- “functions” = formal power series
- “operators” = elements of non-commutative rings

Only algebraic structure is studied.

(& “genericity conditions” often omitted, ...)
§1. What are “integrable systems”?

For systems with finite degrees of freedom,

∃ well established/defined geometric criteria of integrability.

- Frobenius integrability condition
- Liouville integrability condition (for Hamiltonian systems)
  = “existence of sufficiently many conserved quantities”

Examples: Kepler motion, Tops (Euler, Lagrange, Kowalevski)

How about “integrable systems” with infinite degrees of freedom?
Modern theory of integrable systems began with the discovery of remarkable solutions of non-linear partial differential equations = “SOLITONS” in 1960’s.

**Soliton** = particle-like stable solitary wave

Examples of soliton equations:

- **KdV equation (1895):** \( u = u(x, t) \), \( u_t - 3uu_x - \frac{1}{4}u_{xxx} = 0 \).

- **KP equation (1970):** \( u = u(x, y, t) \),
  \[
  \frac{3}{4}u_{yy} - (u_t - 3uu_x - \frac{1}{4}u_{xxx})_x = 0.
  \]

- **Sine-Gordon equation:** \( u = u(x, t) \), \( u_{tt} - u_{xx} - \sin u = 0 \).

- **Toda lattice (1967):** \( u_n = u_n(t) \), \( u_{n,tt} = e^{u_{n-1}-u_n} - e^{u_n-u_{n+1}} \).
• Surprisingly, such soliton equations are solvable in spite of its nonlinearity!
  – inverse scattering method, Lax pairs
  – algebro-geometric solutions
  – Hirota’s bilinear method
  \[ \Rightarrow \text{ various generalisation} \]

• Why are they solvable? \[ \Rightarrow \text{ discovery of} \]
  – infinitely many conserved quantities/ symmetries
  – moduli space of solutions (e.g., \( \infty \)-dimensional Grassmann manifold for KP hierarchy)
  \[ \Rightarrow \text{ relation to algebra (e.g., representation theory of } \infty \text{-dimensional Lie algebras).} \]

Let us examine the KP and the Toda lattice hierarchies as examples.
§2 KP hierarchy and Toda lattice hierarchy

KP hierarchy: integrable nonlinear system for $u_i(t)$ ($i = 2, 3, \ldots$) w.r.t. $t = (t_1, t_2, t_3, \ldots)$. ($x = t_1$, $\partial = \partial/\partial x$.)

The Lax operator: $L = \partial + u_2(t)\partial^{-1} + u_3(t)\partial^{-2} + \cdots$.

Notation: symbols $f(x)\partial^m$ for $m \in \mathbb{Z}$ span an algebra:

$$(f(x)\partial^m)(g(x)\partial^n) = \sum_{r=0}^{\infty} \binom{m}{r} f^{(r)} g^{(m+n-r)}.$$ 

(KP) \[
\frac{\partial L}{\partial t_n} = [B_n, L] \quad (n = 1, 2, \ldots; B_n = (L^n)_{\geq 0}).
\]

Notation: $P = \sum_{n \in \mathbb{Z}} a_n \partial^n \rightarrow P_{\geq 0} := \sum_{n \geq 0} a_n \partial^n$. 


This includes the KP equation for \( u = u_2 \):

\[
\frac{3}{4} u_{tt} u_t - \left( u_{t3} - 3u u_x - \frac{1}{4} u_{xxx} \right)_x = 0
\]

\( \therefore \) First two equations \( \frac{\partial L}{\partial t_2} = [B_2, L] \) and \( \frac{\partial L}{\partial t_3} = [B_3, L] \) are expanded as

\[
\frac{\partial u_2}{\partial t_2} \partial^{-1} + \frac{\partial u_3}{\partial t_2} \partial^{-2} + \cdots = (u''_2 + 2u'_3)\partial^{-1} + (u''_3 + 2u'_4 + 2u_2 u'_2)\partial^{-2} + \cdots.
\]

\[
\frac{\partial u_2}{\partial t_3} \partial^{-1} + \frac{\partial u_3}{\partial t_3} \partial^{-2} + \cdots = (3u'''_3 + 3u'_4 + 6u_2 u'_2 + u''''_2)\partial^{-1} + \cdots.
\]

( \( \cdot' = \partial(\cdot)/\partial x. \) ) Comparing the coefficients of \( \partial^{-1} \) and \( \partial^{-2} \) we have

\[
\frac{\partial u_2}{\partial t_2} = u''_2 + 2u'_3,
\]

\[
\frac{\partial u_3}{\partial t_2} = u''_3 + 2u'_4 + 2u_2 u'_2,
\]

\[
\frac{\partial u_2}{\partial t_3} = 3u'''_3 + 3u'_4 + 6u_2 u'_2 + u''''_2,
\]

Eliminating \( u_3 \) and \( u_4 \) we obtain the KP equation. \( \square \)
• kp hierarchy

= set of compatibility conditions for the linear problem for

\[ \Psi = \Psi(t; z): L\Psi = z\Psi, \quad \frac{\partial \Psi}{\partial t_n} = B_n \Psi. \] (z: spectral parameter)

\[ L \] satisfies (kp) \iff \exists \tau(t) \ (tau function) such that

\[ \Psi(t; z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} e^{\sum t_n z^n}, \]

\[ (t = (t_n)_{n=1,2,...}, \ t - [z^{-1}] = \left( t_n - \frac{z^{-n}}{n} \right)_{n=1,2,...}, \]

and \( \tau(t) \) satisfies a series of bilinear differential equations (the Hirota equations).
• Solutions of the KP hierarchy are parametrised by $\infty$-dimensional Grassmann manifold (the Sato Grassmann manifold).

• Hirota equations = defining equations of the Grassmann manifold (Plücker relations)

• $\infty$-dimensional symmetry:

$GL(\infty)$ acts on the Sato Grassmann manifold $= GL(\infty)/P_{\infty/2}$.

(cf. finite dimensional Grassmann manifold $= GL(N)/P$,

\[
P = \left\{ \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ \hline 0 & & * \\ & \cdots & * \\ & \cdots & * \end{pmatrix} \right\}.
\]
Variants:

• (KP) + constraint $L^2 = \partial^2 + 2u$

  $\implies$ KdV hierarchy, which contains the KdV equation for $u$.

  This has the symmetry of $sl(2, \mathbb{C}[t]) \oplus$ (central extension), i.e., $A_1^{(1)}$-type affine Lie algebra.

• (KP) + constraint $L^* = -\partial L \partial^{-1}$

  (Notation: $(a(x) \partial^n)^* := (-\partial)^n a(x)$ is the formal adjoint operator.)

  $\implies$ BKP hierarchy, which has the symmetry of $so(2\infty + 1)$ ($B_\infty$-type).

• There are CKP and DKP hierarchies corresponding to $C_\infty$ and $D_\infty$ type symmetries, but the definitions are involved.

  (Usually defined by the Hirota bilinear equations.)
Toda lattice hierarchy: $\phi, u_n, \bar{u}_n$: unknown functions of $s$, $t = (t_n)_{n \in \mathbb{Z}, n \neq 0}$.

\[
L = e^{\phi} e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + u_3 e^{-2\partial_s} + \cdots ,
\]

\[
\bar{L}^{-1} = e^{\phi} e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \bar{u}_3 e^{2\partial_s} + \cdots ,
\]

\[
B_n = \begin{cases} 
(L^n)_0 > 0 + \frac{1}{2} (L^n)_0 , & (n > 0), \\
(\bar{L}^{-n})_0 < 0 + \frac{1}{2} (\bar{L}^{-n})_0 , & (n < 0).
\end{cases}
\]

Notation:

- $e^{n\partial_s} f(s) = f(s + n)$: difference operator.
- $A = \sum_{n \in \mathbb{Z}} a_n e^{n\partial_s} \rightarrow A_S = \sum_{n \in S} a_n e^{n\partial_s}$ for $S = "{>0}"$, "{<0}" and "{0}".

Toda lattice hierarchy: (Lax representation)

\[
(Toda) \quad \frac{\partial L}{\partial t_n} = [B_n, L], \quad \frac{\partial \bar{L}}{\partial t_n} = [B_n, \bar{L}], \quad (n \in \mathbb{Z}, n \neq 0).
\]
• Parametrisations of solutions, \( \tau \) function etc. are known.

• \( n = \pm 1 \implies \) the 2d Toda equation:
  \[ \dot{\phi}_{t_1t_{-1}}(s) = e^{\phi(s-1)-\phi(s)} - e^{\phi(s)-\phi(s+1)}. \]

• 2d Toda eq. + constraint \( \phi(s + 2) = \phi(s) \)
  \( (+ \text{ change of variables}) \implies \) Sine-Gordon eq.

• (Toda) + constraint: \( L = \overline{L}^{-1} \)
  \( \implies \) 1d Toda hierarchy (which contains the Toda lattice for \( \phi \)).
§ 4 Dispersionless hierarchies

Replace

- \( \partial, e^{\partial s} \rightarrow \) commutative symbols.
- commutator \([,] \rightarrow \) Poisson bracket \(\{,\}\).

\[ \implies \text{dispersionless KP/Toda lattice hierarchies.} \]

**Dispersionless KP hierarchy:** \( \partial^n \rightarrow w^n, \{w, x\} = 1. \)

\[ \mathcal{L} = w + u_2(t)w^{-1} + u_3(t)w^{-2} + \cdots, \quad \mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}. \]

\[ (\mathcal{P} = \sum_{n \in \mathbb{Z}} a_n w^n \rightarrow \mathcal{P}_{\geq 0} := \sum_{n \geq 0} a_n w^n.) \]

**dKP hierarchy:** \( \frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\} \quad (n = 1, 2, \ldots). \)
dispersionless **Toda lattice hierarchy:** \( e^{n \partial s} \rightarrow w^n, \{w, s\} = w.\)

\[
\mathcal{L} = e^\phi w + u_1 + u_2 w^{-1} + u_3 w^{-2} + \cdots ,
\]

\[
\tilde{\mathcal{L}}^{-1} = e^\phi w^{-1} + \tilde{u}_1 + \tilde{u}_2 w + \tilde{u}_3 w^2 + \cdots ,
\]

\[
\mathcal{B}_n = \begin{cases} 
(L^n)_{>0} + \frac{1}{2} (L^n)_0, & (n > 0), \\
(\tilde{L}^{-n})_{<0} + \frac{1}{2} (\tilde{L}^{-n})_0, & (n < 0).
\end{cases}
\]

\((A = \sum_{n \in \mathbb{Z}} a_n w^n \rightarrow A_S := \sum_{n \in S} a_n w^n \text{ for } S = "> 0", "< 0" \text{ and } "0".)\)

**dToda hierarchy:** \[
\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \frac{\partial \tilde{\mathcal{L}}}{\partial t_n} = \{\mathcal{B}_n, \tilde{\mathcal{L}}\}, \quad (n \in \mathbb{Z}, n \neq 0).\]

For **dKP/dToda hierarchies,** \(\infty\)-dimensional symmetries \((w_\infty\text{-algebra}),\) parametrisation of solutions \(\longleftrightarrow\) canonical transformations are known. ([Takasaki-T.] 1991–1995)
§5 Dispersionless Hirota equations

(Maybe you feel flavour of complex analysis...)

First obtained in [Takasaki-T. (1995)] as a limit of the differential Fay identity (⊂ Hirota eq.) for KP.

Teo’s formulation (2002)

\[ \mathcal{L}(t; w) = w + u_1(t)w^{-1} + u_2(t)w^{-2} + \cdots. \]

\( k(t; z) \): inverse function of \( \mathcal{L}(t; w) \) with respect to \( w \):

\[ \mathcal{L}(t; k(t; z)) = z, \quad k(t; \mathcal{L}(t; w)) = w. \]

Grunsky coefficients \( b_{mn} \) of \( k(t; z) \) (... for the Bieberbach conjecture):

\[ (dH1) \quad \log \frac{k(t; z_1) - k(t; z_2)}{z_1 - z_2} = - \sum_{m,n=1}^{\infty} b_{mn}(t)z_1^{-m}z_2^{-n}. \]
In other words,

\[ \mathcal{L}^n + \sum_{m=1}^{\infty} nb_{nm}(t) \mathcal{L}^{-m} = \text{(polynomial in } w) = (\mathcal{L}^n)_{\geq 0}. \]

In particular

\[ (dH2) \quad k(t; z) = z + \sum_{m=1}^{\infty} b_{1,m} z^{-m}. \]

**Theorem**

\[ \mathcal{L}(t; w): \text{ solution of } dKP \]

\[ \iff \text{ There exists } \mathcal{F}(t) \text{ such that } \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} = -mn b_{mn}(t). \]
(dH1&2) rewritten in terms of $\mathcal{F}(t) \Rightarrow$

\[
\text{dispersionless Hirota eq.:
}\]

\[
(e^{D(z_1)} D(z_2) \mathcal{F} = -\frac{\partial_1 (D(z_1) - D(z_2)) \mathcal{F}}{z_1 - z_2},
\]

which $\mathcal{F}(t)$ should satisfy. (Notations: $D(z) := \sum \frac{z^{-n}}{n} \frac{\partial}{\partial t_n}$.)

Remark: $\tau$ of KP (with $\hbar$) = $\exp(\hbar^{-2} \mathcal{F} + O(\hbar^{-1}))$.

(∃ similar theorem for dToda.)
§6 Dispersionless KP and Löwner equation

Unexpected relation of the (chordal) Löwner equation and the dispersionless KP hierarchy was found by

- Gibbons-Tsarev (1999): for $t_1$ and $t_2$.
- Mañas-Martínez Alonso-Medina (2002): proof by “$S$ function” ($\equiv \log$ (solution of the auxiliary linear problem of KP))).

(Radial (i.e., original) Löwner equation corresponds to the dispersionless Toda.)
Chordal Löwner equation:

\[ H = \{ \text{Im} \, z > 0 \} : \text{the upper half plane.} \]

∪

\[ K_\lambda (\lambda \in [0, a]): \text{growing hull of } H, \, K_0 = \emptyset. \]

\[ g(\lambda; z) : H \setminus K_\lambda \sim H: \text{conformal mapping normalised as} \]

\[ g(\lambda; z) = z + a_1(\lambda)z^{-1} + O(z^{-2}) \quad (z \to \infty), \quad g(0; z) = z. \]

\[ \implies \exists \, U(\lambda) \text{ s.t.} \]

\[ \frac{\partial g}{\partial \lambda}(\lambda; z) = \frac{1}{g(\lambda; z) - U(\lambda)} \frac{da_1}{d\lambda} : \text{Chordal Löwner equation.} \]
One variable reduction of dKP

Theorem

$\mathcal{L}(t; w)$ is a solution of dKP such that:

$\exists$ functions $\lambda(t)$ & $f(\lambda, w)$: $\mathcal{L}(t; w) = f(\lambda(t), w)$.

$\implies$

(i) $f(\lambda, w)$ is the inverse function of a solution $g(\lambda, z)$ of the chordal Löwner eq. $(f(\lambda, g(\lambda, z)) = z, \ g(\lambda, f(\lambda, w)) = w.)$

(ii) $\lambda(t)$ satisfies $\frac{\partial \lambda}{\partial t_n} = \frac{\partial \Phi_n}{\partial w}(\lambda; U(\lambda)) \frac{\partial \lambda}{\partial t_1} \ (n = 1, 2, \ldots)$

Here, $\Phi_n(\lambda; w) = (f(\lambda, w)^n)_{\geq 0}$: Faber polynomial of $g$.

(Polynomial part of $f(\lambda, w)^n$ w.r.t. $w$.)
Conversely:

**Theorem**

\( g(\lambda, z) \): solution of chordal Löwner equation.

\( f(\lambda, w) = w + O(w^{-1}) \): inverse function of \( g \),
i.e., \( f(\lambda, g(\lambda, z)) = z, \ g(\lambda, f(\lambda, w)) = w \).

\( \lambda(t) \): solution of

\[
\frac{\partial \lambda}{\partial t_n} = \frac{\partial \Phi_n}{\partial w}(\lambda; U(\lambda)) \frac{\partial \lambda}{\partial t_1} \quad (n = 1, 2, \ldots)
\]

\( \Rightarrow \mathcal{L}(t, w) := f(\lambda(t), w) \) is a solution of dKP.

Remark: The equation for \( \lambda(t) \) is solved implicitly by the relation

\[
t_1 + \sum_{n=2}^{\infty} t_n \frac{\partial \Phi_n}{\partial w}(\lambda; U(\lambda)) = R(\lambda).
\]

\( R(\lambda) \): arbitrary generic function. (Tsarev’s generalised hodograph method.)
§7 Other examples

- mKP hierarchy $\leftrightarrow$ chordal Löwner-like equation
  
  (Mañas-Martínez Alonso-Medina)

- Toda hierarchy $\leftrightarrow$ radial Löwner equation (T.-Teo-Zabrodin, ...)

- BKP hierarchy $\leftrightarrow$ quadrant Löwner equation (T.)

- DKP hierarchy $\leftrightarrow$ annulus Löwner (Goluzin-Komatu) equation
  
  (Akhmedova-Zabrodin)
dBKP hierarchy: dKP + constraint: $\mathcal{L}(w) = -\mathcal{L}(-w)$.

Quadrant Löwner equation:

$$\frac{\partial g}{\partial \lambda} = \frac{g}{V^2 - g^2} \frac{d u}{d \lambda}.$$
QUESTION

WHY do Löwner type equations give solutions of dispersionless integrable hierarchies?

Thank you for your attention.
References (mainly those cited in the talk)

On dispersionless integrable hierarchies and Löwner equations:


On dispersionless integrable hierarchies:
