Convergence of loop erased random walks on a planar graph to a chordal SLE(2) curve

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International Workshop on Conformal Dynamics and Loewner Theory
2014/11/23
Introduction (1)

- Model

- Loop erasure
**Introduction (2)**

- curve in $D \rightarrow$ curve in $\mathbb{H}$
- curve $\rightarrow$ dynamics of domain
- Represent by Loewner equation
\( \gamma : [0, \infty] \rightarrow \mathbb{C} : \) a simple curve, \( \gamma(0) = 0, \gamma(\infty) = \infty, \gamma(0, \infty) \subset \mathbb{H} \), 
\( g_t : \mathbb{H} \setminus \gamma[0, t] \rightarrow \mathbb{H} : \) conformal map, \( |g_t(z) - z| \rightarrow 0 \) (\( z \rightarrow \infty \)).

If \( \gamma \) is parametrized by half plane capacity \( \lim_{z \rightarrow \infty} z(g_t(z) - z) = 2t \), 
\( g_t \) satisfies the following differential equation

\[
\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z,
\]

where \( U(t) := g_t(\gamma(t)) \) and \( U(t) \) is a \( \mathbb{R} \)-valued continuous function.

We call \( U(t) \) the driving function of \( \gamma \).

Rem. We can consider that a curve \( \gamma \) is described by the driving function \( U(t) \).
Candidate for scaling limits

We consider a candidate for scaling limits of the driving function of discrete random curves. Let $\gamma$ be the scaling limit of some discrete random curve $\gamma_\delta$ connecting two distinct boundary points $a$ and $b$ of $D$. Since there are several conjectures in critical systems, we assume that $\gamma$ satisfies the following properties.

- Domain Markov property
- Conformal invariance

Let $\phi : D \to \mathbb{H}$: conformal map, $\phi(a) = 0, \phi(b) = \infty$. Then, the driving function $U(t)$ of $\phi(\gamma)$ satisfies the following properties.

- Stationary increment
- Independent increment
- Scale invariance

Therefore, $U(t)$ must be a Brownian motion $\sqrt{\kappa}B_t$ of variance $\kappa$. 
Schramm-Loewner evolution

We construct a candidate for scaling limits. Let \( \kappa > 0 \), \( B_t \): 1-dim standard Brownian motion with \( B_0 = 0 \).

**chordal SLE_\kappa**

A chordal Schramm-Loewner evolution with parameter \( \kappa > 0 \) (chordal SLE_\( \kappa \)) is the random family of conformal map \( g_t \) obtained from the chordal Loewner equation driven by \( \sqrt{\kappa}B_t \)

\[
\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z,
\]

The following proposition is very important and basic in SLE theory.

**Proposition (existence of chordal SLE_\( \kappa \) curve)**

With probability 1, we can define the non-self crossing random curve \( \gamma \) which generates SLE_\( \kappa \). We call \( \gamma \) a chordal SLE_\( \kappa \) curve in \( \mathbb{H} \) from 0 to \( \infty \).
SLE in simply connected domains

We define SLE in any simply connected domain. 
\( \gamma : \) a chordal SLE\(_{\kappa} \) curve in \( \mathbb{H} \) from 0 to \( \infty \) 
\( D \subset \subset \mathbb{C} : \) simply connected domain, \( a \in \partial D, \ b \in \partial D, \) 
\( \phi : D \rightarrow \mathbb{H} : \) conformal map, \( \phi(a) = 0, \phi(b) = \infty. \) 
Although \( \phi \) is not unique, the distribution of \( \phi^{-1}(\gamma) \) is independent 
of the choice of the map up to time change. 
We consider SLE\(_{\kappa} \) curves in \( D \) as unparametrized curves.

definition

chordal SLE\(_{\kappa} \) curve in simply connected domains

We call \( \phi^{-1}(\gamma) \) a chordal SLE\(_{\kappa} \) curve in \( D \) from \( a \) to \( b. \)

metric on the space of unparametrized curves

\[
d_{\mathcal{U}}(\gamma_1, \gamma_2) := \inf_{\alpha} \left[ \sup_{0 \leq t \leq 1} d_*(\gamma_1(t), \gamma_2 \circ \alpha(t)) \right].
\]

where \( d_* \) is the spherical metric on \( \widehat{\mathbb{C}} \) and the infimum is taken 
over all reparametrization \( \alpha. \)
We consider properties that SLE curves are expected to have. Let $\mu_D(a, b)$: the law of a chordal $SLE_\kappa$ curve in $D$ from $a$ to $b$.

The following two properties immediately follow from the definition of SLE.

- **domain Markov property**
  $\mu_D(a, b)(\cdot |\gamma[0, t]) = \mu_{D \setminus \gamma[0, t]}(\gamma(t), b)$

- **conformal invariance**
  $f : D \rightarrow f(D)$: conformal map.
  $f \circ \mu_D(a, b) = \mu_{f(D)}(f(a), f(b))$

These properties are important to characterize SLE curves.
The scaling limit of discrete models

- $\kappa = 2$
  - loop-erased random walk (LERW)

- $\kappa = \frac{8}{3}$ (conjecture)
  - self-avoiding walk

- $\kappa = 3$
  - critical Ising model

- $\kappa = 4$
  - harmonic explorer, Gaussian free field

- $\kappa = \frac{16}{3}$
  - FK Ising model (FK percolation, $q = 2$)

- $\kappa = 6$
  - critical percolation

- $\kappa = 8$
  - uniform spanning tree Peano curve
chordal and radial
I will return to talk about LERW.
Known results (radial)

We will introduce known results for radial.

Lawler, Schramm, Werner (2004)

- $G$: square lattice (triangular lattice),
- LERW starting from an inner point $\Rightarrow$ radial SLE$_2$ w.r.t. $d_U$

In above paper, they construct the basic idea of proof of convergence to a SLE curve. So, it is the origin of the research on SLE and scaling limit.

Yadin and Yehudayoff extend to more general graphs.

Yadin, Yehudayoff (2011)

- $G$: planar irreducible graph + invariance principle,
- LERW starting from an inner point $\Rightarrow$ radial SLE$_2$ w.r.t. $d_U$
Known results (chordal)

We will introduce known results for chordal.

**Zhan (2008)**

*G*: square lattice,
LERW connecting two boundary points $\Rightarrow$ chordal SLE$_2$ w.r.t. $d_U$

I extend Zhan’s result in a similar setting to Yadin and Yehudayoff. In the rest of this talk, I will talk about my main result precisely.
A planar-irreducible graph $G = (V, E)$ is a directed weighted graph, where $0 \in V \subset \mathbb{C}$ is a set of vertices, and $E : V \times V \to [0, \infty)$ is a set of edges. We define a planar-irreducible graph $G$ that satisfies the following conditions:

- $G$ is a planar graph, i.e., every two edges do not intersect except for vertices.
- For any compact set $K \subset \mathbb{C}$, $\# \{ v \in V : v \in K \} < \infty$.
- For any $u \in V$, $\sum_{w \in V} E(u, w) < \infty$.
- Let $p(u, v) := \frac{E(u, v)}{\sum_{w \in V} E(u, w)}$. The Markov chain $S(\cdot)$ on $V$ with the transition probability $p(u, v)$ is irreducible.

We call this Markov chain $S(\cdot)$ a natural random walk on $G$. 
Notation

For $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$, let $s_0 := \max\{k \geq 0 : \omega_0 = \omega_k\}$, $s_m := \max\{k \geq 0 : \omega_{s_{m-1}+1} = \omega_k\}$, $l := \min\{m \geq 0 : \omega_{s_m} = \omega_n\}$.

**loop erasure**

$L[\omega] := (\omega_{s_0}, \omega_{s_1}, \ldots, \omega_{s_l})$.

**time-reversal**

$\omega^- := (\omega_n, \omega_{n-1}, \ldots, \omega_0)$.

**dual walk**

Suppose that there exists an invariant measure $\pi$ for a natural random walk $S(\cdot)$ on $G$ such that $0 < \pi(v) < \infty$ for any $v \in V$. Then, we can define the dual walk $S^*(\cdot)$ with the following transition probability $p^*$.

$$p^*(u, v) := \frac{\pi(v)}{\pi(u)} p(v, u).$$
Invariance principle

For $\delta > 0$, the graph $G_\delta = (V_\delta, E_\delta)$ defined by

$V_\delta = \{\delta u : u \in V\}, \quad E_\delta = \{(\delta u, \delta v) : E(u, v) > 0\}.$

Let $S^x_\delta(\cdot)$ be a natural random walk on $G_\delta$ starting at $x \in V_\delta$.

In this talk, invariance principle mean that the following.

invariance principle
A natural random walk trajectory weakly converges to a 2-dim Brownian motion trajectory locally uniformly for starting points.
$D \subseteq \mathbb{C}$: a bounded simply connected domain, $a \in \partial D$, $b \in \partial D$. 
$\partial D$ is locally connected and locally analytic at $a$ and $b$.

$G = (V,E)$: a planar irreducible graph,
$\Gamma_{\delta}^{a,b}$: a natural random walk on $G_\delta$ started at $a$ and stopped on exiting $D$ and conditioned to hit $\partial D$ at $b$,
$\gamma_{\delta}^{a,b}$: the loop erasure of $\Gamma_{\delta}^{a,b}$ (LERW),
$\eta_{a,b}$: a chordal SLE$_2$ curve in $D$ from $a$ to $b$.

**Theorem (S,2014)**

Suppose that $S^x_\delta$ and $(S^*)_\delta^x$ satisfy invariance principle. Then,

$$\gamma_{\delta}^{a,b} \Rightarrow \eta_{a,b}^x (\delta \to 0) \text{ w.r.t } d_U$$
Outline of proof

**step 1** Estimate for driving function $U(t)$
- In order to estimate the driving function of LERW, we must find a "nice" martingale observable for LERW which converges to some conformal invariant.
- By using martingale observable, we estimate expectation and variance of increment of driving function $U(t)$.

**step 2** Convergence w.r.t. driving function $U(t)$

**step 3** Convergence w.r.t. $d_U$
Let \((\gamma^b,a)^{-} = \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)\),
\(\gamma_{[0,j]}\) is a linear interpolation of \((\gamma_0, \gamma_1, \ldots, \gamma_j)\),
\(\phi : D \rightarrow \mathbb{H}\) be a conformal map with \(\phi(a) = 0, \phi(b) = \infty\),
\(U(t) : \) the driving function of \(\phi(\gamma)\),
\(g_t : \) the Loewner chain driven by \(U(t)\), \(t_j = \text{hcap}(\phi(\gamma_{[0,j]}))\),
\(U_j := U(t_j), \phi_j := g_{t_j} \circ \phi, D_j := D \setminus \gamma_{[0,j]}\).
Martingale observable

For \( \forall \epsilon > 0, m = m(\epsilon) := \inf\{ j \geq 1 : t_j \geq \epsilon^2 \text{ or } |U_j - U_0| \geq \epsilon \}. \)

\( w \in V_\delta \cap D, A = \phi^{-1}([-1, 1]), \)

\( H_j^{(\delta)}(x, \cdot) \): the hitting probability of RW starting at \( x \) in \( D_j \).

martingale observable for LERW

Let

\[
M_j := \frac{H_j^{(\delta)}(w, \gamma_j)}{H_j^{(\delta)}(b, \gamma_j)} H_0^{(\delta)}(b; A).
\]

Then, \( M_j \) is a martingale and

\[
M_j = -\frac{2}{\pi} \text{Im} \left( \frac{1}{\phi_j(w) - U_j} \right) + O(\epsilon^3), \quad 0 \leq j \leq m
\]
Because $M_j$ is a martingale and $m$ is a bounded stopping time,

$$E[M_m - M_0] = 0$$

By substituting

$$M_j = -\frac{2}{\pi} \text{Im} \left( \frac{1}{\phi_j(w) - U_j} \right) + O(\epsilon^3), \quad 0 \leq j \leq m,$$

we get

$$E \left[ \text{Im} \left( \frac{1}{\phi_m(w) - U_m} \right) - \text{Im} \left( \frac{1}{\phi(w) - U_0} \right) \right] = O(\epsilon^3). \quad (1)$$

We consider Taylor expansion of the left hand side of this equation.
Let \( f(u, v) = 1/(u - v) \). Using

\[
t_m = O(\epsilon^2), \quad U_m - U_0 = O(\epsilon),
\]

we Taylor-expand \( f(\phi_m(w), U_m) - f(\phi(w), U_0) \) with respect to \( \phi_m(w) - \phi(w) \) and \( U_m - U_0 \), up to \( O(\epsilon^3) \). Observing imaginary part of this Taylor expansion, we get by (1)

\[
\text{Im}\left(\frac{1}{(\phi(w) - U_0)^2}\right)\mathbb{E}[U_m - U_0] + \text{Im}\left(\frac{1}{(\phi(w) - U_0)^3}\right)\mathbb{E}[(U_m - U_0)^2 - 2t_m] = O(\epsilon^3).
\]

By two different choices of \( w \), we get the following estimates

\[
\mathbb{E}[U_m - U_0] = O(\epsilon^3),
\]

\[
\mathbb{E}[(U_m - U_0)^2 - 2t_m] = O(\epsilon^3).
\]
Now, we can estimate for expectation and variance of increment of the driving function $U(t)$ at time 0. Because LERW has the domain Markov property, we may consider at the time 0 in another domain $D \setminus \gamma[0, t]$ instead of at time $t$ in $D$. 
Because we should estimate uniformly, we introduce a domain class $\mathcal{D}$.

$D$: a simply connected domain, $\partial D$ is locally connected, $a, b$: two distinct points on $\partial D$,

$\phi: D \to \mathbb{H}$: a conformal map with $\phi(a) = 0, \phi(b) = \infty$.

Let $p = \phi^{-1}(i)$, $\text{rad}_p(D) := \inf\{|z - p| : z \notin D\}$.

$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

$\psi: D \to \mathbb{D}$: a conformal map with $\psi(b) = 1, \psi(p) = 0, \psi(a) = -1$.

**class $\mathcal{D}$**

Let $\mathcal{D} = \mathcal{D}(r, R, \eta)$ be the collection of all quadruplets $(D, a, b, p)$ such that

- $\text{rad}_p(D) \geq r$
- $D \subset R\mathbb{D}$
- $\psi^{-1}$ has analytic extension in $\{z \in \mathbb{C} : |z - 1| < \eta\}$
key Lemma

For any $r > 0$, $R > 0$, $\eta > 0$. there exists a constant $C > 0$ and a number $\epsilon_0 > 0$ such that for each positive $\epsilon < \epsilon_0$, there exists $\delta_0 > 0$ such that if $(D, a, b, p) \in D(r, R, \eta)$ and $0 < \delta < \delta_0$, then we get the following estimates

$$|E[U_m - U_0]| \leq C\epsilon^3,$$

and

$$|E[(U_m - U_0)^2 - 2t_m]| \leq C\epsilon^3.$$
Step 2  Convergence w.r.t. driving function $U(t)$

- By Key Lemma and Skorokhod embedding theorem, we can prove that the driving function $U(t)$ weakly converges to $\sqrt{2}B_t$

Step 3  Convergence w.r.t. $d_U$

- We improve to convergence w.r.t. the metric $d_U$ by using Sun and Sheffield’s sufficient condition. Then, we need convergence of $\gamma$ and $\gamma^-$ w.r.t. driving function.
H. Suzuki, Convergence of loop erased random walks on a planar graph to a chordal SLE(2) curve, Kodai Math J. 37, (2014), 303-329


