$L^d$-Loewner chains with quasiconformal extensions

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Ordinary differential equations for evolution families

- Radial Loewner equations for $\varphi_{s,t} = f_t^{-1} \circ f_s$:
  \[
  \frac{d}{dt} w_t = G(w_t, t) \quad \text{with} \quad G(z, t) := -zp(z, t)
  \]

- Chordal Loewner equations (by transforming everything from $\mathbb{H}^+$ to $\mathbb{D}$):
  \[
  \frac{d}{dt} w_t = G(w_t, t) \quad \text{with} \quad G(z, t) := (1 - z)^2p(z, t)
  \]

- Berkson-Porta representation for semigroups $\{\phi_t\} \subset \text{Hol}(\mathbb{D})$:
  \[
  \frac{d}{dt} w_t = G(w_t) \quad \text{with} \quad G(z) := (z - \tau)(\bar{\tau}z - 1)p(z)
  \]

$\implies$ Unified treatment of the above differential equations!!
Evolution family of order $d$

**Definition 7.1**

A family of holomorphic self-maps of the unit disk $(\varphi_{s,t})_{0 \leq s \leq t < \infty}$, is an **evolution family of order $d$** with $d \in [1, \infty]$, or in short an $L^d$-evolution family, if

1. $\varphi_{s,s}(z) = z$,
2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < \infty$,
3. for all $z \in \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{z,T} \in L^d([0, T], \mathbb{R})$ such that
   \[
   |\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\zeta)d\zeta
   \]
   for all $0 \leq s \leq u \leq t \leq T$.

We denote the family of all evolution families of order $d$ by $\mathcal{EF}^d$. 
Herglotz vector fields of order $d$

**Definition 7.3**

A weak holomorphic vector field of order $d \in [1, \infty)$ on $\mathbb{D}$ is a function $G : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ with the following properties:

1. For all $z_0 \in \mathbb{D}$, the function $G(z_0, t)$ is measurable on $t \in [0, \infty)$,
2. For all $t_0 \in [0, \infty)$, the function $G(z, t_0)$ is holomorphic on $\mathbb{D}$,
3. For any compact set $K \subset \mathbb{D}$ and for all $T > 0$, there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|G(z, t)| \leq k_{K,T}(t) \quad (1)$$

for all $z \in K$ and for almost every $t \in [0, T]$.

Furthermore, $G$ is said to be a Herglotz vector field of order $d$ if $G(\cdot, t)$ is the infinitesimal generator of a semigroup of holomorphic functions for almost all $t \in [0, \infty)$.

- $\mathbb{H}V^d$: a family of all Herglotz vector field of order $d$
Theorem 7.5

Let \( d \in [1, \infty] \) be fixed. Then, for any \( \varphi_{s,t} \in \text{EF}^d \), there exists an essentially unique \( G \in \text{HV}^d \) such that

\[
\varphi_{s,t}(z) = G(\varphi_{s,t}(z), t)
\]

for all \( z \in \mathbb{D} \) and almost all \( t \in [0, \infty) \), where \( \dot{\varphi}_{s,t} := \frac{\partial \varphi_{s,t}}{\partial t} \).

Conversely, for any \( G \in \text{HV}^d \), a unique solution of (2) with the initial condition \( \varphi_{s,s}(z) = z \) is an evolution family of order \( d \).

It determines one-to-one correspondence between \( (\varphi_{s,t}) \in \text{EF}^d \) and \( G \in \text{HV}^d \).
Definition 7.6

A Herglotz function of order $d \in [1, \infty]$ on the unit disk $\mathbb{D}$ is a function $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ with the following properties:

1. For all $z_0 \in \mathbb{D}$, the function $p(z_0, t)$ belongs to $L^d_{\text{loc}}([0, \infty), \mathbb{C})$ on $t \in [0, \infty)$,
2. For all $t_0 \in [0, \infty)$, the function $p(z, t_0)$ is holomorphic on $\mathbb{D}$,
3. $\Re p(z, t) \geq 0$ for all $z \in \mathbb{D}$ and $t \in [0, \infty)$.

Then, $\mathbb{H}F^d$ denotes the family of all Herglotz functions of order $d$.

Theorem 7.8

Let $G \in \mathbb{H}V^d$. Then there exist an essentially unique $p \in \mathbb{H}F^d$ and a measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$ s.t.,

$$G(z, t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z, t).$$ (3)

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$.

Conversely, for a given $p \in \mathbb{H}F^d$ and a measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$, the equation (3) determines an essentially unique $G \in \mathbb{H}V^d$. 
(essentially) 1-to-1 correspondence between;

\[(\varphi_{s,t}) \in EF^d \quad \text{with the initial condition} \quad \varphi_{s,s}(z) = z\]

\[(B) : G(z, t) = (z - \tau(t))(\tau(t)z - 1)p(z, t)\]
**Loewner chains of order \( d \)**

**Definition 7.9**

A family of holomorphic maps of the unit disk \((f_t)_{t \geq 0}\) is called a **Loewner chain of order** \( d \) with \( d \in [1, \infty) \), or in short an **\( L^d \)-Loewner chain**, if

1. \( f_t : \mathbb{D} \to \mathbb{C} \) is univalent for each \( t \in [0, \infty) \),
2. \( f_s(\mathbb{D}) \subset f_t(\mathbb{D}) \) for all \( 0 \leq s < t < \infty \),
3. for any compact set \( K \subset \mathbb{D} \) and all \( T > 0 \), there exists a non-negative function \( k_{K,T} \in L^d([0, T], \mathbb{R}) \) such that

\[
|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\zeta) d\zeta
\]

for all \( z \in K \) and all \( 0 \leq s \leq t \leq T \).

**\( LC^d \):** a family of all Loewner chains of order \( d \)

- There exist \( (f_t) \in LC^d \) s.t. \( \Omega[(f_t)] := \bigcup_{t \geq 0} f_t(\mathbb{D}) \neq \mathbb{C} \).
  - In fact \( f_s(\mathbb{D}) \) is allowed to be equal to \( f_t(\mathbb{D}) \) for some \( s < t \) (even for all \( s < t \))
- For any compact subset \( K \subset \mathbb{D} \), there exists \( (f_t) \in LC^d \) s.t. \( f_s(K) \not\subset f_t(K) \) for some \( s < t \),
Theorem 7.10

For any \((f_t) \in \text{LC}^d\), if we define

\[ \varphi_{s,t}(z) := (f_t^{-1} \circ f_s)(z) \quad (z \in \mathbb{D}, 0 \leq s \leq t < \infty) \]

then \((\varphi_{s,t}) \in \text{EF}^d\). Conversely, for any \((\varphi_{s,t}) \in \text{EF}^d\), there exists a \((f_t) \in \text{LC}^d\) such that the following equality holds

\[ (f_t \circ \varphi_{s,t})(z) = f_s(z) \quad (z \in \mathbb{D}, 0 \leq s \leq t < \infty). \]

We can deduce that a Loewner chain of order \(d\) satisfies the differential equation

\[ \dot{f}_t(z) = f'_t(z)(z - \tau(t))(1 - \tau(t)\overline{z})p(z, t), \]

where \(\tau : [0, \infty) \rightarrow \overline{\mathbb{D}}\) is a measurable function and \(p \in \text{HF}^d\). 


\( \mathcal{L}[(\varphi_{s,t})] \): a family of \((f_t) \in \mathbb{LC}^d \) associated with \((\varphi_{s,t}) \in \mathbb{EF}^d \) satisfying \( f_0 \in \mathcal{S} \)

**Theorem 7.11**

Let \((\varphi_{s,t}) \in \mathbb{EF}^d \). Then there exists a unique \((f_t) \in \mathcal{L}[(\varphi_{s,t})] \) such that \( \Omega[(f_t)] \) is \( \mathbb{C} \) or an Euclidean disk in \( \mathbb{C} \) whose center is the origin. Furthermore;

- The following 4 statements are equivalent;
  - 1. \( \Omega[(f_t)] = \mathbb{C} \),
  - 2. \( \mathcal{L}[(\varphi_{s,t})] \) consists of only one function,
  - 3. \( \beta(z) = 0 \) for all \( z \in \mathbb{D} \), where
    \[
    \beta(z) := \lim_{t \to +\infty} \frac{|\varphi'_{0,t}(z)|}{1 - |\varphi_{0,t}(z)|^2},
    \]
  - 4. there exists at least one point \( z_0 \in \mathbb{D} \) such that \( \beta(z_0) = 0 \).

- On the other hand, if \( \Omega[(f_t)] \neq \mathbb{C} \), then the Euclidean disk is written by
  \[
  \Omega[(f_t)] = \left\{ w : |w| < \frac{1}{\beta(0)} \right\}
  \]
  and the other \((g_t) \in \mathcal{L}[(\varphi_{s,t})] \) has an expression
  \[
  g_t(z) = \frac{h(\beta(0)f_t(z))}{\beta(0)} \quad (h \in \mathcal{S}).
  \]
In 1972, Becker applied for Loewner’s method to derive a quasiconformal extension criterion.

**Theorem (Becker 1972)**

Let \( k \in [0, 1) \) be a constant. Suppose that \( (f_t) \) is a (classical) radial Loewner chain for which the Herglotz function \( p \) in the Loewner PDE satisfies

\[
p(z, t) \in U(k) := \left\{ w \in \mathbb{C} : \left| \frac{w - 1}{w + 1} \right| \leq k \right\} \subseteq \mathbb{H}
\]

for all \( z \in \mathbb{D} \) and almost all \( t \geq 0 \). Then the function \( F \) defined by

\[
F(z) := \begin{cases} 
  f_0(z), & z \in \mathbb{D}, \\
  f_{\log |z|} \left( \frac{z}{|z|} \right), & z \in \mathbb{C} \setminus \mathbb{D},
\end{cases}
\]

is a \( k \)-quasiconformal mapping of \( \mathbb{C} \).
Main result

Theorem (H. 2014)

Let $d \in [1, \infty)$ and $k \in [0, 1)$. Let $(f_t) \in \mathcal{LC}^d$ and $p \in \mathcal{HF}^d$ associated with $(f_t)$. If $p$ satisfies

$$p(z, t) \in U(k)$$

for all $z \in D$ and almost all $t \in [0, 1)$, then

1. $f_t$ has a $k$-quasiconformal extension to $\hat{\mathbb{C}}$ for each $t \in [0, \infty)$.  
2. $\Omega[(f_t)] = \mathbb{C}$.

In this theorem, any superfluous assumption is not imposed on $\tau$. 
Theorem (Gumenyuk and H, 2014)

Let \( d \in [1, \infty) \) and \( k \in [0, 1) \). Let \( (f_t) \in \text{LC}^d \) and \( p \in \text{HF}^d \) associated with \( (f_t) \). If \( \tau \in \overline{D} \) is constant and \( p \) satisfies

\[
p(z, t) \in U(k)
\]

for all \( z \in D \) and almost all \( t \in [0, \infty) \), then \( f_t \) has a \( k \)-quasiconformal extension to \( \hat{\mathbb{C}} \) for each \( t \in [0, \infty) \).

Then we can also prove it for the case when \( \tau \) is of the form

\[
\tau(t) = \sum_{i=1}^{n} \tau_i \cdot \chi_{I_i}(t),
\]

where \( \tau_i \in \overline{D}, n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty, I_i := [t_{i-1}, t_i) \) and \( \chi_{I_i} \) is a characteristic function.
Theorem (H. 2014, Roth 1997)

Let $G \in \mathbb{H}V^d$. Consider the family $\{G(z, t)\}$ such that

1. $\{G(\cdot, t)\}$ forms a normal family for almost every fixed $t \in [0, \infty)$.
2. $\{G_n(z, t)\}_{n \in \mathbb{N}} \subset \{G(z, t)\}$ is a sequence converging weakly to $G \in \mathbb{H}V^d$.

Then, a sequence of evolution families $\{(\varphi_{s,t}^n)\}_{n}$ of order $d$ associated with $\{G_n\}_{n}$ converges locally uniformly to $(\varphi_{s,t})$ associated with $G$ on $(z, t) \in \mathbb{D} \times [s, \infty)$.

Proposition (Gumenyuk and H, 2014)

Let $(f_t) \in \mathbb{L}C^d$. Let $p \in \mathbb{H}F^d$ and $\tau$ be a measurable function associated with $(f_t)$. Suppose that $\tau \in \overline{\mathbb{D}}$ is a constant, and there exist uniform constants $C_1, C_2 > 0$ such that

$$C_1 < \text{Re}p(z, t) < C_2$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$. Then $\Omega[(f_t)] = \mathbb{C}$.
Definition

A family \( \{g_t\}_{t \geq 0} \) of holomorphic maps of the unit disk is called a \textbf{decreasing Loewner chain of order} \( d \) with \( d \in [1, \infty) \) if it satisfies the following conditions:

1. \( g_t \) is univalent on \( \mathbb{D} \) for each \( t \in [0, \infty) \),
2. \( g_0(z) = z \) and \( g_s(\mathbb{D}) \supset g_t(\mathbb{D}) \) for all \( 0 \leq s < t < \infty \),
3. for any compact set \( K \subset \mathbb{D} \) and all \( T > 0 \), there exists a non-negative function \( k_{K,T} \in L^d([0, T], \mathbb{R}) \) such that

\[
|g_s(z) - g_t(z)| \leq \int_s^t k_{K,T}(\zeta) d\zeta \tag{4}
\]

for all \( z \in K \) and all \( 0 \leq s \leq t \leq T \).

- We denoted by \( \text{DLC}^d \) a family of all decreasing Loewner chain of order \( d \).
- \( \partial_t g_t(z) = (z - \sigma(t))(\overline{\sigma(t)}z - 1) \partial_z g_t(z)q(z, t) \)
- \( \Lambda[(g_t)] := \bigcap_{t \geq 0} g_t(\mathbb{D}) \)
Let $d \in [1, \infty]$. A family $\{\omega_{s,t}\}_{0 \leq s \leq t}$ of holomorphic self-maps of the unit disk $\mathbb{D}$ is called a reverse evolution family of order $d$ with $d \in [1, \infty]$ (or in short, an $L^d$-reverse evolution families) if the following conditions are fulfilled:

1. $\omega_{s,s}(z) = z$,
2. $\omega_{s,t} = \omega_{s,u} \circ \omega_{u,t}$ for all $0 \leq s \leq u \leq t < \infty$,
3. for all $z_0 \in \mathbb{D}$ and for all $T_0 > 0$ there exists a non-negative function $k_{z_0,T_0} \in L^d([0,T_0], \mathbb{R})$ such that

$$|\omega_{s,u}(z_0) - \omega_{s,t}(z_0)| \leq \int_u^t k_{z_0,T_0}(\zeta) d\zeta$$

for all $0 \leq s \leq u \leq t \leq T_0$.

$\text{REF}^d$: a family of all reverse evolution family of order $d$. 
Theorem (H, 2014)

Let $d \in [1, \infty]$ and $k \in [0, 1)$. Let $(f_t) \in \text{LC}^d$ and $(p, \tau) \in \text{BP}$ associated with $(f_t)$. We denote by $T^* \in [0, \infty]$ the smallest number such that $p(\mathbb{D}, t) \in i\mathbb{R}$ for almost all $t \in (T^*, \infty)$. Suppose that $T^* \neq 0$ and $p \in \text{HF}^d$ satisfies

$$|p(z, t) - q(z, t)| \leq k \cdot |p(z, t) + q(z, t)|$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, where $q \in \text{HF}^d$. Let $(\omega_{s,t}) \in \text{REF}^d$ associated with $(q, \tau) \in \text{BP}$ and $(g_t) \in \text{DLC}^d$ associate with $(\omega_{s,t})$. Then, $f_t$ and $g_t$ has continuous extensions to $\overline{\mathbb{D}}$ for each $t \in [0, T^*)$, and $\Phi$ defined by

$$
\begin{cases}
\Phi(z) = f_0(z), & z \in \mathbb{D}, \\
\Phi \left( \frac{1}{g_t(e^{i\theta})} \right) = f_t(e^{i\theta}), & \theta \in [0, 2\pi) \quad \text{and} \quad t \in [0, T^*),
\end{cases}
$$

is a $k$-quasiconformal mapping on $\Delta[(g_t)]$ onto $\Omega[(f_t)]$.

- $\Delta[(g_t)] := \left\{ \frac{1}{w} : w \in \widehat{\mathbb{C}} \setminus \Lambda[(g_t)] \right\}$
Theorem (H, 2014)

Let \( d \in [1, \infty) \) and \( k \in [0, 1) \). Let \((g_t) \in \text{DLC}^d\) and \((q, \tau) \in \text{BP}\) associated with \((g_t)\). If \( q \) satisfies

\[
\left| \frac{q(z, t) - 1}{q(z, t) + 1} \right| \leq k
\]

for all \( z \in \mathbb{D} \) and almost all \( t \in [0, \infty) \), then \( g_t \) has a \( k \)-quasiconformal extension to \( \hat{\mathbb{C}} \) for each \( t \in [0, \infty) \). Further, \( \Lambda[(g_t)] \) consists of one point in \( \overline{\mathbb{D}} \).
Thank you for your attention!!